Minicourse 4:
An Introduction to Fluid Dynamics

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Contents

0 Introduction

My goal for this week is to introduce you to several topics in fluid dynamics, and give you an appreciation of the beauty of the field, its diversity, and most importantly its interplay with various areas of mathematics. Starting with elementary vector fields, by the end I hope you have a sense for a variety of fluid flows, ranging from inviscid, potential, viscous, to highly viscous creeping flows. To study these flows we will use mathematics in several areas, such as Calculus, Vector Analysis, Complex Variables, Ordinary differential equations, Partial differential equations, Linear algebra, Numerical methods. I will review what I feel is necessary from these areas, but please interject and interrupt me whenever you would like to see more explanations.

To understand and visualize many of these flows we will use MATLAB in the afternoon laboratories. I will give you an introduction to this programming language and will give you detailed instructions on how to prepare your codes. Also, there are several of you that have some experience with MATLAB and by working in groups we can help each other. A useful summary of MATLAB commands is posted on the web, at `http://www.math.unm.edu/ nitsche/courses/375` [3]

Overall, my main goal is not to drill and bore you, but to have FUN, while learning something about this area, which (in my humble opinion) is applied math at its best! To start with, let me show you and discuss some sample fluid flows (I will show figures taken from [6]):

- **Flow past cylinder** (Viscous flow past cylinder, R=1.54 9.6-10,000 Inviscid potential flow past cylinder, Highly viscous creeping flow past cyliner)
- **Flow past block** (Creeping, Turbulent)
- **Flow past airfoil** (Symmetric flow past airfoil, higher Re, boundary layer separation past on airfoil, creeping flow past airfoil)
- **Separation, vortex generation** (Flow past vertical plate, Flow out of circular tube, Separation at corner, Separation on triangular wing)
- **Vortex ring** (Closeup, Leapfrogging)
- **Instability** (Vortex ring, Trailing vortices, Kelvin-Helmholtz)
- **Circulations in atmosphere and ocean** (Tornadoes, Tidal vortex)

The topics I will introduce are:
Day 1: Basic concepts of velocity fields. Point vortices.
Day 2: Inviscid flow (governing equations, streamfunction, method of images)
Day 3: Potential flow (potential function, conformal mapping)
Day 4: Viscous flow (equations, boundary layers, exact solutions)
Day 5: Creeping flow (equations, exact solutions)

1 Basic concepts

We begin today introducing some basic concepts and looking at some apparently randomly made-up velocity fields. On Day 2 we will derive the equations that real-life fluid flows satisfy (approximately). The holy grail in fluid dynamics is to find solutions to these equations, that is, velocity fields that satisfy them given certain initial and boundary conditions. Such solutions, if we can find them, can help better understand fluid flow and help to obtain better designs in engineering applications.

We will then show that the velocity field induced by a set of *point vortices* is a solution of the *inviscid planar Euler Equations*, and use them to construct solutions that satisfy certain boundary conditions.

To start with however, we will simply inspect velocity fields without making any direct connection to real fluid flows.

1.1 Velocity fields

Let D be a region in two or three-dimensional space filled with a fluid. Let $\mathbf{x} \in D$. It is described by its Cartesian coordinates $\mathbf{x} = (x, y, z) = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$. Alternatively, it could be described by cylindrical coordinates $\mathbf{x} = (r, \theta, z) = r\mathbf{e}_r + \theta\mathbf{e}_\theta + z\mathbf{e}_z$, spherical coordinates $\mathbf{x} = (\rho, \theta, \psi) = \rho\mathbf{e}_\rho + \theta\mathbf{e}_\theta + \psi\mathbf{e}_\psi$, or in terms of any other basis of $\Re^3$. So, if you write $\mathbf{x} = (1, 2, 3)$ we don't know what point this refers to unless the basis is specified. But, unless otherwise mentioned, we will imply Cartesian coordinates.

Imagine a fluid particle at position $\mathbf{x}$ at time t. It moves with a well-defined velocity $\mathbf{u}(\mathbf{x}, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)$. The velocity $\mathbf{u}$ is a **vector field**. At a given time t, it is a function that assigns to each position $\mathbf{x}$ a vector $\mathbf{u}$. Here are some example vector fields and their graphs

Example 1: $\mathbf{u}(x, y, z) = (U, 0, 0)$, some positive constant U

Example 2: $\mathbf{u}(x, y, z) = (x, y, z)$

Example 3: $\mathbf{u}(x, y, z) = (-y, x, 0)$

Example 4: $\mathbf{u}(x, y, z) = \dfrac{1}{2\pi}\dfrac{(-y, x, 0)}{x^2 + y^2}$ (this flow is induced by what is called a **point vortex**)

Exercise 1: Sketch graph of (a) $\mathbf{u}(x, y, z) = \dfrac{(-y, x, 0)}{x^2 + y^2}$, (b) $\mathbf{u}(x, y, z) = -\dfrac{(x, y, 0)}{\sqrt{x^2 + y^2}}$.

The above examples are very simple velocity fields which we can easily draw by hand. Why? Besides there simple form:

- All but the second examples are **2-dimensional** vector fields. The velocity (i) does not depend on z, and (ii) has zero component in the third dimension. Thus the velocity is fully described by its graph in the 2d plane (even though velocity exists at every point in space).
- For convenience all examples chosen are independent of time. Time independent velocities are called **steady** velocity fields. In such fields it is easy to view particle trajectories (see below). Flows that are not steady are called **unsteady**.

1.2 Streamlines, particle trajectories, streaklines

Suppose you freeze a velocity $\mathbf{u}(\mathbf{x}, t)$ at a fixed time and look at the instantaneous vector field at that instant. Then you can visualize curves $\mathbf{x}(s)$ that are tangent to the velocity at every point in space. That is, they satisfy that their tangent vector equals $\mathbf{u}(\mathbf{x}, t)$ at the given time. If in addition we specify one point on such a curve, the curves $\mathbf{x}(s)$ are given by the unique solution to the initial value problem (IVP)

$$\frac{d\mathbf{x}}{ds} = \mathbf{u}(\mathbf{x}, t = \textit{constant}) \ , \quad \mathbf{x}(s_0) = \mathbf{x}_0 \ . \tag{1.1}$$

These curves are called **integral curves** of the vector field, or **streamlines** of the flow. Let's draw some streamlines of the above vector fields. If a flow is steady, the streamlines do not change in time. If a flow is unsteady, the streamlines change with time. The streamlines at a fixed time t are called the **instantaneous streamlines** at time t.

Now, suppose a fluid particle at time t has position $\mathbf{x}(t)$. Then it moves with velocity $\mathbf{u}(\mathbf{x}, t)$. That is, it satisfies the ordinary differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t) \ , \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1.2}$$

This implies that at every instant in time its motion is tangent to the velocity field. The solution to this IVP is the **particle trajectory** of the particle initially at x_0.

If the velocity field is steady (time-independent) as in the above examples, then the particle trajectories are simply the streamlines of the velocity and are easy to visualize. However, if the velocity is not steady, that is, the velocity at a fixed point $\mathbf{x}$ changes in time, then the particle trajectories are not equal to the streamlines (the solutions to 1.1 and 1.2 are not the same) and are harder to visualize. We can visualize both streamlines and trajectories for any given field $u(\mathbf{x}, t)$ by solving the odes (1.1,1.2) numerically. Will do this in afternoon.

Finally, you may have heard of the word **streakline**. Suppose you have a possibly unsteady flow, such as the flow behind an oscillating rudder. If you insert dye into the flow at a fixed point x_0 in space at all times, the dye will show you the position at time t of all the particles which at time $t < t_0$ where at x_0. This is referred to as a streakline.

A streakline is also used to denote the position at time $t > 0$ of all particles which at time $t = 0$ lay on a curve. This is what you would see if you inject dye along a curve at time 0 and observed where this curve moves to at time $t > 0$. For us, what is important is: streaklines are not trajectories (nor streamlines) but are sample of a whole set of trajectories at a given instant, and reflect what you would see experimentally by injecting dye into the fluid. Of course, if the flow is steady, streaklines=trajectories=streamlines.

1.3 Divergence, vorticity, circulation, decomposition of velocity field, strain rates

The **divergence** of a vector field $\mathbf{u} = (u, v, w)$ is defined as

$$\text{div}\,\mathbf{u} = \nabla \cdot \mathbf{u} = u_x + v_y + w_z \tag{1.3}$$

Let's compute the divergence of the sample vectorfields in §1.1. Does anyone know the meaning of this scalar function associated to the vector field $\mathbf{u}$? The meaning of $\nabla \cdot \mathbf{u}$ can be deduced from the **Divergence theorem**,

$$\int_V \nabla \cdot \mathbf{u}\, dV = \int_D \mathbf{u} \cdot \mathbf{n}\, dS \tag{1.4}$$

where V is a closed volume bounded by D, and $\mathbf{n}$ is the outward unit normal vector. In words: "the volume integral of the divergence of $\mathbf{u}$ equals the flux (or flow rate) of $\mathbf{u}$ out of D." (Why flux? Draw figure showing that RHS = volume of fluid leaving domain per time interval.) Now imagine V to be a small volume within which $\nabla \cdot \mathbf{u}$ is approximately constant. Taking the limit as the volume size $\to 0$ you see that the divergence is the net outflux per unit volume.

