UNIVERSITY OF NEW MEXICO

N-Impact Trajectories in Rectangular Billiards

Dusty Brooks

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Introduction

My research examined the possible periodic trajectories of a point-mass inside rectangular billiards, with the goal of discovering if and when finite and infinite trajectories are possible. A billiard is a type of dynamical system consisting of a closed boundary in the real plane, a ball that travels inside the boundary, and the possible trajectories of the ball. The ball starts its trajectory at some initial point and travels in a specified direction. The trajectory of the ball is the path along which the ball travels inside the boundary of the billiard. An impact occurs whenever the ball collides with the boundary of the billiard. I considered only totally elastic collisions. In totally elastic collisions the ball does not lose any momentum, whereas in inelastic collisions the ball loses momentum with each impact (Berger, 675). This simplification allows for the consideration of infinite trajectories. Additionally, impacts are not defined on corners. We treat corners of the billiard boundary like corners of a pool table. If the trajectory of the ball meets a corner then the trajectory terminates at that corner (Tabachnikov, 2). In my research I treated the ball like a point-mass. As such, I was able to ignore complications like mass, momentum, and friction, allowing me to study the periodicity of trajectories using geometry and algebra. Given an initial point and direction, we define the trajectory of a point-mass inside a billiard boundary to be periodic if the point-mass returns to the initial point within a finite number of impacts. If the point-mass never returns to the initial point within a finite number of impacts, then the trajectory is not periodic.

The trajectory in a billiard changes after each impact. When the point-mass impacts the boundary, it bounces off continuing the trajectory so that "the angle of incidence equals the angle of reflection" (Hasselblatt, Katok, 1017). This means that each successive impact can be mapped by reflecting the trajectory by the line perpendicular to the boundary at the point of impact. This method, however, makes it very difficult to map a significant number of impacts. Instead of reflecting the trajectory, we can reflect the billiard domain, and continue the trajectory in a straight line. This unfolding process allows the trajectory of the point-mass in the billiard domain to be studied as a line in the real plane (Hasselblatt, Katok, 1019). In particular, this method works well for rectangular billiards, as the reflections of rectangles tile the real plane completely. A billiard trajectory is based on the initial conditions of boundary, initial location of the point-mass, and the initial direction of travel. My goal was to find out when periods can exist and if periods can exist with a specific number of impacts. Whether a periodic trajectory will occur relies on the slope of the trajectory before the first impact which gives the initial direction of travel. If the slope of the trajectory is rational in a rectangular billiard domain, the trajectory will be periodic. The opposite is true for irrational slopes. If the slope is irrational in a rectangular billiard domain, then the trajectory will never repeat (Berger, 680). Therefore, while looking for periodic trajectories with a specific number of distinct impacts, I assumed the slope of the trajectory was rational.

Overview

I begin by introducing some special cases of simple trajectories. A two impact periodic trajectory illustrates the importance of our simplifications. I use a four impact periodic trajectory to explain two methods for mapping periodic trajectories. Here I define a notion of equivalence between both methods which will be used throughout to determine if a trajectory is periodic. I then give a geometrical proof that a three impact periodic trajectory inside a square billiard is impossible. I also show an important property of totally elastic collisions, namely that the point-mass cannot travel along the boundary after an impact.

I then move onto squares. Using properties of the square tiling of the real plane, our previous condition about totally elastic impacts, and the linear representation of a billiard trajectory, I show that the terminating point of a periodic trajectory must have an even $x-coordinate$. I similarly show that this point must also have an even $y-coordinate$, using properties of reflections by successive integer lines along the $y-axis$. I then use the midpoint of a general billiard trajectory to eliminate the case of a terminating point of the form $(x_0, y_0 - \alpha)$. This leaves points of the form $(x_0, y_0 + \alpha)$ with $x_0, y_0 \in 2\mathbb{Z}$ as the only possible terminating points of periodic trajectories. Using these claims, I then show that periodic trajectories in square billiard domains must have an even number of distinct impacts.

To generalize the claims I proved for squares, I expand the same claims for rectangles. Similar methods will provide all of these proofs using scaling coefficients. I then exploit the benefits of representing our trajectories as lines by using the slope-intercept equation of general trajectory lines in rectangles. I examine how different initial conditions for the coordinates of the terminating point can be used to determine when trajectories will impact corners, and implicitly how corners can be avoided when specifying a trajectory.

To conclude, I give the reader a brief review of topics regarding billiards in triangles for those more ambitiously interested in the study of billiard dynamics as well as a description of billiards in circles.

Special Cases

In order to familiarize myself with the basic properties of billiards I began by examining some special cases. Their simplicity is deceptive, as even marginally more complex trajectories require more advanced and less intuitive tools to fully examine. The first is perhaps the most basic trajectory inside a square billiard, a periodic trajectory containing only two distinct points of impact. The second is the case of a four impact periodic trajectory. The third is the impossible case of a three impact period inside a square.

Two Impact Periodic Trajectory

We start with a simple square billiard domain, and an initial point placed for purely aesthetic reasons on the midpoint of one side. The trajectory is no more or less trivial if the initial point is placed at any other non-vertex location along that side. Additionally, we define the trajectory of the point-mass by the direction perpendicular to the side containing the initial point.

