Section: Topology of the Real Numbers

1. Let $S \subseteq \mathbb{R}$. Mark each statement as True or False. Explain briefly.
   (a) $\text{int}S \cap \text{bd}S = \emptyset$
   (b) $\text{int}S \subseteq S$
   (c) $\text{bd}S \subseteq S$
   (d) $S$ is open iff $S = \text{int}S$
   (e) If $x \in S$, then $x \in \text{int}S$ or $x \in \text{bd}S$.
   (f) Every neighbourhood is an open set.
   (g) The union of any collection of open sets is open.
   (h) The union of any collection of closed sets is closed.
   (i) The intersection of any collection of open sets is open.
   (j) The intersection of any collection of closed sets is closed.
   (k) The set $\mathbb{R}$ of real numbers is neither open nor closed.
   (l) $\text{bd}S = \text{bd}(\mathbb{R}\setminus S)$
   (m) $\text{bd}S \subseteq \mathbb{R}\setminus S$
   (n) $S \subseteq S' \subseteq \text{cl}S$
   (o) $S$ is closed iff $\text{cl}S \subseteq S$.
   (p) $S$ is closed iff $S' \subseteq S$.
   (q) If $x \in S$ and $x$ is not an isolated point of $S$, then $x \in S'$.

2. Find the interior and the boundary of each set.
   (a) $\left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$
   (b) $[0,3] \cup (3,5)$
   (c) $\left\{ r \in \mathbb{Q} : 0 < r < \sqrt{2} \right\}$
   (d) $\left\{ r \in \mathbb{Q} : r \geq \sqrt{2} \right\}$
   (e) $[0,2] \cap [2,4]$  

3. Classify each of the following sets as open, closed, neither or both. Find their closure.
   (a) $\left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$
   (b) $\mathbb{N}$
   (c) $\mathbb{Q}$
   (d) $\bigcap_{n=1}^{\infty} (0, \frac{1}{n})$
   (e) $\left\{ x : |x - 5| \leq \frac{1}{2} \right\}$
(f) \( \{ x : x^2 > 0 \} \)

4. Let \( S \) and \( T \) be subsets of \( \mathbb{R} \). Find a counterexample for each of the following.
   (a) If \( P \) is the set of all isolated points of \( S \), then \( P \) is a closed set.
   (b) Every open set contains at least two points.
   (c) If \( S \) is closed, then \( \text{cl}(\text{int}S) = S \).
   (d) If \( S \) is open, then \( \text{int}(\text{bd}S) = S \).
   (e) \( \text{bd}(\text{cl}S) = \text{bd}S \)
   (f) \( \text{bd}(\text{bd}S) = \text{bd}S \)
   (g) \( \text{bd}(S \cup T) = (\text{bd}S) \cup (\text{bd}T) \)
   (h) \( \text{bd}(S \cap T) = (\text{bd}S) \cap (\text{bd}T) \)

5. Let \( S \) be a bounded infinite set and let \( x = \text{sup}S \). Prove: If \( x \notin S \), then \( x \in S' \).

6. Prove: If \( x \) is an accumulation point of the set \( S \), then every neighbourhood of \( x \) contains infinitely many points of \( S \).

7. Let \( S \) and \( T \) be subset of \( \mathbb{R} \). Prove the following
   (a) \( \text{cl}(\text{cl}S) = \text{cl}S \)
   (b) \( \text{cl}(S \cup T) = (\text{cl}S) \cup (\text{cl}T) \)

8. Mark each statement as True or False. Explain briefly.
   (a) A set \( S \) is compact iff every open cover of \( S \) contains a finite subcover.
   (b) Every finite set is compact.
   (c) No infinite set is compact.
   (d) If a set is compact, then it has a maximum and a minimum.
   (e) If a set has a maximum and a minimum, then it is compact.
   (f) Some unbounded sets are compact.
   (g) If \( S \) is compact and \( x \) is an accumulation point of \( S \), then \( x \in S \).

9. Show that each subset of \( \mathbb{R} \) is not compact by describing an open cover for it that has no finite subcover.
   (a) \([1,3)\)
   (b) \([1,2) \cup (3,4]\)
   (c) \( N \)
   (d) \( \{ \frac{1}{n} : n \in N \} \)