In particular, if $\nabla \cdot \mathbf{u} > 0$ at a point, then there is a positive net outflux out of small neighbourhood, that is, the flow is **expanding**. If $\nabla \cdot \mathbf{u} < 0$, the flow is **compressing**. A flow for which $\nabla \cdot \mathbf{u} = 0$ everywhere is called **incompressible**. Check the divergence of sample vector fields.

The **vorticity** of a vector field is defined as the curl of $\mathbf{u}$,

$$\omega = \nabla \times u \tag{1.5}$$

The meaning of the vorticity can be deduced from **Stokes Theorem**,

$$\int_C \mathbf{u} \cdot \mathbf{T}\, ds = \int_D (\nabla \times \mathbf{u}) \cdot \mathbf{n}\, dS \tag{1.6}$$

where D is a surface bounded by C, $\mathbf{T}$ is a unit vector tangent to C, $\mathbf{n}$ is a unit vector normal to S, and $\mathbf{T}$ points counterclockwise with respect to $\mathbf{n}$ ($\mathbf{T}, \mathbf{n}$ satisfies right hand rule). If we now consider a small circular surface D of radius R *normal to the vorticity* on which the vorticity is approximately constant, then the magnitude of the RHS is $|\omega|\pi R^2 = u_{av}2\pi R$, where u_{av} is the average counterclockwise velocity component. Therefore the flow has an average component that rotates around the vector ω with angular velocity

$$\frac{u_{av}}{R} \approx \frac{|\omega|}{2}$$

The approximation becomes $=$ in the limit as $R \to 0$. Thus, vorticity measures the amount of **rotation** of the flow.

A flow in which $\nabla \times \mathbf{u} = 0$ everywhere is called **irrotational**.

The meaning of the vorticity is also well elucidated by the following. Consider the velocity $\mathbf{u}(\mathbf{y})$, where $\mathbf{y} = \mathbf{x} + \mathbf{h}$ is near a basepoint $\mathbf{x}$. That is, $h = |\mathbf{h}|$ is small. (Here, for simplicity, I omitted the time dependence of $\mathbf{u}$.) Using Taylor series expansion about $\mathbf{x}$ can show that

$$\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) + D(\mathbf{x})\mathbf{h} + \frac{1}{2}\omega(\mathbf{x}) \times \mathbf{h} + O(h^2) \tag{1.7}$$

where D is a symmetric matrix.

Exercise 2: Show (1.7). (a) write down Taylor series in 1 variable; (b) write down Taylor series in 3 variables; (c) write (b) in vector form $\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) + (\nabla \mathbf{u})\,\mathbf{h} + O(h^2)$ (d) split matrix $\nabla \mathbf{u}$ into symmetric and antisymmetric parts

Thus, locally, near $\mathbf{x}$, the velocity is a sum of 3 components. Lets look at each component, by looking at the particle motion due to each component separately. Let $\mathbf{x}$ be a fixed vector, as before. We want to look at the velocity at $\mathbf{y}$, namely $d\mathbf{y}/dt$. Note that since $\mathbf{x}$ is constant

$$\frac{d\mathbf{y}}{dt} = \frac{d\mathbf{h}}{dt}$$

So the question is, what is the solution to the three parts

$$\frac{d\mathbf{h}}{dt} = \mathbf{u}(\mathbf{x}) \tag{1.8a}$$

$$\frac{d\mathbf{h}}{dt} = D(\mathbf{x})\mathbf{h} \tag{1.8b}$$

$$\frac{d\mathbf{h}}{dt} = \frac{1}{2}\omega(\mathbf{x}) \times \mathbf{h} \tag{1.8c}$$

(For (1.8b) we need to diagonalize D. Use linear algebra results for symmetric matrices.) The results of above shows that to first order in $\mathbf{h}$, the velocity is a sum of a translation, a deformation and a rotation. The matrix D is called the **deformation matrix** (or **rate of strain tensor**), its eigenvalues and eigenvectors are the **principal strainrates** of **principal axes of strain**, respectively.

Lets investigate the change of volume of a small parcel of fluid under the strain field D in a small time Δt. Result: to first order in Δt, volume is amplified by factor $1+(\lambda_1+\lambda_2+\lambda_3)\Delta t$. This again indicates significance of divergence of the flow. (Why?)

Back to vorticity. For any oriented curve C, the line integral $\Gamma_C = \int_C \mathbf{u}\cdot\mathbf{T}\,ds$ is called the **circulation** of the flow around C. Note that the orientation of the curve matters, since it determines the sign of the tangent vector, and $\Gamma_{-C} = -\Gamma_C$. By Stokes Theorem the circulation equals the integral normal vorticity in a surface bounded by the curve.

Finally, some comments about planar 2-dimensional flows of the form $\mathbf{u}(x,y,z) = (u(x,y), v(x,y), 0)$. Since the velocity does not depend on z we commonly drop the z-dependence and write $\mathbf{u}(x,y) = (u(x,y), v(x,y))$. However, remember that these flows are defined at all point in space. Planar 2d simply means invariance and zero flow in z-direction. The vorticity of a planar flow is

$$\omega = \nabla \times \mathbf{u} = (0, 0, v_x - u_y) \tag{1.9}$$

Thus it is a vector that points in the z-direction. This makes sense since the flow rotates in a plane normal to this vector and it can only rotate in the x-y plane. In planar flow the convention is to let $\mathbf{n} = (0,0,1)$ and $\mathbf{T}$ point in the counterclockwise direction (so the pair $\mathbf{n}, \mathbf{T}$ satisfies the right hand rule). As a result the circulation in the plane is defined with counterclockwise orientation and, by Stokes Theorem,

$$\oint_C \mathbf{u}\cdot\mathbf{T}\,ds = \oint_C u\,dy - v\,dx = \int_D v_x - u_y dA \tag{1.10}$$

The small circle around the integral sign denotes counterclockwise orientation of a closed curve. You may recognize this planar version of Stokes Theorem as Greens Theorem that we covered in vector calculus (Math 264).

Note: we refer to these 2d flows as planar to distinguish them from for example axisymmetric or flows on the sphere, in which there are also only two dimensions.

Exercise 4: Compute the vorticity of the sample vector fields above.

Note that in Example 4, even though the flow is rotating, the vorticity is zero everywhere, except at the origin! As we know, vorticity measures local rotation, and in this case an infinitesimal parcel of fluid away from the origin circles about the origin but without rotating about its axis. However, the circulation about any curve enclosing the origin is nonzero.

Exercise 5: (a) Compute the circulation around the unit circle centered at the origin. (b) Use Stokes Theorem to show that the circulation around any other curve inclosing the origin is the same, namely 1.

By Stokes theorem, this implies that the vorticity is zero everywhere except at the origin, but its integral over any region enclosing the origin is 1. This is the definition of the δ-function. The "δ-function" is not really a function. It is defined by what happens when you integrate it:

$$\int_D \delta(x,y)dA = \begin{cases} 0 \text{ if } (0,0) \notin D \\ 1 \text{ if } (0,0) \in D \end{cases} \tag{1.11}$$

It was introduced and used mainly by physicists (Poisson 1815, Fourier 1822, Cauchy 1823, 1827, Kirchoff 1882,1891, Heaviside 1893, 1899, Paul Dirac 1926) but was dismissed by many mathematicians as non-rigorous, until the theory of distributions was developed (Sobolev 1935) which makes it a rigourous mathematical object (not a function, but a distribution) (Schwartz 1945-50). (I got this from an article by Balakrishnan on the web, Resonance, Aug 2003.