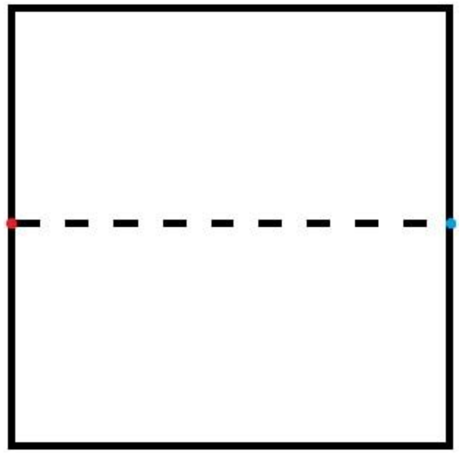

Figure 1 A two-impact periodic trajectory inside a square

The point-mass travels along the perpendicular line from the initial red point to the blue midpoint on the opposite side, and then returns along the same path. It will always return along the same trajectory because the trajectory line is perpendicular to the sides of impact. It is also worth noting that when the point-mass impacts the boundary, its trajectory after that impact is on the interior of the billiard domain. It cannot travel along the boundary. I show this in Lemma 1.1. This means that in the above trajectory, the point-mass will only impact the boundary at the red and blue points.

Even in this simple case, our simplifications are important. If we tried to replicate this trajectory on a physical pool table it would inevitably fail. At each impact the ball would lose momentum, so the periodic trajectory would not continue forever. Additionally, since the ball, pool table, and the surface beneath the pool table would all have imperfections, however small, we would never be able to accomplish a perfectly straight trajectory, and at each impact the ball would deviate slightly from its previous course. Mass, friction, and momentum all make this perfectly simple trajectory realistically impossible. However, with our billiard domain in the real plane and a point-mass for our billiard ball, we can consider this possibility and the infinite movement of our point-mass between the red and blue points along the same trajectory forever.

Four Impact Periodic Trajectory

A four impact trajectory is slightly more interesting. It can take the shape of a square or a rectangle inside the square billiard domain.

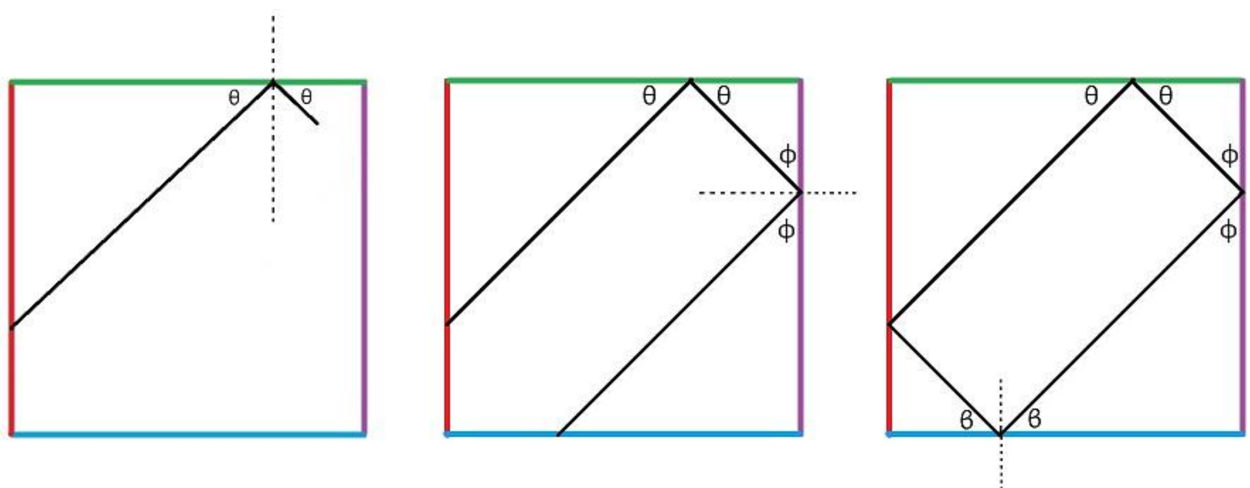

Figure 2 Four impact periodic trajectory inside a square

At each impact, we reflect the trajectory line about the line perpendicular to the side of impact at the point of impact. This method of mapping the trajectory insures that at each impact, the angles of incidence and reflection are the same. Above, the angle approaching the first impact is θ and the angle as the point-mass leaves the first impact is also θ. This ideal periodic trajectory is also only possible in our ideal billiard domain, whereas it could not occur on a real pool table.

With the four impact periodic trajectory the process of mapping the trajectory using reflections of the trajectory lines is clean and makes sense. However, this process becomes much more convoluted as the number of impacts in a trajectory are added and trajectory lines begin to cross. This complication necessitates an alternative method for mapping the trajectories inside a square billiard domain. Instead of reflecting the trajectory line, we reflect the boundary of the billiard domain by the line containing the impact point. This is easiest to see with a square trajectory.

