UNM Math Stats Honors Application

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A Numerical semigroup is a subset of the natural numbers S with the binary operation addition that is commutative, associative and has an identity. An important feature of a numerical semigroup is its finite complement with the natural numbers which we call its gaps, G(S). Numerical semigroups can be denoted minimally by generating elements $a_1, \ldots, a_n$ with $\gcd(a_1, a_2, \ldots, a_n) = 1$ such that

$$< a_1, a_2, \ldots, a_n >= \{x \in \mathbb{N} | x = j_1 a_1 + ... + j_n a_n, j_1, \ldots, j_n \in \mathbb{N}\}.$$

Example:

$S1 =< 2, 3 >= \{0, 2, 3, 4, \ldots\}$ $\quad$ $G(S1) = \{1\}$

$S2 =< 4, 7 >= \{0, 4, 7, 8, 11, 12, 14, \ldots\}$ $\quad$ $G(S2) = \{1, 2, 3, 5, 6, 9, 10, 13\}$

$S3 =< 3, 5, 7 >= \{0, 3, 5, 6, 7, 8, \ldots\}$ $\quad$ $G(S3) = \{1, 2, 4\}$

Generally elements of $G(S)$ can be ordered $b_1, b_2, \ldots, b_n$ where we have b_n called the frobenius of S, $F(S)$. It is known when S is generated by two elements a and b that $F(S)$ can be determined by the formula

$$F(S) = (a - 1)(b - 1) - 1.$$

The conductor, $C(S)$, is $F(S) + 1$ and clearly when generated by two elements can be determined by a slight variation on the above formula.

Any numerical semigroup generates a numerical semigroup ring $k[[t^S]]$ who's elements are $\sum_{s \in S} c_s t^s$.

Now we briefly discuss ideals and more specifically monomial ideals in the semigroup ring $k[[t^S]]$. If I is an ideal of a commutative ring A then I is a subgroup of the additive structure of A satisfying $xa \in I$ for all $a \in A$ and $x \in I$. If M is a the maximal ideal of A than $M := (t^S)$.

We define multiplication, $IJ := \sum_{k=1}^{n} i_k j_k$, and the colon ideal $I : J := \{x \in R | xJ \subseteq I\}$.

Example: $S = k[[t^2, t^5]]$ and we have a monomial ideal $I \in S$,
$I = (t^4, t^5)$, whose elements are sums involving only the powers of t, $\{t^4, t^5, t^6, t^7, t^8, \ldots\}$.
For $J = (t^6, t^9) \in S$, IJ is the set of sums only involving the powers $\{t^{10}, t^{11}, t^{12}, t^{13}, \ldots\}$.
For $J = (t^8) \in S$, IJ is the set of sums only involving the powers $\{t^{12}, t^{13}, t^{14}, t^{15}, \ldots\}$.
For maximal ideal $M = (t^2, t^5)$, MI is the set of sums only involving the powers $\{t^6, t^7, t^8, t^9, \ldots\}$.

If I is a monomial ideal then I is generated by t^s for some $s \in S$. Monomial ideals in our semigroup ring $k[[t^S]]$ correspond to ideals in S. Monomial ideals I, J in a semigroup ring are equivalent to ideals K, L in a numerical semigroup S with addition $K + L$ equivalent to multiplication IJ and subtraction $K - L$ equivalent to the colon ideal $I : J$. For a numerical semigroup S, the maximal ideal is $M = S \setminus \{0\}$. In general we will denote numerical semigroups with $<,>$ and ideals of a numerical semigroup with $(,)$.

Example: $S =< 3, 5 >= \{0, 3, 5, 6, 8, 9, 10 \ldots\}$ and $I = (3) = \{3, 6, 8, 9, 11, 12, 14, \ldots\}$.
For $J = (5, 6)$, $J + I = \{8, 9, 11, 12, 13, \ldots\}$.
For $J = (11)$, $J + I = \{14, 17, 19, 20, 22, 23, 25, \ldots\}$.
For maximal ideal $M = (3, 5) = \{3, 5, 6, 8, 9, 10 \ldots\}$, $M + I = \{6, 8, 9, 10, \ldots\}$.

Now we define basically full ideals of semigroup rings. An ideal I in a semigroup ring A is basically full if $I = MI : M$. Again we can translate the idea of basically full into numerical semigroups by saying that for S, a numerical semigroup, an ideal I is basically full if $I = (M + I) - M$.

Example: For a numerical semigroup $S =< 3, 4 >$and ideals $I = (4)$ and $K = (6, 7)$
we have $I = \{4, 7, 8, 9, \ldots\}$
$K = \{6, 7, 9, 10, 11, 12, 13, 14, 16, 17, 18, \ldots\}$
and $M = (3, 4) = \{3, 4, 6, 7, \ldots\}$.

$M + I = \{7, 8, 10, 11, 12, \ldots\}$ and
$(M + I) - M = \{3, 4, 6, 7, \ldots\} = (3, 4) \neq I$, hence I is not basically full.

$(M + K) = \{9, 10, 11, 12, \ldots\}$ and
$(M + K) - M = \{6, 7, 9, 10, 11, 12, \rightarrow\} = K$, hence K is a basically full ideal and in fact all ideals of numerical semigroups generated by two consecutive numbers are basically full.

We will also need to understand semi-prime closure operations, first in semigroup rings and then in numerical semigroups. In A, a semigroup ring, a semi-prime closure operation is a function f on A such that for ideals $K \supset L$ we have the following.

1.) $f(K) \supset K$
2.) If $K \supset L$ then $f(K) \supset f(L)$.
3.) $f(K) = f(f(K))$
4.) $f(KL) \supset f(K)f(L)$

In S, numerical semigroup, a semi-prime closure operation f on S will have the following for ideals $I \supset J$.

1.) $f(I) \supset I$
2.) If $I \supset J$ then $f(I) \supset f(J)$.
3.) $f(I) = f(f(I))$
4.) $f(I + J) \supset f(I) + f(J)$

Example: For a numerical semigroup S our semi-prime closure is a map that takes us from ideals of S, I, to $I \cup (M \backslash I)$. If our $S =< 2, 5 >$ and we have $I = (4)$, $f(I) = (4) \cup \{5, 7\} = (2, 5)$.

For ideals I, J in S, if $I \supset J$we can show that our 4 requirements for a semi-prime closure hold.

1.) Clearly $f(I) \supset I$.
2.) Since $f(J) = J \cup (M \backslash J)$ and $f(I) = I \cup (M \backslash I)$ and $(M \backslash I) \supset (M \backslash J)$, clearly $f(I) \supset f(J)$
3.) $f(I) = I \cup (M \backslash I)$, and $f(f(I)) = (I \cup (M \backslash I)) \cup (M \backslash I \cup (M \backslash I))$
by DeMorgans laws we have $f(f(I)) = (I \cup (M \backslash I)) \cup (M \backslash I) = I \cup (M \backslash I) = f(I)$
4.) $f(I) = I'$, $f(J) = J'$ and $f(I + J) = f(K \supset I, J) = K' \supset I', J'$ and $K' \supset I' + J'$

The goal of this research is to first redefine the notion of basically full in numerical semigroups then classify monomial ideals which are basically full. We will then classify the basically full closure on numerical semigroups and use this classification to classify the monomial ideals in the semigroup ring which is itself basically full. We suspect that basically full ideals are related to the conductor C(S), the Frobenious F(S), and the gaps G(S).