1.4 Motion of point vortices

The flow in Example 4 induced by a δ-function of vorticity is denoted by a point vortex (vorticity is concentrated in a point). We already know that the flow induced by a point vortex is incompressible, irrotational except at the vortex, decays like 1/R, where R is the distance to the vortex (show this). The circulation of a point vortex is called the **point vortex strength**. By translation, the flow induced by a point vortex at (x_0, y_0) of strength Γ is

$$\mathbf{u}(x,y) = \frac{\Gamma}{2\pi} \frac{(-(y-y_0), x-x_0)}{(x-x_0)^2 + (y-y_0)^2} \tag{1.12}$$

By superposition, the flow induced by 2 vortices of strength Γ_1, Γ_2 at positions $(x_1, y_1), (x_2, y_2)$ is

$$\mathbf{u}(x,y) = \frac{\Gamma_1}{2\pi} \frac{(-(y-y_1), x-x_1)}{(x-x_1)^2 + (y-y_1)^2} + \frac{\Gamma_2}{2\pi} \frac{(-(y-y_2), x-x_2)}{(x-x_2)^2 + (y-y_2)^2} \tag{1.13}$$

and the flow induced by N vortices of strength Γ_k and position (x_k, y_k), $k = 1, \ldots, N$ is

$$\mathbf{u}(x,y) = \sum_{k=1}^{N} \frac{\Gamma_k}{2\pi} \frac{(-(y-y_k), x-x_k)}{(x-x_k)^2 + (y-y_k)^2} \tag{1.14}$$

where (x, y) is not one of the point vortices.

What if (x, y) is one of the point vortices? How do the points evolve under their self-induced velocity?? The self-induced velocity of a point vortex is defined to be zero (it doesnt move in its radially symmetric flow field). As a result, it only moves due to the velocity field induced by the other point vortices. Thus, the flow induced by N vortices (as before, of strength Γ_k and position (x_k, y_k), $k = 1, \ldots, N$) at one of the points, say (x_j, y_j) is

$$\mathbf{u}(x_j,y_j) = \sum_{\substack{k=1 \\ k\neq j}}^{N} \frac{\Gamma_k}{2\pi} \frac{(-(y_j-y_k), x_j-x_k)}{(x_j-x_k)^2 + (y_j-y_k)^2} \tag{1.15}$$

Analyze: what is self-induced motion of one pt vortex, how do 2 pt vortices move in self-induced field (to start with, look at $\Gamma_1 = \pm\Gamma_2$)?

Fact: The motion of $N \leq 3$ vortices is never chaotic (the flow is said to be integrable). However, the motion of particles in the flow induced by 3 or more vortices can be chaotic.

1.5 LAB PROJECT

- Streamfunction
  (a) Derive streamfunction for incompressible flow, and describe relation to instantaneous streamlines.
  (b) Find streamfunction for set of pt vortices

- PROJECT 1: Plot instantaneous streamlines for an range of pt vortices
  (a) 2 pts, counterclockwise
  (b) 3 pts, of your choice
  (c) any other of your choice

  turn in plots that specify position and stregths of pts

- PROJECT 2 plot vortex motion by solving set of odes
  (a) 2 pts, $\Gamma_1 = -\Gamma_2$
  (b) 2 pts, $\Gamma_1 < 0 < \Gamma_2$
  (c) 2 pts, $\Gamma_1 = \Gamma_2$
  (d) 2 pts, $0 < \Gamma_1 < \Gamma_2$
  (e) 3 pts, of your choice
  (f) any other of your choice

  Turn in plots that specify position and stregths of pts

Sample MATLAB code to plot streamlines

```
[x,y]=meshgrid([-2:.01:2],[-2:.01;2]); %what does this do?
z=psi(x,y);
contour(x,y,z), pause    %what does this do?
axis equal

xx=0
yy=0:.1:1;
v=psi(xx,yy);
contour(x,y,z,v)   %what does this do?
axis equal
```

2 Inviscid, Incompressible Flow

In this section we derive equations that model the behaviour of real fluid flows. The model does not account for the effects of viscosity and is thus a good approximation for flows in which these effects are negligible. Viscous effects may be negligible for example far from walls bounding the fluid domain, if the viscosity is very small, or in superfluids, where viscosity is practically zero. We will discuss the effects of viscosity, particularly near walls, later in §4.

The equations modelling fluid flows are based on physical principles that real flows satisfy. The equations in this section were derived by Euler in the 1750s. For incompressible flows, which is mainly what we will consider, he obtained the equations in their present form. For compressible equations, one needs a further equation which was obtained only much later.

We will then show that the planar point vortex flows discussed in the previous section are solutions to the Euler Equations (and thus approximate real fluids which are approximately planar and have negligible viscosity).

2.1 The Euler Equations (for incompressible flow)

2.1.1 The Material derivative

We first derive a differentiation operator we need to be familiar with to understand the following equations.

Suppose $\mathbf{x}(t) = (x(t), y(t), z(t))$ is the trajectory of a fluid particle, and $f(\mathbf{x}(t), t)$ is the value of some quantity assigned to the particle. For example, it could be the density or the temperature at that particle (scalar functions), or the velocity at that particle (a vector valued function). Note that the function $f(\mathbf{x}(t), t) = F(t)$ is a function of time only. We would like to know how this quantity changes in time, that is, we want to find dF/dt. This derivative measures the change of f in time on the particle $\mathbf{x}(t)$. For example, if f is temperature, then dF/dt would be the rate of change of temperature that you would feel if you were sitting on the particle.

Such quantities that are described by their values on a moving particle are called **Lagrangian variables**. Their derivative with respect to time is called the **material derivative** since it denotes changes on a material particle, and is often denoted by Df/DT or df/dt. Using the chain rule and using the fact that $d\mathbf{x}/dt = \mathbf{u}(\mathbf{x}(t), t)$ we find that

$$
\begin{aligned}
\frac{dF}{dt} &= \frac{D}{Dt}\big[f(x(t), y(t), z(t), t)\big] \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t} \\
&= \frac{\partial f}{\partial x}u(x,y,z,t) + \frac{\partial f}{\partial y}v(x,y,z,t) + \frac{\partial f}{\partial z}w(x,y,z,t) + \frac{\partial f}{\partial t} \\
&= \big[(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla)f\big](x(t), y(t), z(t), t)
\end{aligned} \tag{2.1}
$$

The resulting formula for the differentiation operator D/Dt (or d/dt) can be abreviated as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla \tag{2.2}$$

For example, the acceleration of a particle at $\mathbf{x}(t)$ is the time derivative of its velocity

$$\frac{D\mathbf{u}}{Dt} = (\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla)\mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} \tag{2.3}$$

The basic idea is simple (chain rule), just make sure you understand when to use ∂ and when you use d. In (2.3) the brackets around $\mathbf{u}\cdot\nabla$ denote that you take this dot product first. Otherwise you may read it as $\mathbf{u}\cdot(\nabla\mathbf{u})$ and it is a little confusing what the gradient of a vector is.

2.1.2 Conservation of mass

The physical principle is: *mass is neither created nor destroyed.* How do we obtain an governing equation from this? Consider a fixed region W in the fluid. The boundary of W is a surface denoted by ∂W. The physical principle states that *the rate of change of mass in W = the rate at which mass crosses ∂W* or, more precisely

$$\textit{the rate of increase of mass in } W = \textit{the rate at which mass enters } \partial W \tag{PPI}$$

The mass in W is

$$\lim_{||\Delta V||\to 0} \sum_k \rho_k \Delta V_k = \int_W \rho(\mathbf{x}, t) dV(\mathbf{x}) = m(t) \tag{2.4}$$

that is, it is the volume integral of density over the region W. Here we are assuming that there is a well-defined density $\rho(x, t)$ at every point x. This is called the **continuum assumption**. We will also assume that the velocity and density are sufficiently smooth to apply the main operations of calculus.

The mass in W can depends on time, since the density at a point depends on time. For example if the mass in W increases the density inside has to increase. The left hand side of the physical principle PPI is the rate of change of the mass

$$\frac{dm}{dt} = \frac{d}{dt}\int_W \rho(\mathbf{x},t)dV(\mathbf{x}) = \int_W \frac{\partial\rho}{\partial t}(\mathbf{x},t)dV(\mathbf{x}) \tag{2.5}$$

(The time derivative enters as a partial derivative since the region of integration W is time independent.)

To find an expression for the right hand side of PPI we look at a small piece of the boundary, ΔS, and assume that ρ, $\mathbf{u}$ and $\mathbf{n}$ are constant on this piece. This assumption holds in the limit as $\Delta S \to 0$. Then (show figure in class) we see that the volume of fluid entering through ΔS in time Δt is $-\mathbf{u}\cdot\mathbf{n}\Delta S\Delta t$. So the mass entering in this time is $-\rho_k\mathbf{u}_k\cdot\mathbf{n}_k\Delta S\Delta t$. Dividing by Δt and taking the limit as $\Delta t \to 0$ (to get the rate at which mass enters) and summing over all ΔS, then taking limit as $\Delta S \to 0$ get

$$-\lim_{||\Delta_S||\to 0}\sum_k \rho_k\mathbf{u}_k\cdot\mathbf{n}_k\Delta S_k = -\int_{\partial W}(\rho\mathbf{u}\cdot\mathbf{n})(\mathbf{x},t)dS(\mathbf{x}) \tag{2.6}$$

According to PPI, we set (2.6) equal to (2.5) to get

$$\int_W \frac{\partial\rho}{\partial t}(\mathbf{x},t)dV(\mathbf{x}) = -\int_{\partial W}(\rho\mathbf{u}\cdot\mathbf{n})(\mathbf{x},t)dS(\mathbf{x}) \tag{2.7}$$

or, after applying the Divergence Theorem

$$\int_W [\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})](\mathbf{x},t)dV(\mathbf{x}) = 0 \tag{2.8}$$

Equation (2.8) states that the integral is zero for any region of integration W. If we assume the integrand is continuous, this implies that the integrand must be identically zero.

