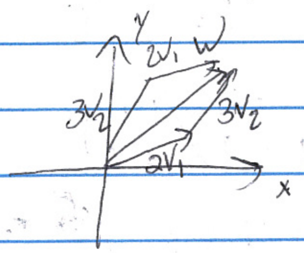
Change of Basis

(Df) Let $\{v_1, \dots, v_n\}$ be a basis of a F VS V.
~~Then for any $w \in V$, $w = w_1 v_1 + \dots + w_n v_n$, $w_i \in F$ is unique~~

Then, $\forall w \in V$ $\exists$ a unique linear combination $w = w_1 v_1 + \dots + w_n v_n$, $w_i \in F$, $1 \le i \le n$.

So we can write $w = (w_1, \dots, w_n)$. $(w_1, \dots, w_n)$ is called the coordinate vector of w and the w_i's the coordinates.

(R) Note that the coordinate vector depends on the order of the basis, e.g., for a basis $\{v_1, v_2\}$ and vect w,



So $w = (2, 3)$ if $\{v_1, v_2\}$
or
$(3, 2)$ if $\{v_2, v_1\}$

(Q) Suppose we have two bases $\{v_1, \dots, v_n\}$ and $\{u_1, \dots, u_n\}$, and a vector w in ~~the~~

1. If $w = (a_1, \dots, a_n)$ wrt $\{v_i\}$, then what are coords wrt $\{u_i\}$?
2. If $w = (b_1, \dots, b_n)$ wrt $\{u_i\}$, coords wrt $\{v_i\}$?

Let $V = (v_1, \dots, v_n)$, $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, Then $Va = w = Ub$

$U = (u_1, \dots, u_n)$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

Then $U^{-1} V a = U$

So $T = U^{-1}V$ is a matrix that transforms coordinates in V to coords in U. It is called a transition matrix ~~(or passive transformation)~~

(R) Suppose $VE = U$. E is called a change of basis (or passive transformation). Note that $E = V^{-1}U = (U^{-1}V)^{-1}$. So the coordinates of a vector change inversely to how the basis changes. Such vectors are called <u>contravariant</u>.

(Ex) Find the transition matrix from $\{u_1, u_2\} \to \{e_1, e_2\}$

$u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $u_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ $V = I$

Sol $T = I^{-1}U = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$

Also $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \xrightarrow{-2e_1+e_2} \begin{pmatrix} 1 & 0 \\ 2 & +1 \end{pmatrix} \xrightarrow{-2e_2+e_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$UE = I \Rightarrow E = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & +2 \\ -2 & +1 \end{pmatrix} = T^{-1}$

(Ex) Let $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $u_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ $u_3 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

(a) Find trans mat corr to $E \to U$.

(b) Find coords of $\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ wrt U.

Sol (a) $T = U^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$ $U^{-1}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

Ex) (a) Find trans mat for $\{1, x, x^2\} = U$ to $\{1, 1+x, 1+x+x^2\} = V$.

~~(b) Find mat F st VF = U~~

Sol $U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad V = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$$T = V^{-1}U = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

(b)

$\cancel{F = T^{-1} = V} \quad \cancel{UT^{-1} = UV = V \Rightarrow}$

The four fundamental subspaces

Let A be an $m \times n$ matrix.

We will consider four fund. subsp def by A:

1. The column space of A, $C(A)$
2. The null space of A, $N(A)$
3. The row spaces of A, $C(A^T)$
4. The left null space of A, $N(A^T)$.

Col space

D) Put $A = (a_1, \dots, a_n)$, $a_i \in \mathbb{R}^m$. Then $C(A) := \operatorname{span}(a_1, \dots, a_n) \subset \mathbb{R}^m$. Moreover, the rank of A, $r(A) := \dim C(A)$.

Ex) $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 6 \end{pmatrix} \xrightarrow{\text{row ops}} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 2 \end{pmatrix} = U$ $\quad C(A) = \operatorname{span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} x+y \\ 2x+3y \end{pmatrix}$

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$\quad C(U) = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} x+y \\ -y \end{pmatrix}$

$\xrightarrow{\text{col ops}} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$