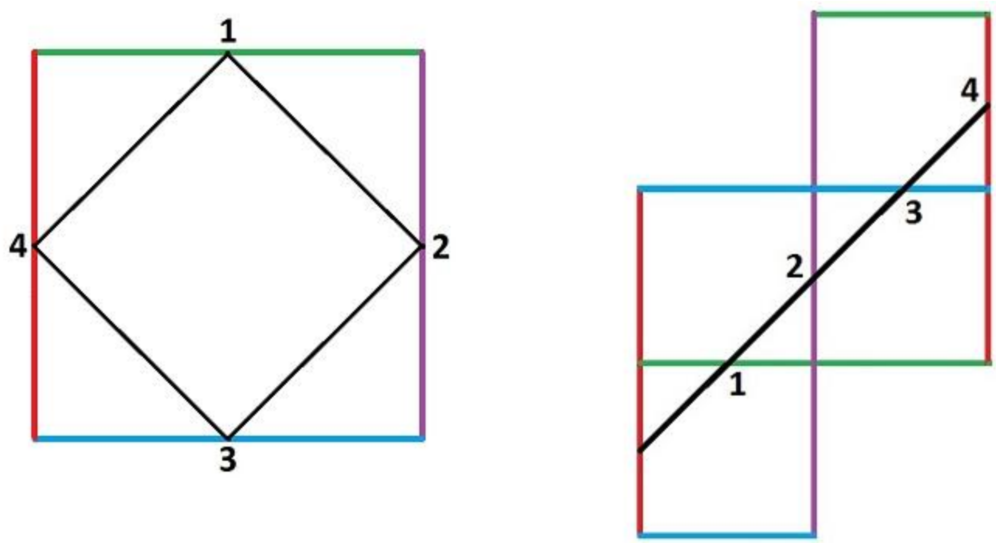

Figure 3 A square trajectory mapped by both methods

Whereas with our original method, we map the trajectory with reflections of the trajectory lines and draw it in the domain as it would occur, with this new method we reflect the boundary itself and continue the trajectory as a straight line. This offers the advantage of studying the trajectory as a straight line, eliminating the mapping difficulty as the number of impacts increases. In the above trajectory the first impact is on the midpoint of the green line in both representations of the trajectory. Similarly, the second impact is on the midpoint of purple line, and so on. The impacts occur in the same order and the same locations relative to the initial orientation of the square. Visually, a periodic trajectory is more difficult to identify using this method for mapping the trajectory, but the benefit of using the equation of a line to analyze trajectories is well worth this sacrifice in intuition.

In order to determine whether an impact corresponds to a specific location on our billiard boundary when we use this method of mapping our trajectory we need to define a notion of equivalence. A point on our original billiard boundary is equivalent to a point in the plane if the two points are equal by the composition of reflection isometries about lines of the form $y = w$ and $x = v$, where $v, w \in \mathbb{Z}$. This is the notion of equivalence that we will use for the method of trajectory mapping in which we reflect the billiard domain instead of the trajectory at each impact. The restriction to reflections by integer lines limits our definition of equivalence very specifically. If two points are equivalent by this type of isometry, they correspond to the same point on the boundary of our unit square. If two points are not equivalent by this type of isometry, they do not correspond to the same point on the boundary of our unit square. Instead of returning to the initial point, periodic trajectories will eventually impact a point equivalent to the initial point. In the simple example of a periodic trajectory above, impact 4 is at (0, 0.5) in the first representation and (2,2.5) in the second representation if our unit squares are place in the real plane. The points (0,0.5) and (2,2.5) are equal by the composition of reflection isometries about integer lines. We reflect the point (0,0.5) about the lines $y = 1$, $x = 1$, and then $y = 2$ to get the point (2,2.5). Similarly, we could reflect (2,2.5) about the lines $y = 2, x = 1,$ and then $y = 1$ to get the point (0,0.5). These points are equivalent so they both correspond to the initial location on the boundary of our billiard domain, meaning the trajectory is periodic.

Three Impact Periodic Trajectory

The next simple case worth investigating is that of the impossible three impact periodic trajectory. The beauty in this case comes from the ease with which we can dismiss the possibility of a periodic trajectory with exactly three distinct points of impact.

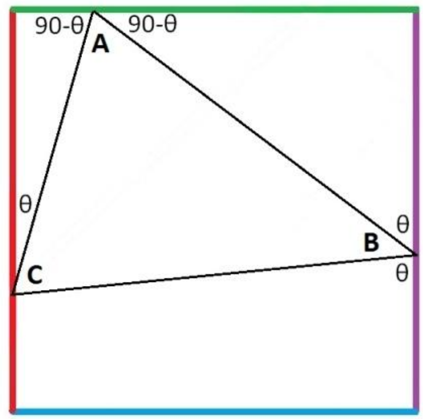

Figure 4 An impossible three impact periodic trajectory

If we suppose that a three impact periodic trajectory is possible, this trajectory must necessarily be triangular if it does not impact a corner, since the point-mass cannot travel along the boundary, as shown in Lemma 1.1. If θ is the angle of the first trajectory line, then the subsequent angles must be as pictured since the sum of the angles of a triangle is 2π. This gives us the following,

$$A = 180 - 2(90 - \theta) = 2\theta$$
$$B = 180 - 2\theta$$
$$C = 180 - A - B$$

If we plug A and B into our equation for C we get that,

$$C = 180 - 2\theta - 180 - 2\theta = 0$$

This is clearly impossible if A,B, and C are actually the interior angles of a triangle. Therefore, our assumption that a three impact periodic trajectory was possible is false. A three point periodic trajectory cannot exist in a rectangular billiard. Unfortunately, this type of intuitive, geometrical reasoning is not sufficient for cases involving more than four distinct points of impact.

