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[AN INTRODUCTION TO THE BLACK-SCHOLES PDE MODEL]

This paper will serve as background and proposal for an upcoming thesis paper on nonlinear Black-Scholes PDE models. The thesis paper will be linked to the Research Experience for Undergraduates (REU) program and my mentor will be Dr. Jens Lorenz.

Foundations of the Black-Scholes Model

There are many instances in which distinct relationships and patterns do not exist and the only way to describe the exhibited behavior is by terming it random.

A Scottish scientist, Robert Brown, is documented as the pioneer of observing random behavior when he noticed that the motion of pollen floating in water did not follow any distinct pattern, independent of change in water current. This observation and its subsequent research came to be known as Brownian motion and propelled mathematicians to the creation of stochastic calculus—a sub-discipline in mathematics. Stochastic calculus defines the rates of change of functions in which one or more terms are random.

Fischer Black and Myron Scholes, finance professors at MIT, employed these tools in their research to discover an effective and reliable model to price derivative securities known as options. Their work led them to a partial differential equation (PDE) that could be transformed into the exact same one that describes diffusion or heat in physics. Black and Scholes used the already known solution to the heat PDE for their option pricing model. This work was published in the *Journal of Political Economy* in 1973.

During this time, Robert Merton, a financial economist at MIT, was also working on a model to price options and came up with roughly the same conclusions as Black and Scholes. Merton's paper was published in the *Bell Journal of Economics and Management Science* during the same time as the Black-Scholes publication.

Twenty four years later, Scholes and Merton were awarded the Nobel Prize for Economic Science; Black was also recognized for his contributions but he had passed away two years prior to the bestowment of the award.

This research has engendered significant attention to the field of financial asset pricing and financial mathematics.

Option Securities

An option is a contract between a buyer and a seller that gives the buyer the right to purchase or sell an underlying asset at a specified price and time.

The price it costs to purchase this contract is known as the premium and it is what the Black-Scholes model derives. The exercise or strike price is the specified price at which the buyer of the options contract can buy or sell a certain quantity, typically 100 shares, of the underlying asset. Most option contracts expire in less than a year's time.

Options are a zero-sum investment; the contract is either valuable or worthless at the expiration date. If the option is worthless the investor will allow the contract to expire and will have a loss equivalent to the premium paid plus other transaction fees.

A call option is the right to buy an underlying asset at a fixed price, the exercise price, and a put option is the right to sell an underlying asset at the exercise price. The most efficient way to define these two option contracts is with a functional example.

We will assume that a buyer has purchased a call stock option with an exercise price of $100 and an expiration date of December 31. Let us also assume that the price of the underlying asset, the stock, is trading at $90. The buyer of the call contract can purchase, we will assume the typical amount, 100 shares of stock for the strike price until the expiration date. The seller of this option contract would be obligated to sell 100 shares of the stock if the buyer decides to exercise their right during the duration of the contract.

Clearly, the buyer of the call option contract would not exercise this option if the price of the underlying asset remained below the strike price, “out of the money”. However, if the price of the stock increased above the exercise price, the option would be “in the money” and the buyer would exercise the option or sell the valuable contract to another buyer in the market. We value the zero-sum call option investment using a maximum function.

$$C(S_T, 0, X) = Max(0, S_T - X)$$

$where,$

$C = Call\ option\ function$
$S_T = Price\ of\ underlying\ asset\ at\ expiration$
$X = Exercise\ price$

Therefore, a buyer of a call option is assumed to be “bullish” or optimistic about the prospects of the underlying asset increasing in value. This investor is speculating on an increasing stock price.

Now let us assume that the buyer has a “bearish” or pessimistic outlook for the economy or price of a company’s stock. This investor would purchase a put option contract.

We will assume that the exercise price is $80 and the price of the underlying asset is $90. The contract will expire on December 31 and the seller of the put option is obligated to buy the underlying asset at the strike price until the agreement expires.

The investor of the put option would not exercise this option at the current price level because they can sell the underlying asset in the market at a higher price than the strike price. However, if before December 31, the price of the underlying asset decreases below the exercise price the contract becomes valuable, "in the money", because there exists an obligation for the seller of the contract to buy the shares of the underlying asset at a higher price than what the market is demanding. This type of option is valued with a maximum function as well but the price of the underlying asset and the strike price swap positions in the function.

$$P(S_T, 0, X) = Max(0, X - S_T)$$

$where,$

$P = Put\ option\ function$
$X = Exercise\ price$
$S_T = Price\ of\ underlying\ asset\ at\ expiration$

Figures have been provided in the Appendix to graphically display the minimum values of the call and put option examples given increases and decreases in the underlying asset.