Exercise 1: Proof this by contradiction

As a result we get the law of **conservation of mass**.

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{2.9}$$

which can be rewritten as

$$\frac{D\rho}{Dt} = -(\nabla\cdot\mathbf{u})\rho \tag{2.10}$$

Exercise 2: Deduce (2.10) from (2.9). (You are in fact proving a vector identity. There are many identities often used, some are listed in the appendix as examples. Which one of those did you prove in this exercise?).

Again, this makes perfect sense. If $\nabla \cdot \mathbf{u} > 0$ then the fluid near a material particle expands and the density at the material particle decreases. That is, if $\nabla \cdot \mathbf{u} > 0$ then $D\rho/DT < 0$ (using that the density is always positive, $\rho > 0$).

Equation (2.9) is our first equation of motion. In general we do not know the fluid velocity $\mathbf{u} = (u, v, w)$ nor the density function. These are 4 unknowns and the above is only one equation, which is now enough to solve for the unknowns.

2.1.3 Conservation of momemtum

The next equation of motion (whose derivation we will only outline) is based on the physical principle of conservation of momentum *the rate of change of momentum of W = force applied to it.*, or, more precisely

$$\textit{the rate of increase of momentum of } W = \textit{force applied to it in inward direction.} \tag{PPII}$$

This is Newton's second law, which you may be more familiar with when mass is constant: $F = ma$. (Momentum is mv, so for a nonconstant mass $m(t)$ moving with velocity v this law reads $F = d(mv)/dt$. Here the equation is not that simple because different portions of the fluid move with different velocities.)

We apply this law to a region W_t moving with the fluid. The left hand side of PPII is

$$\frac{d}{dt}\int_{W_t} \rho \mathbf{u} dV = \int_{W_t} \rho \frac{D\mathbf{u}}{Dt} dV \tag{2.11}$$

Proving this equality is not trivial and we will skip it. Having proved the conservation of mass in detail we only wish to indicate how the final equations follow from physical principles.

For the right hand side of PPII we will assume that the force of stress on the boundary of W acts normal to it, and is given by the pressure, so that the

$$\text{force across } \partial W \text{ per unit area} = p(\mathbf{x}(t), t)\mathbf{n}\ .$$

As a result the total inward force on W is

$$-\int\limits_{\partial W_t} p\mathbf{n} dA = -\int\limits_{W_t} \nabla p \, dV \tag{2.12}$$

where again, we used the divergence theorem and skipped some details to obtain the right hand side. By equating (2.11) and (2.12) and assuming the integrands are continuous we obtain the equation of **conservation of momentum**,

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p \tag{2.13}$$

where we ignored other forces such as gravity that could be acting on the body.

2.1.4 Euler equations for incompressible flow

Equations (2.9, 2.13) are a set of 4 equations, but now we have 5 unknowns!, density ρ, velocity $\mathbf{u}$, and pressure p. For incompressible flow, Euler completed the system of equations by $\nabla \cdot \mathbf{u} = 0$. For general flows, the system is completed by enforcing concervation of energy. It turns out that the condition of incompressibility $\nabla \cdot \mathbf{u} = 0$ is equivalent to conservation of kinetic energy. So now we have 5 equations for 5 unknowns that hold in the domain of the fluid D. These need to be complemented by boundary conditions. The boundary conditions $\mathbf{u} \cdot \mathbf{n} = \mathbf{U}_{wall} \cdot \mathbf{n}$ on ∂D, where $\mathbf{U}_{wall}$ is the velocity of the wall, enforce that there is no flow going through the walls. (If wall steady, condition is $\mathbf{u} \cdot \mathbf{n} = 0$.) In summary, Eulers equations for incompressible flow, ignoring other external body forces, are

$$\frac{D\rho}{Dt} = 0 \tag{2.14a}$$

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p \tag{2.14b}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2.14c}$$

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{U}_{wall} \cdot \mathbf{n} \text{ on } \partial D \tag{2.14d}$$

with initial conditions at $t = 0$, $\mathbf{u}(\mathbf{x}, 0)$, given. Here the first 5 equations hold in the interior D. So, our goal: solve these equations!

2.1.5 Streamfunction and vorticity equation for planar flows

We need to introduce an important quantity that exists in incompressible flows. From Vector Analysis you know that for incompressible 2D flows $\mathbf{u}(x, y, t) = (u(x, y, t), v(x, y, t))$ with $\nabla \cdot \mathbf{u} = u_x + v_y = 0$ there exists a function $\psi(x, y, t)$ such that

$$\frac{\partial \psi}{\partial x} = -v, \frac{\partial \psi}{\partial y} = u. \tag{2.15}$$

or, $\mathbf{u} = (\psi_y, -\psi_x) = \nabla^{\perp}\psi$. (To prove: define $\psi(x, y)$ as a line integral $\int_{(0,0)}^{(x,y)} u dy - v dx$, show it is path independent so that this is well-defined, find the partial derivatives.) This function is called the **streamfunction** since for fixed t, its level curves are streamlines of the flow. Why? Let $x(s), y(s)$ be a streamline: $x'(s) = u(x, y, t), y'(s) = v(x, y, t)$, then

$$\frac{d}{ds}\big[\psi(x(s), y(s))\big] = \frac{\partial \psi}{\partial x}\frac{dx}{ds} + \frac{\partial \psi}{\partial y}\frac{dy}{ds} = -vu + uv = 0$$

In particular, on the boundary $\psi = 0$.

Exercise: Find streamfunction for the incompressible examples on page 4. and plot its level curves. Write down streamfunction for N point vortices.

Furthermore,

$$\Delta\psi = -v_x + u_y = -w \ . \tag{2.16}$$

So if the vorticity is known, streamfunction is solution to a Poisson equation with Dirichlet boundary conditions $\psi = 0$ on ∂D. This solution can be written down in terms of Green's functions of the Poisson equation, which is known for many domains D. (In particular, if $D = \Re^2$, then $\psi = -\frac{1}{2\pi}\int \omega(\mathbf{x}') \log|\mathbf{x}-\mathbf{x}'|\, d\mathbf{x}' \approx -\frac{1}{2\pi} \sim \Gamma_k \log|\mathbf{x}-\mathbf{x}_k$, that is, approximately a sum of point vortices.) So if you know the vorticity, then you can find ψ, therefore can find $\mathbf{u}$. Together with the equation (2.18) derived below, this forms a basis of many numerical methods. A streamfunction also exists in 3D, although we skip the details here.

Our last remark regards the evolution of vorticity. For simplicity, suppose the flow is **homogeneous**, that is, $\rho = \rho_0$ is constant, and take the curl of the balance of momentum equation,

$$\nabla \times \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p\right]$$

Using the identities

$$\begin{aligned} \nabla \times (\nabla f) &= 0 && (2.17a)\\ \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) &= \mathbf{u}\times \text{curl}\ \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} && (2.17b)\\ \nabla \times (\mathbf{u}\times\omega) &= \mathbf{u}\ \text{div}\ \omega - \omega\ \text{div}\ \mathbf{u} + (\omega\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\omega && (2.17c) \end{aligned}$$

it follows that

$$\frac{D\omega}{Dt} = -(\omega\cdot\nabla)\mathbf{u} \tag{2.18}$$

This equation holds in 3D. It says that the vorticity of a material particle can be stretched by the term on the right hand side. Thus the right hand side is a called the **stretching term**. Because of this term it is not known whether smooth initial vorticity fields in 3D can blow-up develop and develop singularities in finite time. (This is one of the Million Dollar questions!)

For planar 2D flows, the vorticity evolution equation simplifies significantly since the vorticity vector points in the z-direction and the gradient vector $\nabla = (\partial_x, \partial_y)$ lies in the x, y-plane, so that the operator $\omega\cdot\nabla$ is zero. Thus in 2D

$$\frac{D\omega}{Dt} = 0 \tag{2.19}$$

This means the vorticity of a material particle stays constant. No stretching can occur and as a result it can be shown that smooth solutions remain smooth for all times.

Moreover, for the 2D point vortices we have been looking at, (2.19) implies that their circulation remains constant for all times. Thus the solution of the Euler Equations with point vortex initial data is simply given by the motion of these point vortices in their self-induced velocity field, such as we have been computing yesterday. Summary: the point vortex motion given by (1.15) is solves the Euler Equation with point vortex initial data, in infinite domain, and homogeneous density: 2.14a is satisfied (ρ constant), 2.14b is satisfied (since strengths Γ_j are constant, so the equivalent eqn 2.19 holds), 2.14c is satisfied (induced velocity is incompressible), and 2.14d is satisfied (flow is zero at infinity - flow actually decays as 1/R).