Properties of Totally Elastic Collisions

Lemma 1.1. *If a point-mass impacts a line segment* $\overline{AB}$ *at* n *from a point* r *not on* $\overleftrightarrow{AB}$, *and the trajectory of the point-mass is along* $\overrightarrow{np}$, *then* $\overrightarrow{np} \cap \overleftrightarrow{AB} = \{n\}$.

Proof. Since r is not on $\overleftrightarrow{AB}$, $m\measuredangle rnA = \theta$, for some $\theta > 0$. But by properties of elastic impacts we get that $m\measuredangle rnA = m\measuredangle xnB$ for any $x \in \overrightarrow{np}$ with $x \neq n$. This implies that $m\measuredangle xnB = \theta > 0$. Therefore, $x \notin \overleftrightarrow{AB}$, which means that for any $x \in \overrightarrow{np}$ with $x \neq n$, $x \notin \overrightarrow{np} \cap \overleftrightarrow{AB}$. Therefore, $\overrightarrow{np} \cap \overleftrightarrow{AB} = \{n\}$.

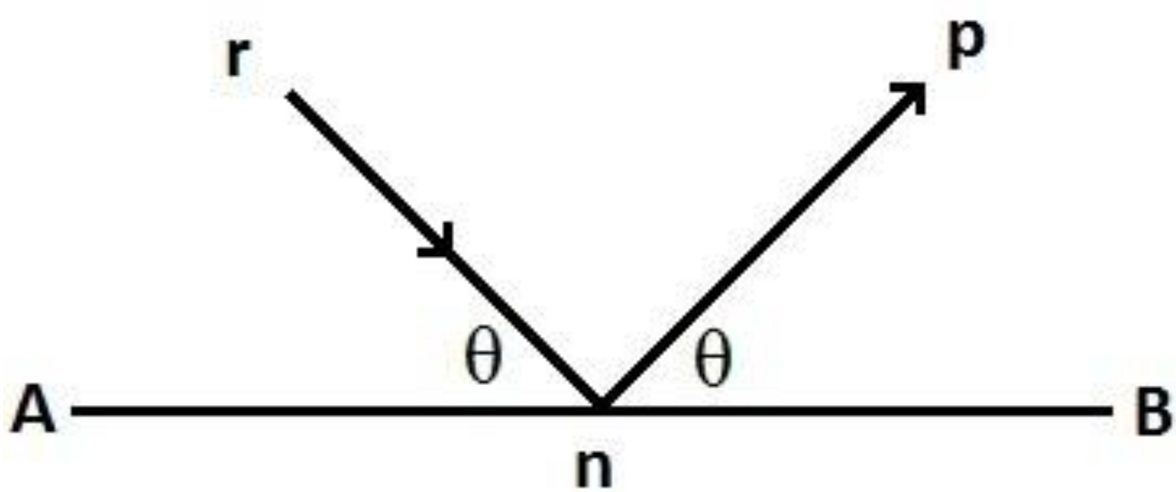

Figure 5 The point-mass impacts the boundary at only one point

Square Billiards

For square billiards consider square ABCD a unit square with A at (0,1), B at (1,1), C at (1,0), and D at (0,0), with the trajectory of a point-mass starting on $\overline{AD}$ at $(0, \alpha)$.

Claim 1.1. *If the trajectory of a point-mass is periodic ending at a point* $(x_0, y_0 \pm \alpha)$ *equivalent to* $(0, \alpha)$, *then* $x_0 \in 2\mathbb{Z}$.