The premium paid to purchase call and put options is what the Black-Scholes model prices. Black and Scholes employed some simplistic economic assumptions in order to derive a linear PDE that would accurately price the premium just described.

Assumptions of the Black-Scholes Model

The model assumes simplified economic conditions in order to engender a linear PDE that can be solved analytically. These assumptions include:

Short-Selling of the Underlying Asset

Short-selling is an investment strategy in which an investor either has a pessimistic outlook for the price of a company's stock or is trying to hedge another investment position. The investor will borrow shares of stock from a broker with the contractual obligation to return an equal quantity of shares at some time in the future. The investor will sell the borrowed shares in the market with the objective of purchasing back the shares in the future at a lower price, returning the borrowed shares to the lending broker and pocketing the difference as profit. This strategy can also be employed as a hedge. For instance, if an investor believes that a particular stock is undervalued, that person could purchase a call option contract on the underlying company anticipating that the market price will rise above the option's strike price and become "in the money". However, if this prediction is incorrect the call option contract will expire worthless. Therefore, selling short the underlying asset will mitigate the loss of the call option premium and hedge the bet.

Underlying Asset Follows a Lognormal Random Walk (trading stocks is continuous)

Brownian motion was engendered from the observation of pollen particles in water, but this concept is similarly ostensible to the stock market. The price of a stock over a given period will fluctuate; how frequent and to what scale this fluctuation will occur is difficult to predict. The price movements in equities do not follow a definite pattern, and therefore, we describe their behavior as random. There is another assumption that we can make about stock price fluctuations.

Stock returns evolve according to a log-normal distribution. This assumption is feasible because of continuous compounding. Stock price movements and their corresponding returns are not discrete, they occur continuously over the length of a given period. The returns from a continuously compounding stock investment can then be expressed using the natural logarithm. This assumption is convenient because it does not allow negative stock prices and it is somewhat consistent with reality.

Arbitrage Opportunities do Not Exist

Arbitrage opportunities are situations in which risk-free profits can be made. This assumes an inefficient market place in which the prices of assets are not always accurately priced. The Black-Scholes model makes the assumption that the market is efficient and that there are no opportunities for risk-free profits because assets are always priced correctly. This is a contentious assumption as there are differing points of view as to how quickly and effectively the market adjusts to information.

Constant Risk-Free Rate and Volatility of the Return on the Stock

The risk-free rate is assumed constant such that we can assume interest rates remain constant. This assumption is not realistic, but it is convenient for the mathematics and the model. Furthermore, research indicates that interest rates have minimal effects on the prices of stock options, making this unrealistic assumption relatively innocuous.

Constant volatility, which is standard deviation, is assumed to be constant as well. However, this assumption is not realistic either. Risky assets are bound to experience volatility over the duration of a given period, where volatility is defined by the spread in the price of the asset relative to its mean. This assumption is included in order to keep the model linear and analytically solvable. Relaxing this assumption will cause the model to become much more complicated.

No Taxes or Transaction Costs

Taxes and transaction costs occur in options trading. They reduce profits and can increase losses. However, in order to focus their efforts on the fundamentals of pricing option premiums, Black, Scholes and Merton assumed these costs away to create an option market of activity, free from the restraints and advanced level of comprehension created by taxes and transaction costs.

Securities are Perfectly Divisible

This assumption allows for securities to be purchased in fractional quantities. However, in the actual market place shares can only be purchased in integer amounts. This assumption is therefore unrealistic.

No Dividend Payments

Dividend payments have an effect on the price of a stock. They can cause minor fluctuations in price when announced and paid. However, not every company pays a dividend and because the price increase/decrease is minor this assumption is not completely unrealistic.

European Options

European options do not allow for early exercise. This means that the buyer of a call or put option must wait until the end of the duration specified by the contract to exercise their right to buy or sell the underlying asset. Early exercise of options is not easily accounted for by the model and thus the derivative instruments are assumed to be European. American options can be exercised early and thus the model is not readily applicable.

