Adeles and the Riemann-Roch Theorem for Number Fields

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Background

As a junior, I gave three talks at our geometry seminar which focused on a new proof I found of a theorem of Serge Lang, namely his Riemann-Roch Theorem for number fields. The background is this: Lang's theorem lies at the intersection of algebraic geometry and algebraic number theory. On the one hand, the Riemann-Roch theorem from algebraic geometry is a very important tool for examining properties of smooth curves and divisors on those curves. It links the intrinsic geometry of smooth curves to extrinsic properties, like possible projective embeddings. On the other hand, the number fields of algebraic number theory behave very much like the function fields of curves in algebraic geometry; these structures are similar, so much so that one can prove a direct analogue of the Riemann-Roch Theorem for them, as Lang did. However, the proofs in each case are very different.

Now John Tate, in his famous 1950 thesis, found a different analogue of the Riemann-Roch Theorem (though note that his theorem came before Lang's) when using harmonic analysis on the adeles in order to reprove some classical results of Hecke. His theorem is an analogue of the Riemann-Roch Theorem because it immediately implies the usual Riemann-Roch Theorem for smooth curves over finite fields.

In the same spirit, my idea was to apply the theorem of Tate in much the same way to obtain an adelic proof of Lang's theorem. Previously, this theorem was proved via different methods which are standard in algebraic number theory, namely via Minkowski theory. The duality between the Minkowski theory and adelic theory is rich.

Also, my proof serves to unify the result of Lang with a recent variant, due to van der Geer and Schoof; To be precise, I apply my method to one choice of adelic Schwartz function to obtain the result of Lang, and I apply it to another to obtain the result of van der Geer and Schoof. Please note there are infinitely many such functions, hence we may have a plethora of Riemann-Roch theorems.

Proposed Work

For my thesis, I would like to pursue these ideas further. Right now, I have put my research on this subject on hold in order to make an intensive study of scheme theoretic algebraic geometry. However, I will resume this direction of research in the fall when I study Arakelov theory with Professor Buium.

Lang's theorem is essentially a one-dimensional case of certain ideas in Arakelov theory. From another angle, Arakelov theory is a theory which includes a two dimensional version of the result of Lang. (There is also a higher dimensional analogue which I will not study, as it requires deep analysis and has not found itself too important in applications yet). I want to see if I can generalize my methods to two dimensions.

There are also ideas involving sheaf cohomology that I would like to pursue in order to find other, perhaps more unified, proofs of Lang's theorem and other results. I will not go into this here, however, but to say that some exploration of these ideas in the past have been somewhat fruitful for me.

Plan

At the moment, the plan is very vague. What I do know is that I will include in my thesis the work which I have already done, along with a summary of the necessary background in algebraic number theory, algebraic geometry, and abstract harmonic analysis. I have this part written already from the previous semester, but it is in no way final. Once it is final, Professor Buium and I believe it is certainly enough for a thesis. However, I want to do more. I mentioned, for instance, that I will study Arakelov theory this coming fall and try to apply my ideas there. The thesis will probably fairly complete early on, and I will add to it over time. I also plan to present my work at a conference, but I am not sure which one yet. This should be decided soon, however. Finally, I plan to defend my thesis at the end of the spring semester of 2015, which is when I also plan to graduate.