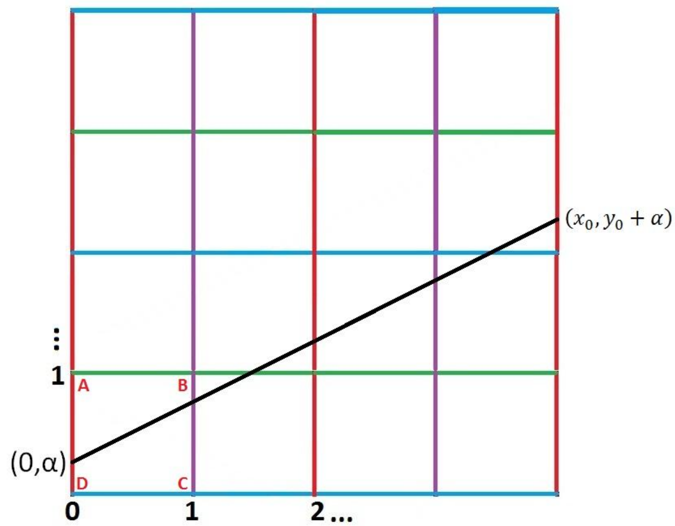

Figure 6 A trajectory mapped as a line in the plane

Proof. By Lemma 1.1, in order for the point-mass to travel from $(0, \alpha)$ back to any point on $\overline{AD}$, including $(0, \alpha)$, it must leave the line $\overline{AD}$, and eventually impact $\overline{BC}$. With the trajectory unfolded in the plane, this means that in order for the point mass to travel from $(0, \alpha)$ on $\overline{AD}$ to a point $(x_0, y_0 \pm \alpha)$ equivalent to $(0, \alpha)$, the trajectory line must cross the line containing $\overline{BC}$. In the plane, the square is originally positioned with $\overline{BC}$ at $x = 1$. Because we are investigating x_0 we are only concerned with what happens when the square is reflected horizontally because vertical reflections do not alter x_0. When the square is horizontally reflected, it is reflected first about the line $x = 1$. This places the side of the square equivalent to $\overline{AD}$ on the line $x = 2$. Then reflection by the line $x = 2$ places the side of the square equivalent to $\overline{BC}$ on the line $x = 3$ since our unfolding process involved only reflections by successive horizontal and vertical integer lines. If we continue reflecting the square by each successive integer line in this manner, the side equivalent to $\overline{AD}$ will always be placed on an even integer line. Similarly, these reflections will always place the side equivalent to $\overline{BC}$ on an odd integer line. Therefore, if $(x_0, y_0 \pm \alpha)$ is the terminating point of the trajectory, and $(x_0, y_0 \pm \alpha)$ is equivalent to $(0, \alpha)$, x_0 cannot be an odd number. Therefore $x_0 \in 2\mathbb{Z}$.

Claim 1.2. *If the trajectory of a point-mass is periodic ending at a point* $(x_0, y_0 \pm \alpha)$ *equivalent to* $(0, \alpha)$, *then* $y_0 \in 2\mathbb{Z}$**.**

Proof. If the trajectory of the point-mass ends at $(x_0, y_0 \pm \alpha)$ and this point is equivalent to $(0, \alpha)$, then it must be possible to reflect the point-mass by integer lines until the reflection of $(0, \alpha)$ is $(x_0, y_0 \pm \alpha)$. If we reflect $(0, \alpha)$ by successive integer lines starting with $y = 1$, then $y = 2$, and so on, then the distance between two successive reflections of $(0, \alpha)$ is either 2α or $2 - 2\alpha$ since the point is always a distance of either α or $1-1\alpha$ from the nearest integer line.

If we reflect once, the distance between the two successive reflections of $(0, \alpha)$ will be $2 - 2\alpha$. If we reflect again, the distance between $(0, \alpha)$ and the new reflected point will be $2 - 2\alpha + 2\alpha$. As we continue to reflect the point, we continue to alternate adding $2 - 2\alpha$ and 2α. Thus, if we reflect an even number of times our new $y - coordinate$ will be $r(2 - 2\alpha) + r2\alpha + \alpha$, for some $r \in \mathbb{N}$. If we reflect an odd number of time, since we alternate adding $2 - 2\alpha$ and 2α and we start with $2 - 2\alpha$, our new $y - coordinate$ will be $r(2 - 2\alpha) + (r - 1)2\alpha + \alpha$, for some $r \in \mathbb{N}$.

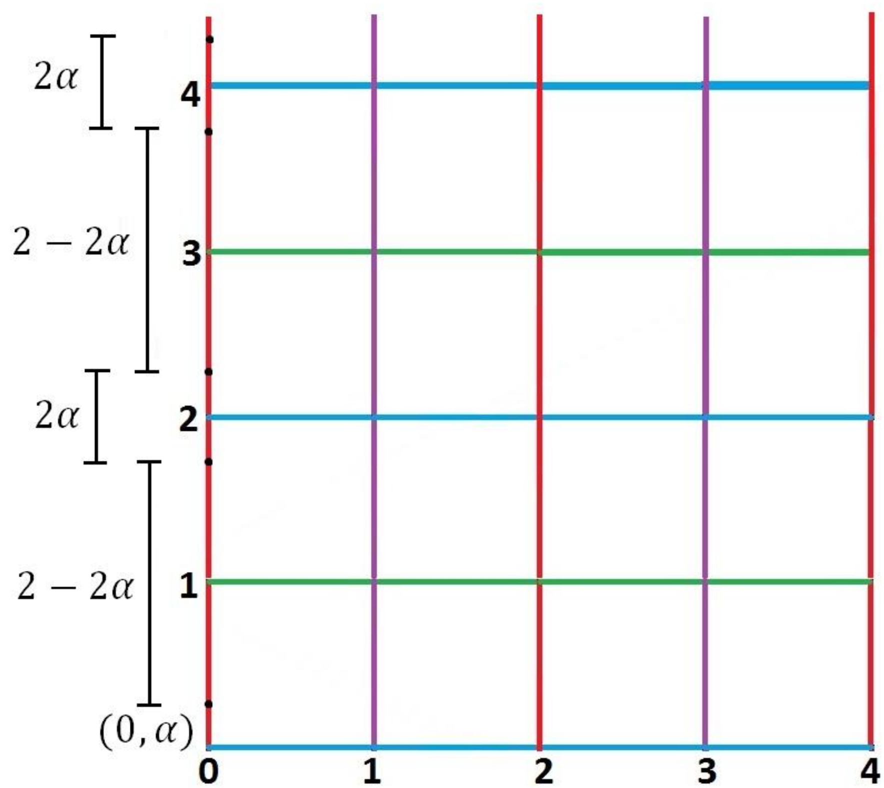

Figure 7 Vertical reflections of the point mass by successive horizontal integer lines