Black-Scholes Model

We will define a standard Brownian motion $W(t)$, on $0 \leq t \leq T$, and a probability space (Ω, F, P). The process driving the stock price is a geometric Brownian motion:

$$dS(t) = \mu S(t) + \sigma S(t) dW(t) \tag{1}$$

Suppose that $C(t) = C(S(t), t)$ is the value at t of an option on the stock $S(t)$. Changes in the value of the option on small intervals, $dC(t)$, are

$$dC(t) = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} dt \tag{2}$$

The term $(\partial C/\partial S)dS$ is stochastic. However, we can eliminate it from equation (2). Suppose we can construct the following portfolio P consisting of a long-position on the call option and a short-position on n units of stock:

$$P(t) = C(S(t), t) - n(t)S(t)$$

We can differentiate P to obtain

(3)

$$dP = dC - ndS(t)$$

Using (2) in combination with (3) we obtain

(4)

$$dP(t) = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right)dt + \left(\frac{\partial C}{\partial S} - n\right)dS$$

Thus, setting n in (4) equal to $\partial C/\partial S$, we have

(5)

$$dP(t) = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right)dt$$

The portfolio in (5) is risk free. Thus, it should earn the risk-free rate of return. Since $dP = rPdt = r[C(S,t) - (\partial C/\partial S)S(t)]dt$, it follows that

(6)

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0$$

or

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = -\frac{\partial C}{\partial t}$$

This result is known as the Black and Scholes partial differential equation. In the absence of arbitrage any derivative security having as an underlying stock S should satisfy (6) (Cerrato, 2012). Mathematically, this is a diffusion or heat equation.

Black-Scholes Formula

$$C = S_0 N(d_1) - Xe^{-r_c T} N(d_2),$$

$where,$

$$d_1 = \frac{\ln(S_0/X) + (r_c + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$N(d_1), N(d_2) = cumulative\ normal\ probabilites$
$\sigma = annualized\ volatility\ (standard\ deviation) of\ the\ continuously\ compounded\ (log)\ return\ on\ the\ stock$
$r_c = continuously\ compounded\ risk - free\ rate$

The Black-Scholes formula is derived in the appendix.

Proposal for REU Project (Funding)

This paper was written to provide the framework for a *Research Experience for Undergraduates* (REU) research project topic that emphasizes financial mathematics.

I would like to propose the continuation of research set forth by Yan Qui, Ph.D. and Jens Lorenz, Ph.D. in the paper *A Nonlinear Black Scholes Equations*, published in the *International Journal of Business Performance and Supply Chain Modeling*. In this paper, the assumption of constant volatility is relaxed and the model becomes non-linear. This is a more realistic approach to price options as volatility is not constant. In fact the more variable the price of the underlying asset results in a more valuable corresponding derivative security.

Therefore, I would like to propose building on the above mentioned paper by first re-creating the original work and then extending the boundary conditions past the 1-periodicity originally analyzed. Furthermore, I would like to simulate our results using MATLAB and C++ in an attempt to estimate solutions using numerical methods.

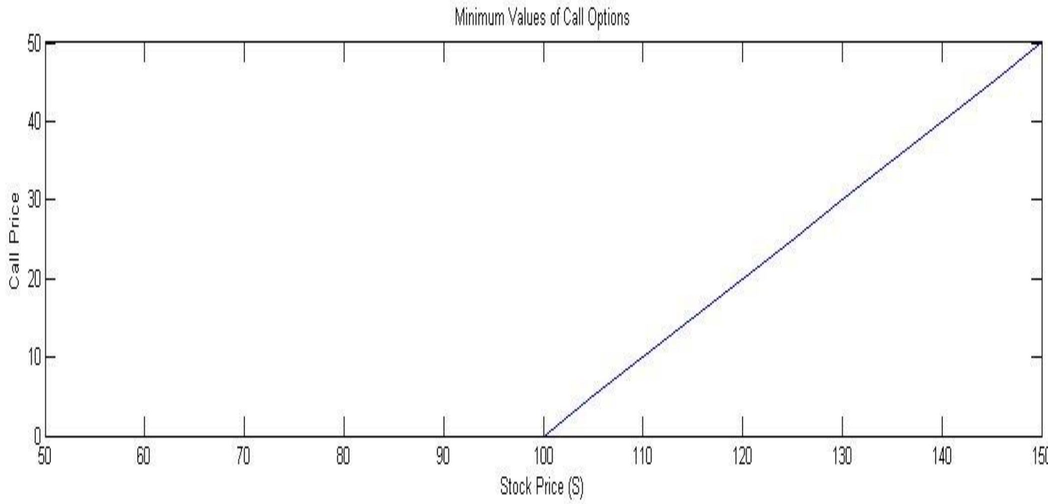
The tentative schedule that I would like to propose for the project breaks the research into halves. The first half will be the fall 2013 semester and will include obtaining the necessary skills to analyze a non-linear PDE. I am a participant in the MTCP Summer Math Camp and I am enrolled in MATH 312 (Partial Differential Equations for Engineering), MATH 401 (Advanced Calculus I), STAT 427 (Advanced Data Analysis I) and STAT 461 (Probability) for the upcoming semester. In addition, I have purchased and rented several texts on mathematical finance and derivative asset pricing. The second half will consist of writing a thesis paper based on the analysis of non-linear Black-Scholes PDEs and presenting these research conclusions at the SUnMaRC conference.