That is, the vortex flows we computed approximates the actual flow induced by approximate vortices such as tornadoes, tidal vortices, vortices behind objects dragging through water, such as behind oars or in your coffeecup behind the spoon.

2.2 The method of images

How about solutions to the Euler Equations in finite domains?

Example: Suppose D is the upper half-plane and the initial condition again consits of N point vortices in D. How do these points move so as to satisfy the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$? Answer: the boundary condition can be enforced by placing N **image vortices** of opposite sign symmetrically across the boundary to the N point vortices in D. Draw Picture! This gives a solution to the Euler Equations in this domain D: 2.14a-c are satisfied by same reasoning as above. 2.14d is also satisfied.

Placing image vorticity outside the domain to yield an incompressible velocity field that satisfies the boundary conditions is referred to as **the method of images**.

2.3 LAB PROJECT

1. Consider two counterrotating point vortices at $(0, \pm R/2)$, $\Gamma_1 = -\Gamma_2$
   (a) Find their translation velocity U.
   (b) Plot streamlines of counterrotating pair in moving reference frame (need to add a component to the streamfunction corresponding to $\mathbf{u} = (-U, 0)$ that exactly cancels the translation velocity of the vortex pair so that in this frame it does not move. As a result streamlines=particle trajectories). Describe your observations.
2. Place an arbitrary set of point vortices in a domain D, and place image vorticity outside D so that the no-though-flow boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ on ∂D is satisfied for
   (a) D: upper half-plane
   (b) D: first quadrant
   (c) D: circle centered on origin with radius a.

   (For b,c: guess the position of the image vortices.)

3. Place n vortices on a line segment as follow $x_j = 0$, $y_j = \cos(j\pi)$, with strength $\Gamma_j = \cos(j\pi)(\pi/n)$, $j = 0, \ldots, N$. Claim: as $N \to \infty$ this flow induces flow past a plate moving with horizontal velocity $U = 1/2$. Plot streamlines in moving reference to confirm.

4. Place n vortices on a circle as follow $x_j = \sin(2j\pi)$, $y_j = \cos(2j\pi)$, with strength $\Gamma_j = \sin(2j\pi)(\pi/n)$. Claim: as $N \to \infty$ this flow induces flow past a cylinder moving with horizontal velocity $U = 1/4$. Plot streamlines in moving reference to confirm.

3 Potential flow

3.1 Potential function. Laplace equation.

An irrotational, incompressible flow is called *potential* flow, that is, in potential flow $\nabla \cdot \mathbf{u} = 0$ and $\nabla \times \mathbf{u} = 0$.

If $\nabla \cdot \mathbf{u} = 0$ in a domain D then, as we already know, there exists a streamfunction ψ such that $\mathbf{u} = \nabla \times \psi$. (In 3D, ψ is a vector. In 2d, ψ is a scalar with $\mathbf{u} = \nabla^{\perp}\psi$.) For this, D has to be simply connected, which means that it contains no holes.

If $\nabla \times \mathbf{u} = 0$ in a simply connected domain D then there exists a **potential function** ϕ such that $\mathbf{u} = \nabla\phi$. Can you prove this? It follows that the circulation around any closed curve C in D is zero. (Why?)

The level curves of ϕ are the potential curves. How are the potential curves and streamlines related??

Exercise: Find the streamfunction and the potential functions for
(a) uniform flow $\mathbf{u} = (U, 0, 0)$
(b) The point vortex flow (which is irrotational and incompressible and thus potential away from the origin)

It follows that

$$\Delta\psi = 0 \text{ in D}, \text{ with } \psi = 0 \text{ on } \Delta D$$

and

$$\Delta\phi = 0 \text{ in D}, \text{ with } \frac{\partial\phi}{\partial n} = U_{wall} \cdot \mathbf{n} \text{ on } \Delta D$$

That is, both ϕ, ψ satisfy the **Laplace equation** (with different boundary conditions, Dirichlet for ψ and Neumann for ϕ). Such functions are called *harmonic*.

The Laplace equation is a fundamental equation that appears in many applications. It has been much studied and many theoretical results exist about its solutions. There are also many numerical methods to solve the Laplace equation. In 2D (and only in 2D!) there is an analytical method based on complex variables to obtain solutions for certain domains. This is what we'll go over the rest of today.

3.2 Review of Complex variables

complex numbers $z = a + ib = r\cos(\theta) + ir\sin\theta = re^{i\theta}$ where $r = \sqrt{a^2 + b^2} = |z|$ and $\theta = \text{atan}(b/a) = arg(z)$. Here we used Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. This follows from Taylor series expansions of these three functions.) So can also write

$$z = |z|e^{iarg(z)} \; .$$

Complex functions $f(z) = u(z) + iv(z), u, v$ real, $z = x + iy$. Examples: z, z^2, polynomials, e^z, $\log z$, $\overline{z}$ (Find real and imaginary components in each case.)

Limits of complex functions. The derivative $f'(z)$. **Cauchy-Riemann Equations**. Expression for the derivative $f'(z) = u_x + iv_x = v_y - iu_y$. If f is differentiable in an (open) neighbourhood of a point, then it is infinitely often differentiable in that region! This is a very strong result. In that case f is said to be **analytic**.

3.3 Complex velocity

$z = x + iy$, define **complex velocity** as $F = u - iv$, where $u = u(x, y)$, $v = v(x, y)$

Exercise: Check that if $\mathbf{u}$ is potential in D, F is analytic in D.

Exercise: Check that (if D simply connected) $W = \phi + i\psi$ satisfies $F = dW/dz$, W analytic. W is called the **complex potential** and ϕ, ψ are called harmonic conjugates.

Exercise: Find complex potential for parallel flow.

Exercise: Find complex potential for pt vortex flow (in domain not including origin)

Exercise: Check that $W(z) = U[z + a^2/z]$ is the complex potential for flow past a circle. Find complex velo and velo at infinity.

3.4 Conformal mapping

An analytic function with nonzero derivative preserves angles between curves. This can be seen as follows: suppose $\zeta = f(z)$ maps $D \to R$ is analytic with $f'(z) \neq 0$. Then

$$\frac{d\zeta}{dz} = f'(z) = |f'(z)|e^{iarg[f'(z)]}$$

or an infinitesimal vector dz gets transformed to

$$d\zeta = f'(z) = |f'(z)|e^{iarg[f'(z)]}dz$$

Note that the rotation angle $arg[f'(z)]$ is independent of dz! (see picture) so any dz gets rotated by same angle and angles between dz1,dz2 are preserved ([4]).

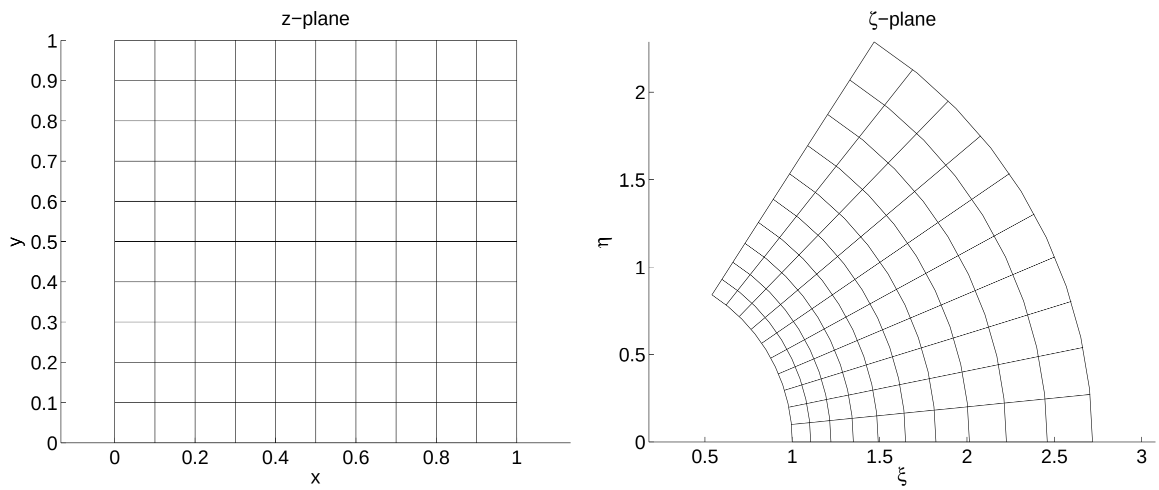


Figure 1: Mapping of square grid under $\zeta = f(z) = e^z$.

