Honors Project Proposal

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July 23, 2013

Abstract

I would like to explore the discrete uncertainty principle and its applications to the distribution of prime numbers and compressed sensing.

Fourier Transform

In order to establish Fourier analysis on a finite group G, we must find analogs to the trigonometric functions used in classical Fourier analysis. Characters will take the place of trigonometric functions. A character of G is a group homomorphism that maps elements of G into the unit circle. The characters of G form a dual group $\widehat{G}$, with the binary operation $\chi_1\chi_2(a) = \chi_1(a)\chi_2(a)$ for $\chi_1, \chi_2 \in \widehat{G}$ and $a \in G$, [5].

Some examples of groups and dual groups include $G = \mathbb{T}, \mathbb{Z}, \mathbb{R}, \mathbb{Z}_n$ and $\widehat{G} = \mathbb{Z}, \mathbb{T}, \mathbb{R}, \mathbb{Z}_n$ respectively.

The Fourier transform, $\widehat{f} : \widehat{G} \to \mathbb{C}$, is defined as

$$\widehat{f}(\chi) = \frac{1}{|G|} \sum_{a \in G} f(a)\chi(a).$$

The Uncertainty Principle

The Uncertainty Principle states that it is impossible for a vector to be localized in time and frequency, or in space and frequency. Probably the best-known application of this principle is in physics, where it states it is impossible to know a particle's position and its momentum simultaneously. However, there are other applications of this well-known principle.

The Discrete Uncertainty Principle

In order to state the discrete version of the Uncertainty Principle, we first need to define some concepts, taken from Professor Pereyra's book on Harmonic Analysis, [5]

Let $L^2(G) = \{f : G \to \mathbb{C}\}$, where G is a group. Let the order of the group be denoted $\#G$. Let the Fourier transform of f be denoted $\widehat{f}$.

Define the support of a function $f \in L^2(G)$ as $supp(f) := \{x \in G : f(x) \neq 0\}$.

The Discrete Uncertainty Principle [6] states that

$$|supp(f)| \times |supp(\widehat{f})| \geq |G| = n.$$

T. Tao has proved a refinement of this principle [7]: If G is a cyclic group of order p, a prime number and again, $f \in L^2(G)$, then

$$|supp(f)| + |supp(\widehat{f})| \geq p + 1.$$

Entropy Uncertainty Principle Closely related to the discrete uncertainty principle is the entropy uncertainty principle:

When $f \in L^2(G)$ and $\| f \|_{L^2(G)} = 1$, then

$$\frac{1}{2|G|}\sum_{x\in G}|f(x)|^2 log|f(x)| + \frac{1}{2}\sum_{\xi\in\widehat{G}}|\widehat{f}(\xi)|^2 log|\widehat{f}(\xi)| \leq 0.$$

This principle can actually be used to prove the discrete uncertainty principle.

Project Proposal

Last semester, I built a Fourier theory on finite abelian groups, using group theory and the ideas developed in classical Fourier analysis over the real numbers. I will use this theory, as well as linear algebra and some more advanced group theory [2], to prove the discrete uncertainty principle and Tao's refinement. Along the way, I will also be using some analysis and calculus to prove the classical version of the uncertainty principle. I will also try to draw as many parallels as I can between the classical and group Fourier analysis, including identifying the extremal functions (the "Gaussians") that give us sharpness for the different versions of the uncertainty principle [4].

Other subjects closely related to the discrete uncertainty principle and Fourier analysis on groups are the entropy uncertainty principle and compressed sensing. I will also prove the entropy uncertainty principle and answer the question brought about by compressed sensing, "If $f \in L^2(G)$, $\Lambda \subseteq \widehat{G}$, and we can measure $\widehat{f}(\xi)$ for all $\xi \in \Lambda$, then can we uniquely reconstruct the function f?"[1]

Of course, in the real numbers, we not only have a Fourier basis, but a Haar basis as well. I will attempt to develop a Haar theory for groups in a similar way that I did for Fourier theory.

Tao and B. Green have proved an application to Tao's refinement [3], stating that there are arbitrarily long arithmetic progressions of primes, that is a set $\{p, p+d, p+2d, \ldots, p+nd\}$ where p and d are both prime numbers. I will also look at the proof of this application and attempt to understand at least part of it.

References

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[4] Hatami, Hamed. "COMP760, Lectures 2,3: Fourier Analysis of Finite Abelian Groups."Cs.mcgill.ca. N.p., n.d. Web. 19 June 2013.

[5] Pereyra, María Cristina., and Lesley A. Ward. Harmonic Analysis: From Fourier to Wavelets. Providence, RI: American Mathematical Society, 2012. Print.

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