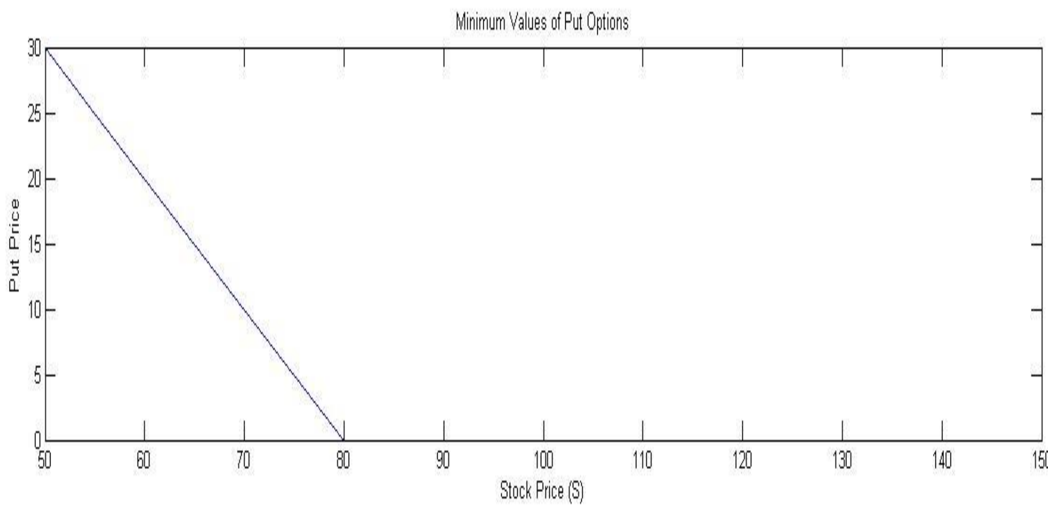
Background

I am an applied mathematics undergraduate student with the fall 2013 and spring 2014 remaining. I would like to participate in the REU project at the University of New Mexico because I intend to pursue doctoral level training in quantitative research. This project would be great exposure and experience for that objective. Professor Jens Lorenz has graciously agreed to mentor me for this endeavor. I will be registering for the Independent Study course taught by Dr. Lorenz and anticipate writing a thesis paper summarizing the results of our research with the intention of graduating with honors. I would also like to submit my REU project proposal to be considered for funding.

Appendix

Minimum Value Option Figures





Black-Scholes Formula Derivation

We start with just two assumptions:

1) The underlying asset follows a lognormal random walk
2) Arbitrage arguments allow us to use a risk-neural valuation approach (Cox-Rubinstein's proof is easiest here), discounting the expected payoff of the option at expiration by the riskless rate and assuming the underlying's return is the risk free rate

Derivation of Black-Scholes for a European call option **c** with strike **K**, discount rate **r**, on stock **S**, with time to maturity **t**, and expectations operator **E**.

Equation 1: The definition of a call option

$$c = r^{-t} E\left[\max\left(0, S(t) - K\right)\right]$$

Equation 2: End of period stock price as a function of its return by definition, where R is the gross rate of return

$$s(t) = R \cdot S(0)$$

Equation 3: rewriting eq(1) in integral form, where h() is the lognormal density function, and labeling S(0) as simply S, (note K and k are the same below)

$$c = r^{-t} \int_{K/S}^{\infty} \left(SR - K\right) h\left(R\right) dR$$

Equation 4: substitute into R an exponential and its normal distribution, where f(u) is the normal density function with a mean of $\mu t = (\ln(r) - \frac{1}{2}\sigma^2)t$ and volatility $\sigma\sqrt{t}$

$$c = r^{-t} \int_{\ln\left(K/S\right)}^{\infty} \left(Se^{u} - K\right) f\left(u\right) du$$