This means that if $(x_0, y_0 - \alpha)$ is equivalent to $(0, \alpha)$, then $y_0 + \alpha = r(2 - 2\alpha) + r2\alpha + \alpha$ or $y_0 - \alpha = r(2 - 2\alpha) + (r - 1)2\alpha + \alpha \, with \, r \in \mathbb{N}$.

$$y_0 - 2\alpha = r(2 - 2\alpha) + (r - 1)2\alpha$$
$$y_0 - 2\alpha = 2r - 2r\alpha + 2r\alpha - 2\alpha$$
$$y_0 = 2r$$

Therefore, $y_0 \in 2\mathbb{Z}$**.**

Similarly, if $(x_0, y_0 + \alpha)$ is equivalent to $(0, \alpha)$, then,

$$y_0 + \alpha = r(2\alpha) + r(2 - 2\alpha) + \alpha$$
$$y_0 = 2r\alpha + 2r - 2r\alpha$$
$$y_0 = 2r$$

Therefore, $y_0 \in 2\mathbb{Z}$**.** So, if $(x_0, y_0 \pm \alpha)$ is the terminating point of the trajectory and is equivalent to $(0, \alpha)$, then $y_0 \in 2\mathbb{Z}$**.**

In order for $(x_0, y_0 \pm \alpha)$ and $(0, \alpha)$ to be equivalent while reflecting by these specific and successive integer lines, $x_0, y_0 \in 2\mathbb{Z}$. This intuitively fits our method for unfolding trajectories by reflecting the billiard domain. However, our notion of equivalence allows for reflections by any integer lines, so we require a further lemma to show that in order for $(0, \alpha)$ and (x_0, y_0) to be equivalent x_0 and y_0 must be even.

Claim 1.3. *If the trajectory of a point-mass which starts on* $\overline{AD}$ *at* $(0, \alpha)$ *ends at a point* $(x_0, y_0 - \alpha)$ *with* $x_0, y_0 \in 2\mathbb{Z}$ *then the point terminates in a corner before the trajectory completes a period.*

Proof. The initial point of the trajectory is $(0, \alpha)$ and the terminating point is $(x_0, y_0 - \alpha)$. Therefore, the trajectory line is the line through these two points. This means the midpoint of this line is in the trajectory. The midpoint of this line is $(\frac{x_0}{2}, \frac{y_0}{2})$, and since $x_0, y_0 \in 2\mathbb{Z}$ that means that $\frac{x_0}{2}, \frac{y_0}{2} \in \mathbb{Z}$ so this point is a corner. Therefore the point mass impacts a corner before a period can be completed.

Claim 1.4. *The trajectory of the point-mass cannot be periodic if the number of impacts, N, is odd.*

Proof. We know that $x_0, y_0 \in 2\mathbb{Z}$. In order for $x_0 \in 2\mathbb{Z}$, the number of horizontal reflections of the billiard boundary must be odd. And in order for $y_0 \in 2\mathbb{Z}$, the number of vertical reflections of the billiard boundary must be even. This difference is because the initial point is on the integer line $x = 0$, but falls between integer lines vertically. Since an even number plus an odd number is an odd number, the total number of reflections to map the trajectory is odd. However, the first impact occurs before we reflect the billiard domain, so we must add the first impact to this odd number of impacts. So, the total number of impacts is even. This will always be the case when $x_0, y_0 \in 2\mathbb{Z}$, which we require for periodic trajectories. Therefore, if the number of impacts, N, in a trajectory is odd, the trajectory cannot be periodic.

Though we know that our unfolding method requires x_0 and y_0 to be even integers, this alternative proof shows x_0 and y_0 must be even based on the conditions of our definition of equivalence of points.

Lemma 1.2 *Reflections about lines $x = v$ and $y = w$ with $v, w \in \mathbb{Z}$ preserve the parity of the coordinates of all integer points.*

Proof. For any point (c, d) with $c, d \in \mathbb{Z}$, reflect the point (c, d) by an integer line $y = w, w \in \mathbb{Z}$. Denote this reflection by $\sigma_{y=w}(c, d)$. By properties of reflections we get that:

$$\sigma_{y=w}(c, d) = \big(c, d + 2(w - d)\big) = (c, 2w - d)$$

Since $w \in \mathbb{Z}$, $2w$ is an even integer. This means that if d is even $2w - d$ will be even, and if d is odd $2w - d$ will be odd. We can similarly reflect by a our point (c, d) by a vertical integer line, $x = v, v \in \mathbb{Z}$.

$$\sigma_{x=v}(c, d) = (c + 2(v - c), d) = (2v - c, d)$$

Since $v \in \mathbb{Z}$, $2v$ is an even integer. If c is odd $2v - c$ will be odd. If c is even $2v - c$ will be even as well.

