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27 December 2011

Undergraduate Research Proposal: P-norm Estimates on Eigenfunctions of the Spherical Laplace Operator

The Legendre Polynomials are a subset of a larger number of functions called Associated Legendre functions that are solutions to the Associated Legendre Differential Equation. These are part of a broad range of functions known as "special functions." The Legendre Polynomials arise in many situations namely, as we shall see, situations involving only angular dependence on the unit sphere. The product of the Associated Legendre functions and a simple exponential forms an important class of functions in analysis known as the spherical harmonics. The spherical harmonics are parameterized by two numbers m and l, which will be introduced shortly. The ultimate goal of my research is to estimate the growth rate of the spherical harmonics with respect to the L^p norm as l tends to infinity.

Investigating the behavior of the spherical harmonics under the action of the Laplace operator gives us a few important results. We would like to look at the Laplace operator in spherical coordinates and more specifically, we will only account for the *angular* portion of the Laplace operator. We call this the "spherical Laplacian differential operator" and give it explicitly as (here θ is the azimuthal angle and ϕ is the zenith)

$$\Delta_{S^2} = \frac{1}{sin\phi}\frac{\partial}{\partial\phi}\left(sin\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{(sin\phi)^2}\frac{\partial^2}{\partial\theta^2}. \qquad (1)$$

We would like to find eigenfunctions of this operator on the sphere. In analyzing both the spherical Laplace operator and the Associated Legendre Differential Equation, we find that we have solved the eigenvalue problem for a function $\psi(\theta,\phi) = g(\phi)f(\theta)$ provided the function g is an "Associated Legendre Polynomial" and f is a simple exponential of the form,

$$f(\theta) = e^{im\theta}. \qquad (2)$$

Here m is a constant to be described shortly. The product of the functions f and g are what we call the spherical harmonics, which we write as

$$Y_l^m(\theta,\phi) = P_l^m(cos\phi)e^{im\theta}. \qquad (3)$$

Where here the $P_l^m(cos\phi)$ are the Associated Legendre Polynomials. Now we should discuss the constants m and l. They are constants appearing in Legendre's Differential Equation and each distinct pair (m,l) gives a different spherical harmonic.

By analyzing both the spherical Laplace operator and Legendre's differential equation, I had solved the eigenvalue problem of the Spherical Laplace Operator to find specifically,

$$\Delta_{S^2}Y_l^m(\theta,\phi) = -l(l+1)Y_l^m(\theta,\phi). \qquad (4)$$

This shows that our spherical harmonics are eigenfunctions of the Laplace Operator on the sphere. We do place some restrictions on the constants m and l; l must be an integer greater than or equal to zero, and we insist that $|m| \leq l$. So should we fix an l, there exists $2l+1$ values for m that give us distinct spherical harmonics. I have also shown that for two distinct pairs, (m,l) and (n,q) the spherical harmonics are orthogonal on the sphere with respect to the L^2 norm.

But why is this important? Well, I have shown that the spherical harmonics are an infinite set of mutually orthogonal eigenfunctions living on the sphere. We can normalize each spherical harmonic so that the L^2 norm is equal to one. In fact, the set $\{Y_l^m(\theta,\phi)|\ l = 0,1,2,\ldots,|m| \leq l\}$ is a complete orthonormal basis of the infinite dimensional space of square integrable functions living on the sphere. That the above set forms a basis implies that any square integrable function, $f(\theta,\phi)$ containing only angular dependence may be expanded in terms of the spherical harmonics, given the proper coefficients. The infinite set of all such $f(\theta,\phi)$ constitutes our function space.

So I have effectively described the function space I will be dealing with. My research will consist of investigating the *properties* of the basis functions of this function space: the spherical harmonics. Namely, I will be looking at the "p-norm" of the spherical harmonics. By definition, the p-norm is given by,

$$||Y_l^m||_{L^p} = \left(\iint |Y_l^m(\theta,\phi)|^p \, sin\phi d\theta d\phi\right)^{1/p}. \quad (5)$$

This so called p-norm is clearly dependent on l, m, and p. I want to look specifically at the cases where l tends to infinity and m=0, m= l, and m=-l. I would like to find the best possible estimate of the growth rate for the spherical harmonics with respect to the L^p norm in these cases. Formally, this corresponds to finding a function μ such that,

$$h(m,l,p) \approx \mu(l,p) * ||Y_l^m||_{L^2} \quad (6)$$

where,

$$||Y_l^m||_{L^p} = h(m,l,p). \quad (7)$$

The pursuit of the p-norms will tell us about the size and concentration properties of the spherical harmonics. We would like to estimate the growth rate of the spherical harmonics with respect to this fixed p. We expect this growth rate to be in powers of l but finding exactly what power for a given m and p will be the goal.