Equation 5: substituting for *u* now using a change in variables to *z* we have

$$c = r^{-t} \int_{\frac{\ln\left(K/S\right) - \mu t}{\sigma\sqrt{t}}}^{\infty} \left(Se^{\mu t + z\sigma\sqrt{t}} - k\right) \phi\left(z\right) dz$$

Equation 6: rearranging

$$c = Sr^{-t} \int_{\frac{\ln\left(K/S\right) - \mu t}{\sigma\sqrt{t}}}^{\infty} e^{\mu t + z\sigma\sqrt{t}} \phi\left(z\right) dz - kr^{-t} \int_{\frac{\ln\left(K/S\right) - \mu t}{\sigma\sqrt{t}}}^{\infty} \phi\left(z\right) dz$$

Equation 7: substitute *(ln(r)- ½ σ²) for μ* and factor out $e^{ln(r)t}$

$$c = S \int_{\frac{\ln\left(Kr^{-t}/S\right)+\sigma^2/2\cdot t}{\sigma\sqrt{t}}}^{\infty} e^{-\frac{1}{2}\sigma^2 t + z\sigma\sqrt{t}} \phi(z)\,dz - Kr^{-t} \int_{\frac{\ln\left(Kr^{-t}/S\right)+\sigma^2/2\cdot t}{\sigma\sqrt{t}}}^{\infty} \phi(z)\,dz$$

Equation 8: multiply the normal density by the exponent

$$c = S \int_{\frac{\ln\left(Kr^{-t}/S\right)+\sigma^2/2\cdot t}{\sigma\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2 - \frac{1}{2}\sigma^2 t + z\sigma\sqrt{t}}\,dz - Kr^{-t} \int_{\frac{\ln\left(Kr^{-t}/S\right)+\sigma^2/2\cdot t}{\sigma\sqrt{t}}}^{\infty} \phi(z)\,dz$$

Equation 9: factor exponent

$$c = S \int_{\frac{\ln\left(Kr^{-t}/S\right)+\sigma^2/2\cdot t}{\sigma\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(z-\sigma\sqrt{t}\right)^2}\,dz - Kr^{-t} \int_{\frac{\ln\left(Kr^{-t}/S\right)+\sigma^2/2\cdot t}{\sigma\sqrt{t}}}^{\infty} \phi(z)\,dz$$

Equation 10: make substitution zhat=z-σ√t

$$c = S \int_{\frac{\ln\left(Kr^{-t}/S\right)+\sigma^2/2\cdot t}{\sigma\sqrt{t}}-\sigma\sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\hat{z}^2}\,d\hat{z} - Kr^{-t} \int_{\frac{\ln\left(Kr^{-t}/S\right)+\sigma^2/2\cdot t}{\sigma\sqrt{t}}}^{\infty} \phi(z)\,dz$$

Equation 11: rearrange integral bound

$$c = S \int_{\frac{\ln\left(Kr^{-t}/S\right)-\sigma^2/2\cdot t}{\sigma\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\hat{z}^2}\,d\hat{z} - Kr^{-t} \int_{\frac{\ln\left(Kr^{-t}/S\right)+\sigma^2/2\cdot t}{\sigma\sqrt{t}}}^{\infty} \phi(z)\,dz$$

Equation 12: using the fact that

$$\int_{a}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{-a} e^{-x^2}\,dx$$

we can switch and negate the integral bounds

$$c = S \int_{-\infty}^{\frac{-\ln\left(Kr^{-t}/S\right)+\sigma^2/2\cdot t}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\hat{z}^2} d\hat{z} - Kr^{-t} \int_{-\infty}^{\frac{-\ln\left(Kr^{-t}/S\right)-\sigma^2/2\cdot t}{\sigma\sqrt{t}}} \phi(z)\,dz$$

Equation 13: using algebra we then get

$$c = S \int_{-\infty}^{\frac{\ln\left(S/K\right)+\left(r+\sigma^2/2\right)\cdot t}{\sigma\sqrt{t}}} \phi(\hat{z})\,d\hat{z} - Kr^{-t} \int_{-\infty}^{\frac{\ln\left(S/K\right)+\left(r-\sigma^2/2\right)\cdot t}{\sigma\sqrt{t}}} \phi(z)\,dz$$

Equation 14: rewrite in Normal Cumulative Density notation to get the familiar Black-Scholes equation

$$c = SN(d_1) - r^{-t}KN\left(d_1 - \sigma\sqrt{t}\right)$$

where,

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$$

QED

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