Such an angle preserving function, or mapping, is called *conformal*. For another illustration, suppose $f : D \to R$ is conformal, then the image of a rectangular grid of lines (with orthogonal angles between them) is a set of curves such that the angles between them are orthogonal. See picture, where I used $f(z) = exp(z)$.

Now, suppose you have two singly connected domains D and R (neither of which are the whole plane). By the **Riemann mapping theorem** there exists an invertible conformal map $z = f(\zeta)$ mapping D onto R. The boundary of D is mapped onto the boundary of R. The mapping is unique if some additional features are specified. For example, it is unique if the image of 3 points on the boundary is specified.

Suppose you know the complex potential $w(\zeta) = \phi + i\psi$ in D, with complex velocity $dw/d\zeta = u - iv$. Then

$$w(\zeta) = w(f(z)) = (w \circ f)(z) = \Phi(z) + i\Psi(z)$$

Since the composition of analytic functions is analytic and Φ, Ψ satisfy the correct boundary conditions in the domain R, it follows that $W = w \circ f$ is the complex potential in the domain R. The complex velocity in R is then given by $dW/dz = U - iV$. Since angles are preserved, streamlines and potential curves are orthogonal to each other in either D or R.

Lets see how this works for an example. See back of complex variables book for examples of conformal functions f.

Exercise: Check that $\zeta = f(z) = \dfrac{z}{a} + \dfrac{a}{z}$ maps the region $|z| > 1, y > 0$ to $y > 0$. $w = U\zeta$ is complex potential in upper half. Find complex potential above circle. Compare to earlier example we already looked at.

Answer to: why $z_0^\star = \frac{z^2}{\bar{z}_0}$??

3.5 Bernoulli's Theorem

Experiment: Blow over paper and between two papers held closely together. Explain your observations.

Bernoulli's Theorem: In stationary isentropic flows, the quantity

$$\frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho_0}$$

is constant along streamlines.

Outline of proof: use vector identities to show that

$$\nabla(\frac{1}{2}|u|^2 + \frac{p}{\rho_0}) = \mathbf{u} \times (\nabla \times \mathbf{u})$$

Let $x(s)$ be a streamline. Then

$$\frac{1}{2}|u|^2 + \frac{p}{\rho_0}\Big|_{x(s_1)}^{x(s_2)} = \int_{x(s_1)}^{x(s_2)} \nabla(\frac{1}{2}|u|^2 + \frac{p}{\rho_0}) \cdot x'(s)\, ds = \int_{x(s_1)}^{x(s_2)} \mathbf{u} \times (\nabla \times \mathbf{u}) \cdot x'(s)\, ds$$

Why?

Explain: your earlier observations in experiment with paper.

Explain: how does an airfoil moving at high speeds induce lift?

Explain: connection with potential flow theory just covered?

3.6 LAB PROJECT

Since we dont have lab on this Day 3 we will do the projects for todays class in the lab for Day 4. Computer project for todays lab i

4 Viscous flow

4.1 The Navier Stokes equations (for incompressible flow)

When we derived the Euler Equations we assumed that the forces across a surface were normal to that surface. This does not allow any transfer of momentum across fluid slabs moving past each other with distinct velocity. Or, for that matter, between a steady wall and fluid moving past it with nonzero velocity. We know in practice that this model is unreasonable. Faster particles in one layer will diffuse and impart momentum to slower particles next to them, slower particle will diffuse, impart momentum and slow down the fluid above. So a better model is to assume that the forces also have nonnormal components, as in

$$\text{force on } S \text{ per unit area} = p(x,t)\mathbf{n} + \sigma(x,t)\mathbf{n}$$

where σ is a matrix. Under some assumptions on σ (symmetric, invariant under rotation, proportional to velocity gradients), and after using divergence theorem as in case of (2.13), **conservation of momemtum** for incompressible flows yields the equation

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mu\Delta\mathbf{u} \tag{4.1}$$

where μd_i are the eigenvalues of σ, where d_i are the eigenvalues of the deformation matrix D. Again, we are skipping a lot of details here. The coefficient μ is called the **viscosity of the fluid**, and measures the transfer of momentum in nonnormal direction due to diffusion.

Comparing equation 4.1 to the inviscid Euler equation 2.14b we see that 4.1 has more derivatives and this requires an additional boundary condition over 2.14d. What is required is that the fluid particles not only move parallel to the wall, but stick to the wall

$$\mathbf{u} = \mathbf{U}_{wall} \text{ on } \partial D \tag{4.2}$$

where $\mathbf{U}_{wall}$ is the wall velocity. For incompressible, viscous, homogeneous ($\rho = \rho_0$) flow, this yields the Navier-Stokes equations

$$\begin{aligned}
\frac{D\mathbf{u}}{Dt} &= -\frac{\nabla p}{\rho_0} + \nu\Delta\mathbf{u} && \text{(4.3a)}\\
\nabla\cdot\mathbf{u} &= 0 && \text{(4.3b)}\\
\mathbf{u} &= \mathbf{U}_{wall} \text{ on } \partial D && \text{(4.3c)}
\end{aligned}$$

Here $\nu = \mu/\rho_0$ is called the **kinematic viscosity**. (Note: if flow homogeneous then conservation of mass autmatically satisfied.)

4.2 Nondimensionalization

You may think that the effect of viscosity on the fluid behaviour depends solely on the value of the parameter ν. However, the size of the structures involved and the characteristics speeds of the fluid matter as well. For example, dragging a spoon through water at a small speed may be similar to dragging the same spoon though oil at much faster speeds. To determine the relevant physical parameters that determine the flow we *nondimensionalize* the flow.

First we find the dimensions of all variables. In this case they involve length and time, or equivalently, lengths and velocity. Namely

$$[\mathbf{u}] = U \ , \quad [\mathbf{x}] = L \ , \quad [t] = T = L/U \ , \quad [p/\rho_0] = L^2/T^2 = U^2$$

Here, the brackets denote "the dimensions of" what is inside, and U,L,T simply denote that the dimensions are those of velocity, length, or time, respectively. Note that I treated p/ρ_0 as one variable, to avoid having to introduce mass. The dimensions of p/ρ follow by ensuring that the equation is dimensionally correct. Similary, one can deduce that

$$[\nu] = UL$$

Of course, these dimensions also follow from the definition of p and ν used in the derivation.

Now we introduce dimensionless variables denoted by tilde. Let U and L be characteristic lengths and speeds of the flow. (I am being slightly sloppy by reusing these letters..) For example, U could be the speed with which the spoon is moving, and L could be the size of the spoon. Or, if you are studying flow past a plane, U could be the speed of the plane, and L could be its size, or the width of the wing. The point is this is a choice you make, which needs to be specified. Now let

$$\tilde{\mathbf{u}} = \mathbf{u}/U \ , \quad \tilde{\mathbf{x}} = \mathbf{x}/L \ , \quad \tilde{t} = Ut/L \ , \quad \tilde{p} = \frac{p}{\rho_0}/U^2 \ ,$$

be dimensionless variables. Then the first equation of 4.3(a) states that

$$\frac{\partial \tilde{u}U}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial t} + \tilde{u}U\frac{\partial \tilde{u}U}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} + \tilde{v}U\frac{\partial \tilde{u}U}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial y} + \tilde{w}U\frac{\partial \tilde{u}U}{\partial \tilde{z}}\frac{\partial \tilde{z}}{\partial z} = -\frac{\partial \tilde{p}U^2}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} + \nu\frac{\partial^2 \tilde{u}U}{\partial^2 \tilde{x}}(\frac{\partial \tilde{x}}{\partial x})^2 \tag{4.4}$$

and similarly for the other 2 equations in 4.3(a). This reduces to

$$\frac{\partial \tilde{u}U}{\partial \tilde{t}}\frac{U}{L} + \tilde{u}U\frac{\partial \tilde{u}U}{\partial \tilde{x}}\frac{1}{L} + \tilde{v}U\frac{\partial \tilde{u}U}{\partial \tilde{y}}\frac{1}{L} + \tilde{w}U\frac{\partial \tilde{u}U}{\partial \tilde{z}}\frac{1}{L} = -\frac{\partial \tilde{p}U^2}{\partial \tilde{x}}\frac{1}{L} + \nu\frac{\partial^2 \tilde{u}U}{\partial^2 \tilde{x}}(\frac{1}{L})^2 \tag{4.5}$$

After dividing by U^2 and multiplying by L get

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} + \tilde{w}\frac{\partial \tilde{u}}{\partial \tilde{z}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{1}{Re}\frac{\partial^2 \tilde{u}}{\partial^2 \tilde{x}} \tag{4.6}$$

where $Re = \frac{LU}{\nu}$ is called the **Reynolds number**. Similarly we can check that equation 4.3(b) reduces to

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0 \tag{4.7}$$

Now we remove the tildes for convenience and state the Navier-Stokes equation in nondimensional variables

$$\begin{aligned} \frac{D\mathbf{u}}{Dt} &= -\frac{\nabla p}{\rho_0} + \frac{1}{Re}\Delta\mathbf{u} && (4.8a) \\ \nabla \cdot \mathbf{u} &= 0 && (4.8b) \\ \mathbf{u} &= \mathbf{U}_{wall} \text{ on } \partial D && (4.8c) \end{aligned}$$

As a result we deduce that the solution to the Navier-Stokes equation remains the same for different flows, up to scaling, as long as the Reynolds number is the same. For example, slow flow past a thin cylinder in water is a small sized version of fast flow past a thick cylinder in water. The nondimensionalization process reveals the relevant (nondimensional) combination of parameters determining the solution of the equation.