Lemma 1.3 No composition of reflections about integer lines will ever map one side of our billiard boundary to the opposite side of the billiard boundary, so if the points $(x_0, y_0 + \alpha)$ and

$(0, \alpha)$ are equivalent, $x_0, y_0 \in 2\mathbb{N}$. This is a result of Lemma 1.2. The coordinates of the points of the line segment $\overline{AD}$ will always have even parity, while the coordinates of the points of the line segment $\overline{BC}$ will always have even parity.

Rectangular Billiards

For rectangular billiards consider rectangle ABCD positioned in the plane such that A is at $(0, \mathrm{a})$, B is at (b, a), C is at $(\mathrm{b}, 0)$, and D is at $(0,0)$ and a point mass starts its trajectory on $\overline{\mathrm{AD}}$ at $(0, \alpha)$ with $\alpha \in (0, \mathrm{a})$ with terminating point of the trajectory $(\mathrm{x}_0, \mathrm{y}_0 + \alpha)$. The equation of the trajectory line in a rectangle is:

$$f(x) = \frac{y_0}{x_0} x + \alpha$$

Rectangular billiards tile the plane uniquely like squares, so proofs similar to those above also show that in rectangles:

Claim 2.1. *If the terminating point is* $(x_0, y_0 \mp \alpha)$ *then* $x_0 \in 2b\mathbb{Z}$.
Claim 2.2. *If the terminating point is* $(x_0, y_0 \mp \alpha)$ *then* $y_0 \in 2a\mathbb{Z}$.
Claim 2.3 *If the terminating point is* $(x_0, y_0 - \alpha)$ *then the point-mass impacts a corner before the trajectory completes a period.*
Claim 2.4 *The trajectory of the point-mass cannot be periodic if the number of impacts, N, is odd.*

However, going through the same type of proofs is unnecessary since we already have these properties for squares. For every point (x, y) in our original square billiard domain, we can define the function $f(x, y) = (bx, ay)$, which just applies the dimensions of our rectangle as scaling coefficients. This function maps our square lattice to the rectangular lattice, mapping points to points, lines to lines, sides of our square to sides of our rectangle, and vertices to vertices. This means the trajectories inside our square are mapped by $f(x, y)$ to trajectories within the rectangular lattice. So, all of the properties that we have for trajectories inside square billiard domains will also hold for rectangular billiard domains.

Claim 2.5 *If the trajectory of the point-mass starting at* $(0, \alpha)$ *and terminating at* $(x_0, y_0 + \alpha)$ *equivalent to* $(0, \alpha)$ *has scaling coefficients such that* $x_0 = bm$ *and* $y_0 = an$*, and* $\frac{\alpha}{a} = \frac{p}{q}$ *with* $p, q \in \mathbb{Z}$ *and* $\gcd(p, q) = 1$*, then the point-mass will impact a corner only if* $q|m$ *and* $p \in 2\mathbb{Z}$ *or if* $\frac{m}{q} \in 2\mathbb{Z}$.

Proof. The equation of the trajectory line is:

$$f(x) = \frac{y_0}{x_0}x + \alpha$$

Suppose the point-mass impacts the corner when $x = sb$, $f(x) = ra$ for some $s, r \in \mathbb{Z}$. Then the equation of the trajectory line becomes:

$$ra = \frac{y_0}{x_0}sb + \alpha = \frac{an}{bm}sb + \alpha = \frac{an}{m}s + \alpha$$

This implies that:

$$r = \frac{ns}{m} + \frac{\alpha}{a}$$

Since $\frac{\alpha}{a} = \frac{p}{q}$ our equation becomes:

$$mr = ns + \frac{mp}{q}$$

We have that $m, n \in 2\mathbb{Z}$ by Claim 2.1 and 2.2, and since $r, s \in \mathbb{Z}$ we also have that $mr, ns \in 2\mathbb{Z}$. This equation is only true if $\frac{mp}{q}$ is an even integer. This can only happen when $q|m$ and $p \in 2\mathbb{Z}$ or $\frac{m}{q} \in 2\mathbb{Z}$, since $\gcd(p, q) = 1$. It is also worth noting that both of these cases can only occur when either both α and a are rational, or both are irrational, else the ratio $\frac{\alpha}{a}$ will not be rational. These are the only trajectories inside rectangular billiards which can terminate at corners. All other trajectories are either periodic or infinite. This rule can also be used to show when segments of infinite trajectories will terminate in a corner. Instead of using a terminating point equivalent to $(0, \alpha)$, since infinite trajectories do not have terminating points like periodic trajectories, we can just use some point along the trajectory and examine the trajectory on the segment from $(0, \alpha)$ to that specified point.