4.3 Boundary layers

The Navier-Stokes equations incorporate particle diffusion, which the Euler-Equations do not. This obviously is important for heavily viscous flows, or more appropriately, flows with small Reynolds numbers. But one may think that for flows with large Reynolds numbers (eg, small viscosities, or large velocities) the $\Delta\mathbf{u}$ term on the right hand side of 4.8(a) is negligible and the solution to the Navier Stokes equation is similar to that of the Euler Equation. However, the differences can be striking. Compare inviscid potential flow past a cylinder with viscous flow, even are $Re \to \infty$. The reason for the difference are the different boundary conditions.

In a brief summary: in inviscid flow the particles move past the bounding walls at nonzero speeds. In viscous flows the particles stick to the wall creating a thin transition layer between the wall velocity and the outer fluid velocity. This is referred to as a boundary layer. The velocity gradients are large in the boundary and thus this creates *vorticity* (which cannot be created in potential flow, as we know from equation 2.19. This vorticity can *separate* from the boundary and change the behaviour of the fluid far from the walls.

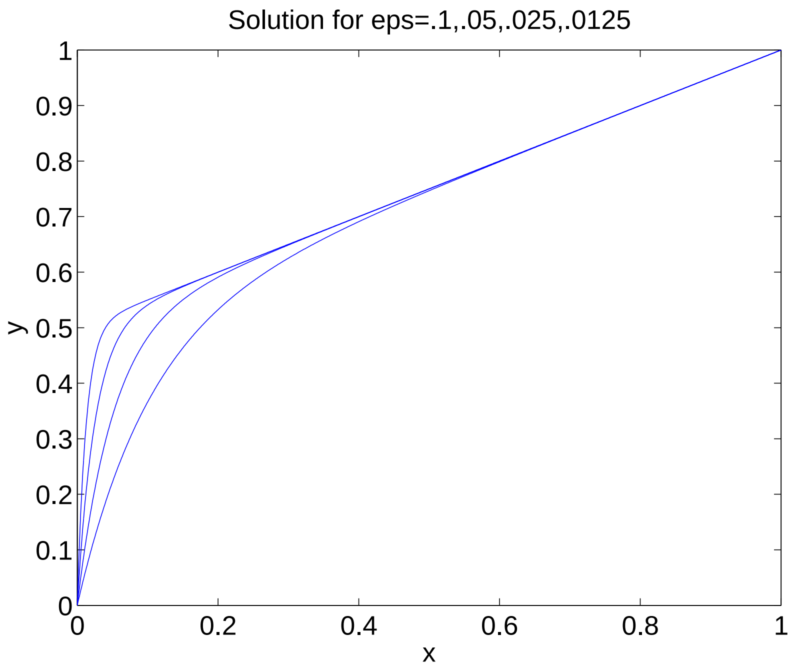
4.3.1 Model Problem

We can more closely analyze the effect of the boundary condition on a model problem. ([1], page 69) Consider the equation

$$\frac{dy}{dx} = a, y(1) = 1$$

Which has solution

$$y(x) = 1 + a(x-1)\ .$$



Now add a higher order viscosity term. The resulting second order equation requires another boundary condition, so consider

$$\epsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} = a, y(1) = 1, y(0) = 0$$

Its solution is

$$y_\epsilon(x) = \frac{1-a}{1-e^{-1/\epsilon}}(1 - e^{-x/\epsilon}) + ax$$

The figure shows that as $\epsilon \to 0$, $y_\epsilon(x)$ approaches $y(x)$ at all x *except* in a thin region near the second boundary. To measure the size of this region we determine for which x the difference $|y(x) - y_\epsilon(x)| = c$ is a certain constant. This equation holds for come constant such that $x/\epsilon = c_0$, that is $x_\epsilon = O(\epsilon)$. Thus the two solutions are as close as we want except in a region whose width shrinks to 0 as epsilon vanishes. In this region, called the **boundary layer**, the difference remains $O(1)$.

4.3.2 Parallel flow past an infinite plate

In many cases we can find exact solutions to NS equations. Consider for example two dimensional flow past a rigid plate on the x-axis, with far-field velocity is $(U, 0, 0)$. Assume the plate is infinitely long, and thus the flow can be expected to be parallel and independent of x,

$$\mathbf{u} = (u(y,t), 0, 0)$$

The streamlines are parallel lines, particles travel along straight lines, and the boundary conditions are

$$u(0,t) = 0, u(\infty, t) = U \tag{4.9a}$$

The dimensional NS equations for homogeneous flow reduce to

$$u_t = \nu u_{yy} \tag{4.9b}$$

the heat equation! The following trick works because there is no length scale in the problem. Let T, L be any numbers and let $\tilde{t} = t/T$, $\tilde{y} = y/L$. Then

$$u_{\tilde{t}} = \frac{\nu T}{L^2} u_{\tilde{y}\tilde{y}} \ , \quad u(\tilde{y} = 0, t) = 0 \ , \quad u(\tilde{y} = \infty, t) = U$$

This initial value problem for $u(\tilde{y}, \tilde{t})$ is exactly the same as the one for $u(y, t)$ if $T/L^2 = 1$. Now, choose $T = t$ and $L = \sqrt{t}$ then it follows that

$$u(\tilde{y}, \tilde{t}) = u(\frac{y}{\sqrt{t}}, 1) = u(y, t) = Uf(\eta) \tag{4.10}$$

where $\eta = \frac{y}{\sqrt{t}}$. This is called a **self-similar** solution. A solution $u(y, t)$ that depends on one parameter only. Thus, if you know the solution at one time you know it at all times, it simply gets rescaled by $\sqrt{t}$.

Much information can be drawn from the form of this solution, even without knowing f. For example, we know that at all times u transitions from 0 to U. If we define the width of the transition region (or boundary layer) to be the value of y for which $u(y, t) = 0.9U = Uf(\eta)$, then it occurs for one fixed value of $\eta = c$ (for some as yet unknown constant), so that $y \sim \sqrt{t}$. This tells you that the boundary layer thickness grows like $\sqrt{t}$.

Lets substitute a solution u of the form 4.10 into the equations 4.9 and find that f satisfies the IVP

$$\nu f" + \frac{\eta}{2} f' = 0 \ , \quad f(0) = 0 \ , \quad f(\infty) = 1$$

which can be solved using separation of variables to get

$$u(y, t) = Uerf(\frac{y}{t\sqrt{\nu t}})$$

where $erf(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-s^2}\, ds$.

We can now plot $f(\eta)$ and $u(y, t)$ for various values of t.

4.3.3 Boundary Layer Equations

Assume boundary layer of thickness δ. Then in boundary layer, $y = O(\delta)$, but assume $u, x = O(1)$. Then $v_y = -u_x = O(1)$. Then $v \approx v(0) + yv_y = O(\delta)$. This motivates:

$$x' = x, y' = \frac{y}{\delta}, t' = t, u' = u, v' = \frac{v}{\delta}$$

Substitute into NS and get

$$u'_{t'} + u'u'_{x'} + v'u'_{y'} = -p'_{x'} + \frac{1}{Re}\Big(u'_{x'x'} + \frac{1}{\delta^2}u'_{y'y'}\Big) \tag{4.11a}$$

$$\delta v'_{t'} + \delta u'v'_{x'} + \delta v'u'_{y'} = -\frac{1}{\delta}p_{x'} + \frac{1}{Re}\Big(\delta v'_{x'x'} + \frac{1}{\delta}v'_{y'y'}\Big) \tag{4.11b}$$

$$u'_{x'} + v'_{y'} = 0 \tag{4.11c}$$

$$u = v = 0 \quad \text{on x-axis} \tag{4.11d}$$

Now we build approximate equations by keeping only the largest order terms. If we want viscous effects and nonviscous effects to remain in the approximation, we must have that $\delta = O(1/\sqrt{Re})$! (Note, this is exactly what we got for the special case treated in the previous example.) The approximate **Prandtl boundary layer equations** are: (in unprimed variables):

$$u_t + uu_x + vu_y = -p_x + \frac{1}{Re}u_{yy} \tag{4.12a}$$

$$0 = -p_y \tag{4.12b}$$

$$u_x + v_y = 0 \tag{4.12c}$$

$$u = v = 0 \quad \text{on x-axis} \tag{4.12d}$$

Exercise: Show that the approximate vorticity $\omega = -u_y$ satisfies $\dfrac{D\omega}{Dt} = \dfrac{1}{R}\omega_{yy}$. That is, it convects downstream and diffuses in the normal direction.