Billiards in Triangles

The natural progression of this research is to examine periodic billiard trajectories inside billiard domains defined by non-rectangular polygons. Triangles provide an interesting avenue of study. When studying triangles there are more conditions to consider than those which must be taken into account for rectangles. Whether a triangle has a right angle, whether it is rational with all interior angles rational multiples of π, or irrational with at least one interior angle an irrational multiple of π, isosceles, or acute or obtuse all make a difference. Each case has to be dealt with separately. Though the existence of periodic orbits in all triangles is an open problem, there are some interesting results for acute, right, rational, and isosceles triangles (Hooper, Schwartz, 1).

A rational polygon is a polygon in which the interior angles are all rational multiples of π (Galperin, Zvonkine, 30). We know that all rational polygons contain periodic trajectories, including our basic squares and rectangles (Troubetzkoy, 1). However, it is unclear which irrational polygons contain periodic trajectories, whether those periodic trajectories can have any number of impacts, and how densely non-periodic trajectories fill in the billiard domain. So when taking rationality into consideration, rational triangles will contain some periodic trajectories, but irrational triangles require further study.

Some triangular billiards, however, are better understood. Though obtuse triangles are still largely a mystery, all acute triangles and right triangles contain periodic trajectories (Hooper, Schwartz, 1). There are even some specific periodic trajectories that occur in these triangles. For example, in every acute triangle there is a three impact periodic trajectory, called the Fagnano trajectory, predictably named after the man who discovered it (Galperin, Zvonkine, 29). Given an isosceles triangle, one can draw a straight line that perpendicularly intersects the opposite side from each vertex. These are the altitude lines of the triangle. We designate the vertices of our triangular trajectory by the points at which these lines intersect the boundary (Baxter, Umble, 479).

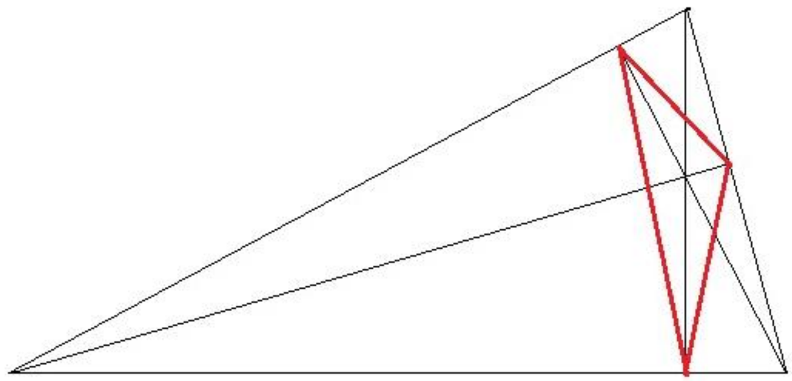

Figure 8 Sketch of a Fagnano Trajectory

Also of note, there are additional six impact periodic trajectories in bands parallel to the Fagnano trajectory (Galperin, Zvonkine, 29). Trajectories inside triangles and rectangles behave similarly to each other, filling in parallel bands within the domain.

For anyone interested in continuing the study of billiards in polygons, triangles provide a rich environment to explore. There are still many unanswered questions, and the likely tools needed to solve them delve much further into the abstract than the simple methods which can be used to study billiards in squares and rectangles.

Billiards in Circles

Billiards in circles are much more intuitive than billiards in triangles and rectangles. There are periodic and infinite trajectories inside circles. If the trajectory is defined by an angle, and the ratio of this angle to π is rational, then the impacts of the trajectory form a regular polygon and the trajectory is periodic (Berger, 713). The periodic trajectory will be a regular polygon because circles are invariant under rotation. If there are six distinct impacts, then there are six rotations which map the trajectory to itself, and thus the trajectory can only be a regular hexagon.

If the ratio of the trajectory angle to π is irrational, then the trajectory is infinite (Berger, 713). More interestingly, however, unlike the square in which infinite trajectories fill in the entire billiard domain densely, infinite trajectories inside the circle cannot fill in the entire billiard domain (Berger, 713). The only trajectory in a circle that travels through the center of the circle is the simple two impact periodic trajectory from one point on the circle to another point, such that the trajectory line is perpendicular to the line tangent to the circle at that point.

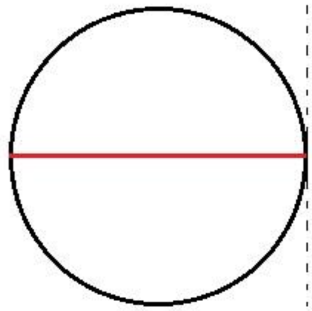

Figure 9 The only trajectory in a circle through the center

Instead of filling in the entire billiard domain, infinite trajectories in circles densely fill in a ring within the circle (Berger, 713). They fill in the ring densely because circles are invariant under rotation. Whereas a periodic trajectory will be mapped to itself by a finite number of rotations of the circle, with an infinite trajectory the trajectory is invariant under an infinite number of rotations. The points of impact will be dense everywhere on the boundary of the circle, so the trajectory lines will be everywhere dense on some ring inside the circle (Berger, 713).

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