Exercise: Show that the exact solution found previously also solves the boundary layer equations. Find the approximate vorticity $\omega = -u_y$ and plot it. Discuss your results (including the sign of ω).

One idea: solve Prantdl boundary equations near wall and inviscid away from wall. Match.

A Vector Identities

The following list of vector identities in from Chorin & Marsden[1].

B Complex variables mappings

References

[1] Chorin, A. J. & Marsden J. E. A mathematical introduction to fluid dynamics, Springer Verlag, 1979.

[2] Lugt, H. J. Vortex flow in nature and technology. Krieger Publishing Company, 1995.

[3] Nitsche, M. MATLAB: A tutorial,
`http://www.math.unm.edu/ nitsche/courses/375/handouts/mattutorial.pdf`

[4] Pozrikidis, C. Introduction to theoretical and computational fluid dynamics, Oxford University Press, 1997.

[5] Sherman, F. S. S. Viscous flow. McGraw Hill, 1990.

[6] Van Dyke, M. An album of fluid motion, Parabolic Press, 1982.

Vector Identities

The following table summarizes some general formulas that are useful for calculations with vector fields in $\mathbb{R}^3$. Some expressions in this table require explanation. In identity **7**, $\mathbf{V} = (\mathbf{F} \cdot \nabla)\mathbf{G}$ has components $V_i = \mathbf{F} \cdot (\nabla G_i)$, for i = 1, 2, 3, where $\mathbf{G} = (G_1, G_2, G_3)$. In identity **13**, the vector field $\nabla^2\mathbf{F}$ has components $\nabla^2 F_i$, where $\mathbf{F} = (F_1, F_2, F_3)$. In identity **20**, $(\mathbf{F} \times \nabla) \times \mathbf{G}$ means ∇ is to operate only on $\mathbf{G}$ in the following way: To calculate $(\mathbf{F} \times \nabla) \times \mathbf{G}$, we define $(\mathbf{F} \times \nabla) \times \mathbf{G} = \mathbf{U} \times \mathbf{G}$ where we define $\mathbf{U} = \mathbf{F} \times \nabla$ by:

$$\mathbf{U} = \mathbf{F} \times \nabla = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ F_1 & F_2 & F_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}.$$

1 $\nabla(f + g) = \nabla f + \nabla g$

2 $\nabla(cf) = c\nabla f$, for a constant c

3 $\nabla(fg) = f\nabla g + g\nabla f$

4 $\nabla\left(\frac{f}{g}\right) = \frac{(g\nabla f - f\nabla)}{g^2}$

5 $\text{div}(\mathbf{F} + \mathbf{G}) = \text{div}\,\mathbf{F} + \text{div}\,\mathbf{G}$

6 $\text{curl}(\mathbf{F} + \mathbf{G}) = \text{curl}\,\mathbf{F} + \text{curl}\,\mathbf{G}$

7 $\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times \text{curl}\,\mathbf{G} + \mathbf{G} \times \text{curl}\,\mathbf{F}$

8 $\text{div}(f\mathbf{F}) = f\,\text{div}\,\mathbf{F} + \mathbf{F} \cdot \nabla f$

9 $\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl}\,\mathbf{F} - \mathbf{F} \cdot \text{curl}\,\mathbf{G}$

10 $\text{div curl}\,\mathbf{F} = 0$

11 $\text{curl}\,(f\mathbf{F}) = f\,\text{curl}\,\mathbf{F} + \nabla f \times \mathbf{F}$

12 $\text{curl}(\mathbf{F} \times \mathbf{G}) = \mathbf{F}\,\text{div}\,\mathbf{G} - \mathbf{G}\,\text{div}\,\mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

13 $\text{curl curl}\,\mathbf{F} = \text{grad div}\,\mathbf{F} - \nabla^2\mathbf{F}$

14 $\text{curl}\,\nabla f = 0$

15 $\nabla(\mathbf{F} \cdot \mathbf{F}) = 2(\mathbf{F} \cdot \nabla)\mathbf{F} + 2\mathbf{F} \times (\text{curl}\,\mathbf{F})$

16 $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\,\nabla f \cdot \nabla g)$

17 $\text{div}(\nabla f \times \nabla g) = 0$

18 $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$

19 $\mathbf{H} \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\mathbf{H} \times \mathbf{F}) = \mathbf{F} \cdot (\mathbf{G} \times \mathbf{H})$

20 $\mathbf{H} \cdot ((\mathbf{F} \times \nabla) \times \mathbf{G}) = ((\mathbf{H} \cdot \nabla)\mathbf{G}) \cdot \mathbf{F} - (\mathbf{H} \cdot \mathbf{F})(\nabla \cdot \mathbf{G})$

21 $\mathbf{F} \times (\mathbf{G} \times \mathbf{H}) = (\mathbf{F} \cdot \mathbf{H})\mathbf{G} - \mathbf{H}(\mathbf{F} \cdot \mathbf{G})$

Figure 2:

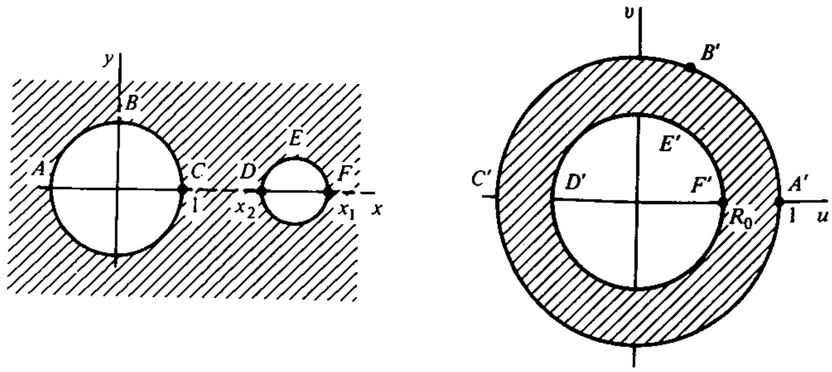


Figure 15. $w = \dfrac{z-a}{az-1}$; $a = \dfrac{1 + x_1x_2 + \sqrt{(x_1^2-1)(x_2^2-1)}}{x_1+x_2}$,

$$R_0 = \frac{x_1x_2 - 1 - \sqrt{(x_1^2-1)(x_2^2-1)}}{x_1 - x_2} \quad (x_2 < a < x_1 \text{ and } 0 < R_0 < 1 \text{ when } 1 < x_2 < x_1).$$

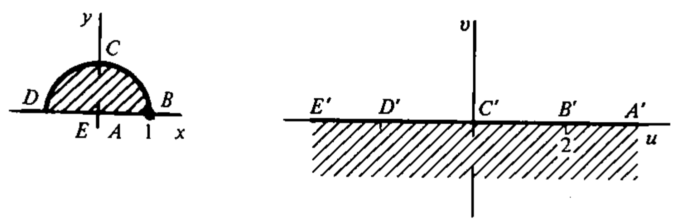


Figure 16. $w = z + \dfrac{1}{z}$.

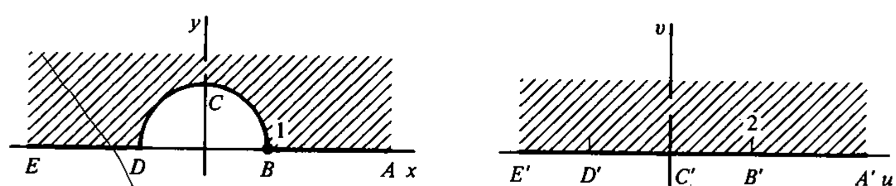


Figure 17. $w = z + \dfrac{1}{z}$.

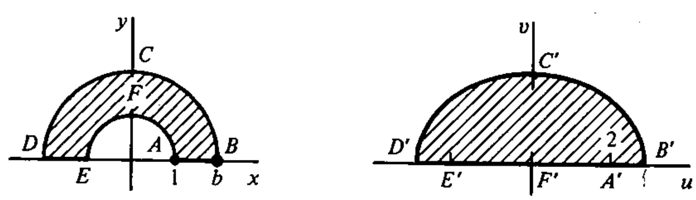


Figure 18. $w = z + \dfrac{1}{z}$; $B'C'D'$ on ellipse $\dfrac{u^2}{(b+1/b)^2} + \dfrac{v^2}{(b-1/b)^2} = 1$.

Figure 3: