Representations, Classification and Contractions of 3-Dimensional Lie Algebras

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Introduction

This purpose of this paper is to summarize research done in the theory of Lie algebras, with attention directed towards contractions of Lie algebras. It is written in the style of an introduction to these topics for undergraduates familiar with (though not necessarily proficient in) linear and abstract algebra. It is separated into three sections: section (1) describes the basic definitions and properties of *Lie algebras* in an abstract setting; section (2) narrows the discussion to specific *linear Lie algebras* and their properties, which are the focus of the remaining section; section (3) defines and describes a particular type of a general process known as *contractions* between linear Lie algebras, focusing on those properties of Lie algebras that remain invariant under this process.

1 Abstract Lie Algebras

1.1 The General Definition of a Lie Algebra

A Lie algebra $\mathfrak{g}$ of dimension n is a vector space $\mathbf{V}$ of dimension n over a field $\mathbb{F}$ with a binary operation $\mu : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{V}$ that is

(A1) *bilinear*:

$$\mu(rx + sy, z) = r\mu(x, z) + s\mu(y, z)$$
$$\mu(x, ry + sz) = r\mu(x, y) + s\mu(x, z)$$

for all x and y in $\mathbf{V}$, and for all r and s in $\mathbb{F}$;

(A2) *alternating*:

$$\mu(x, x) = 0$$

for all x in $\mathbf{V}$;

and satisfies the (A3) *Jacobi Identity*:

$$\mu(x, \mu(y, z)) + \mu(y, \mu(z, x)) + \mu(z, \mu(x, y)) = 0$$

for all x, y, and z in $\mathbf{V}$. We will sometimes write $\mathfrak{g} = (\mathbf{V},\mu)$, where the underlying field is implicitly understood.

Note that (A3) implies non-associativity. A very important implication of (A2) is *anti-commutativity.* This can be demonstrated by applying (A2) to the vector $x + y$:

$$\begin{aligned} \mu(x+y, x+y) &= \mu(x, x+y) + \mu(y, x+y) \\ &= \mu(x,x) + \mu(x,y) + \mu(y,x) + \mu(y,y) \\ &= \mu(x,y) + \mu(y,x) \\ &= 0 \end{aligned}$$

so that

$$\mu(x,y) = -\mu(y,x)$$

We must also note that if $char(\mathbb{F}) \neq 2$, then anti-commutativity implies (A2) by considering $\mu(x,x)$:

$$\begin{aligned} \mu(x,x) = -\mu(x,x) &\implies 2\mu(x,x) = 0 \\ &\implies \mu(x,x) = 0, \end{aligned}$$

where the fact that the characteristic of $\mathbb{F}$ is not 2 is used to establish the final implication. Thus, if the field is of characteristic zero, as in the case of the real and complex fields, then anti-commutativity is logically equivalent to (A2). This fact will be used extensively later on.

As a special example, consider the transformation defined by $\mu(x,y) = 0$ for all x and y in $\mathbf{V}$. This always trivially defines a Lie algebra for any vector space $\mathbf{V}$ over any field, known as the *abelian* algebra. This name is adopted because it is the only case where the operator commutes, since we have $\mu(x,y) = 0 = \mu(y,x)$ for all x and y.

It is important to note that this abstract definition of a Lie algebra makes no mention of what the vector space $\mathbf{V}$ is, nor does it explicitly define μ in any computational sense. For the moment, this level of generality is adequate to define some simple algebraic notions, and to demonstrate some basic propositions. For the purposes of this paper, we shall consider *only* finite-dimensional Lie algebras, the infinite-dimensional cases being far beyond the scope of this paper. Furthermore, the underlying field $\mathbb{F}$ of the vector space $\mathbf{V}$ will only be specified if necessary. Most of the general results in this paper hold regardless of the field.

1.2 Structure Constants

Since Lie operators are linear mappings of vector spaces, if we let $\mathfrak{g} = (\mathbf{V},\mu)$, and we fix a basis $\{x_1, ..., x_n\}$ for $\mathbf{V}$, then the transformation μ is completely defined by the images of $\mu(x_i, x_j)$ for i, j = 0,1,2,...,n. Expressing these images as linear combinations of the basis vectors, we see that

$$\mu(x_i, x_j) = \sum_{k=1}^{n} c_{ij}^k x_k$$

where the subscript ij indicates that the pair (x_i, x_j) is the pre-image. The coefficients $c_{ij}^k \in \mathbb{F}$ are known as the **structure constants** (or **structure coefficients**) of the Lie algebra $\mathfrak{g}$.

In principle, simply knowing these structure constants is enough to completely determine $\mathfrak{g}$, so the converse is also true: if we specify a set of constants, we can construct a Lie algebra from them, provided that they satisfy

$$c_{ij}^k + c_{ji}^k = c_{ii}^k = 0$$
$$\sum_{k=1}^n (c_{ij}^k c_{kp}^q + c_{jp}^k c_{kp}^q + c_{pi}^k c_{kj}^q) = 0.$$

This forces the algebra to satisfy (A2) and (A3), and intuitively we can see that two Lie algebras are essentially the same if a suitable change of basis in one or both algebras yields the same structure contants[14, 9]. This will be made precise in Section 1.4. While it is not typical to construct Lie algebras in this way, structure constants can be used in a variety of ways to simplify calculations, as will be seen later.

1.3 Ideals of a Lie Algebra

A subset $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called a subalgebra of $\mathfrak{g}$ if and only if $\mathfrak{h}$ is a vector subspace of $\mathbf{V}$, and $\mu(x,y)$ is an element of $\mathfrak{h}$ whenever x and y are elements of $\mathfrak{h}$. In other words, a subset of a Lie algebra is a subalgebra if it is closed under the Lie operator.

An *ideal* of a Lie algebra is defined similarly to other algebraic structures. An ideal of a Lie algebra is a subalgebra with the property that $\mu(x,y) \in \mathfrak{h}$ whenever $x \in \mathfrak{h}$ and $y \in \mathfrak{g}$. It is important to realize that due to (A2), we could have just as easily defined an ideal as $\mu(x,y) \in \mathfrak{h}$ whenever $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$. This paper will utilize both as is convenient without explicitly mentioning which convention is being adopted. Ideals are among the most important substructures in Lie algebra theory, so it is beneficial to mention several examples and some general properties.

The zero subalgebra is simply the vector subspace containing only the zero vector (it is clearly a subalgebra by (A2)). Both it and $\mathfrak{g}$ itself are ideals of $\mathfrak{g}$:

$$\mu(0, x) = \mu(y - y, x) = \mu(y, x) - \mu(y, x) = 0,$$

for all x in $\mathfrak{g}$, and since the Lie operator is a mapping, it is trivial that $\mathfrak{g}$ is an ideal of itself.

Another important example of an ideal is the **center** of $\mathfrak{g}$, denoted by $Z(\mathfrak{g})$:

$$Z(\mathfrak{g}) = \{z \in \mathfrak{g} \mid \mu(x, z) = 0, \ \forall x \in \mathfrak{g}\}.$$

This is well-defined since we always have $0 \in Z(\mathfrak{g})$. To see that $Z(\mathfrak{g})$ is an ideal, we note that if $x \in Z(\mathfrak{g})$ and $y \in \mathfrak{g}$, then by the definition of the center we must have $\mu(x,y) = 0$, so for any z in $\mathfrak{g}$, we have that $\mu(\mu(x,y),z) = \mu(0,z) = 0$. If $\mathfrak{g}$ is abelian, then it follows immediately that $Z(\mathfrak{g}) = \mathfrak{g}$.

If $\mathfrak{i}$ and $\mathfrak{j}$ are both ideals of $\mathfrak{g}$, then their *sum*

$$\mathfrak{i} + \mathfrak{j} = \{x + y \mid x \in \mathfrak{i}, \ y \in \mathfrak{j}\}$$

is also an ideal. This is easily seen by noting that for $z \in \mathfrak{i} + \mathfrak{j}$, if we take any $w \in \mathfrak{g}$, then

$$\mu(w, z) = \mu(w, x + y) = \mu(w, x) + \mu(w, y)$$

where $x \in \mathfrak{i}$ and $y \in \mathfrak{j}$. This then implies that $\mu(w,x) \in \mathfrak{i}$ and $\mu(w,y) \in \mathfrak{j}$, so that $\mu(w,z) \in \mathfrak{i} + \mathfrak{j}$. The *intersection* of two ideals, $\mathfrak{i} \cap \mathfrak{j}$ is also an ideal: if $x \in \mathfrak{i} \cap \mathfrak{j}$, then for $y \in \mathfrak{g}$, $\mu(x,y)$ is in both $\mathfrak{i}$ and $\mathfrak{j}$, since x is. Then $\mu(x,y) \in \mathfrak{i} \cap \mathfrak{j}$. Note that the sum of $\mathfrak{i}$ and $\mathfrak{j}$ contains both $\mathfrak{i}$ and $\mathfrak{j}$, while the intersection of $\mathfrak{i}$ and $\mathfrak{j}$ is contained in both $\mathfrak{i}$ and $\mathfrak{j}$.

Furthermore, we have that their *product*

$$\mathfrak{ij} = \{\sum \mu(x_i, y_i) \mid x_i \in \mathfrak{i},\ y_i \in \mathfrak{j}\}$$

is likewise an ideal. To see this, let $x \in \mathfrak{g}$ and $z \in \mathfrak{ij}$. Then $z = \sum \mu(y_i,w_i)$, where $y_i \in \mathfrak{i}$ and $w_i \in \mathfrak{j}$ for each i, so that

$$\begin{aligned}
\mu(x,z) &= \mu(x, \sum \mu(y_i, w_i)) \\
&= \sum \mu(x, \mu(y_i, w_i)) \\
&= \sum(-\mu(y_i, \mu(w_i, x)) - \mu(w_i, \mu(x, y_i))) \\
&= \sum(\mu(y_i, \mu(x, w_i)) + \mu(\mu(x, y_i), w_i)) \\
&= \sum \mu(y_i, \mu(x, w_i)) + \sum \mu(\mu(x, y_i), w_i)
\end{aligned}$$

where $\sum \mu(y_i,\mu(x,w_i))$ and $\sum \mu(\mu(x,y_i),w_i)$ are in $\mathfrak{ij}$, since $\mu(x,w_i)$ are in $\mathfrak{j}$ for each i and $\mu(x,y_i)$ are in $\mathfrak{i}$ for each i.

Using the notion of an ideal, we can form the *quotient algebra* $\mathfrak{g}/\mathfrak{i}$. This is simply the quotient space of $\mathbf{V}$ over the vector subspace $\mathfrak{i}$, with the Lie operator μ_I defined on the cosets by

$$\mu_I(x + \mathfrak{i}, y + \mathfrak{i}) = \mu(x, y) + \mathfrak{i},$$

where μ is the Lie operator over $\mathbf{V}$. This is well-defined: for $x_1 + \mathfrak{i} = x_2 + \mathfrak{i}$ and $y_1 + \mathfrak{i} = y_2 + \mathfrak{i}$, we have

$$\begin{aligned}
\mu(x_1, y_1) &= \mu(x_2 + p, y_2 + q) \\
&= \mu(x_2, y_2) + (\mu(x_2, p) + \mu(p, y_2) + \mu(p, q))
\end{aligned}$$

for p,q in $\mathfrak{i}$. If $\mathfrak{i} = \mathfrak{g}$, then $\mathfrak{g}/\mathfrak{i}$ has only a single element and behaves like the zero algebra, and if $\mathfrak{i}$ is the zero subalgebra, then $\mathfrak{g}/\mathfrak{i}$ is the set of singlets of the elements of $\mathfrak{g}$ (it is *isomorphic* to $\mathfrak{g}$—see below). Since these cases are uninteresting, it is typically assumed that $\mathfrak{i} \neq \mathfrak{g}$ (in which case we say that the ideal is *proper*) and that $\mathfrak{i} \neq 0$ (in which case we say that the ideal is *non-trivial*). Any Lie algebra that has no proper, non-trivial ideals is called **simple**. Since many claims involving simple Lie algebras would fail in the case of abelian algebras, it is also assumed that $\mathfrak{g}$ is non-abelian—otherwise, many proofs would need to be modified in order to accomodate it[9]. Generally, in order to form an interesting quotient algebra, we usually assume that $\mathfrak{g}$ is not simple.

There are other very important ideals, and ideals are in general powerful theoretical tools. However, since the majority of this paper deals with linear Lie algebras, they will be discussed in detail in Section 2.4.

1.4 Mappings between Lie Algebras

Let φ be a linear mapping from $\mathfrak{g}$ to $\mathfrak{g}_0$, where μ and μ_0 are the Lie operators on $\mathfrak{g}$ and $\mathfrak{g}_0$, respectively. Suppose φ satisfies the familiar homomorphism property with respect to the Lie operators

$$\varphi(\mu(x, y)) = \mu_0(\varphi(x), \varphi(y)),$$

or in matrix terms

$$A\mu(x, y) = \mu_0(Ax, Ay)$$

for all x and y in $\mathfrak{g}$. Then φ is called a **homomorphism** of Lie algebras. If φ is surjective, then φ is called an **epimorphism**; if φ is injective, then φ is called a **monomorphism**; and if φ is bijective, then φ is called, predictably, an **isomorphism**, which we denote by $\simeq$. The notion of a Lie algebra isomorphism requires that $\mathbf{V}$ and $\mathbf{V}_0$ be isomorphic as vector spaces, and if we represent φ as a vector space isomorphism with the non-singular matrix A, then we can rewrite the matrix representation of the map as

$$\mu(x, y) = A^{-1}\mu_0(Ax, Ay).$$

Note that this is valid only for Lie algebra *isomorphisms*. This will become extremely important in Section 3.

Recall the definitions of the image and kernel of a linear map:

$$Im(\varphi) = \{y \in \mathfrak{g}_0 \mid \exists\ x \in \mathfrak{g}\ s.t.\ \varphi(x) = y\}$$

and

$$Ker(\varphi) = \{x \in \mathfrak{g} \mid \varphi(x) = 0\}.$$

We want to know under what conditions φ preserves the structure given by the Lie operator. We find that for any homomorphism φ, the following propositions hold:

Proposition 1.1: *$Ker(\varphi)$ is an ideal of $\mathfrak{g}$.*

Proof: Let $x \in Ker(\varphi)$ and $y \in \mathfrak{g}$. Since $\varphi(x) = 0$ and φ is a homomorphism, we have that

$$\varphi(\mu(x, y)) = \mu_0(\varphi(x), \varphi(y)) = \mu_0(0, \varphi(y)) = 0,$$

so $\mu(x, y) \in Ker(\varphi)$. ■

Proposition 1.2: *$Im(\phi)$ is a subalgebra of $\mathfrak{g}_0$.*

Proof: The proof is immediate from the definition of a subalgebra and the fact that φ is a homomorphism:

$$\mu(\varphi(x), \varphi(y)) = \varphi(\mu(x, y)) \in Im(\varphi).■$$

Proposition 1.3: *$Ker(\varphi) = 0$ if and only if φ is a monomorphism.*

Proof: Suppose $Ker(\varphi) = 0$, and let $\varphi(\mu(x_1,y_1)) = \varphi(\mu(x_2,y_2))$. Then we have that

$$\varphi(\mu(x_1, y_1)) - \varphi(\mu(x_2, y_2)) = \varphi(\mu(x_1, y_1) - \mu(x_2, y_2)) = 0$$

so that $\mu(x_1,y_1)$ - $\mu(x_2,y_2) \in Ker(\varphi)$. But since $Ker(\varphi) = 0$, we must have that $\mu(x_1,y_1) - \mu(x_2,y_2) = 0$, so $\mu(x_1,y_1) = \mu(x_2,y_2)$, so φ is a monomorphism.

Now suppose φ is a monomorphism. Then since $\varphi(0) = 0$, we have at once that $Ker(\varphi) = 0$. ■

The standard isomorphism theorems hold as well. We state and prove them for later use. In what follows, μ is the Lie operator on $\mathfrak{g}$.

Theorem 1.1 (*First Isomorphism Theorem*): *Let* φ*:* $\mathfrak{g} \longrightarrow \mathfrak{g}_0$ *be a homomorphism of Lie algebras. Then* $\mathfrak{g}/Ker(\varphi) \simeq Im(\varphi)$*.*

Proof: Define γ: $\mathfrak{g}/Ker(\varphi) \longrightarrow Im(\mathfrak{g})$ by $\gamma(x + Ker(\varphi)) = \varphi(x)$. This is well-defined: If $x_1 + Ker(\varphi) = x_2 + Ker(\varphi)$, then we have $x_1 + k_1 = x_2 + k_2$ or $x_1 - x_2 = k_2 - k_1 = k_0$, so that $x_1 - x_2 \epsilon Ker(\varphi)$. But then we have $\varphi(x_1 - x_2) = \varphi(x_1) - \varphi(x_2) = 0$, so that $\varphi(x_1) = \varphi(x_2)$. This then implies that $\gamma(x_1 + Ker(\varphi)) = \gamma(x_2 + Ker(\varphi))$.

Now let $y \epsilon Im(\varphi)$. Then $y = \varphi(x) = \gamma(x + Ker(\varphi))$ for some x in $\mathfrak{g}$, so γ is surjective. The coset $Ker(\varphi)$ acts as identity in $\mathfrak{g}/Ker(\varphi)$, so for each and only those x in $Ker(\varphi)$, we have that $\gamma(x + Ker(\varphi)) = \gamma(Ker(\varphi)) = \varphi(x) = 0$, so $Ker(\gamma) = Ker(\varphi)$. Thus γ is injective, and so bijective.

To see that γ satisfies the homomorphism property, let μ_K and μ_0 denote the Lie operators on $\mathfrak{g}/Ker(\varphi)$ and $\mathfrak{g}_0$, respectively. We see that from the definition we have

$$\begin{aligned}
\mu_0(\gamma(x_1 + Ker(\varphi)), \gamma(x_2 + Ker(\varphi))) &= \mu_0(\varphi(x_1), \varphi(x_2)) \\
&= \varphi(\mu(x_1, x_2)) \\
&= \gamma(\mu(x_1, x_2) + Ker(\varphi)) \\
&= \gamma(\mu_K(x_1 + Ker(\varphi), x_2 + Ker(\varphi))).
\end{aligned}$$

This demonstrates that γ is an isomorphism, so that $\mathfrak{g}/Ker(\varphi) \simeq Im(\varphi)$. ■

Corollary 1.1: *If* $\mathfrak{i} \subset Ker(\varphi)$*, then there exists a unique homomorphism* γ*:* $\mathfrak{g}/\mathfrak{i} \longrightarrow \mathfrak{g}_0$ *such that* $\gamma \circ \pi = \varphi$*, where* π *is the canonical map from* $\mathfrak{g}$ *to* $\mathfrak{g}/\mathfrak{i}$ $(\pi(x) = x + \mathfrak{i})$*.*

Proof: Let $\mathfrak{i} \subset Ker(\varphi)$, and define γ as before, except with $\mathfrak{i}$ in place of $Ker(\varphi)$. Since $\mathfrak{i}$ is in the kernel of φ, γ is still well-defined. It is clear that $(\gamma \circ \pi)(x) = \gamma(\pi(x)) = \gamma(x + \mathfrak{i}) = \varphi(x)$. The explcit construction of γ guarantees uniqueness. ■

This corollary is sometimes called the *universal property of the quotient* and, like the theorems themselves, applies generally to quotient algebras.

Theorem 1.2 (*Second Isomorphism Theorem*)*: If* $\mathfrak{i}$,$\mathfrak{j} \subset \mathfrak{g}$*, then* $(\mathfrak{i} + \mathfrak{j})/\mathfrak{j} \simeq \mathfrak{i}/(\mathfrak{i} \cap \mathfrak{j})$*.*

Proof: Define γ: $\mathfrak{i} \longrightarrow (\mathfrak{i} + \mathfrak{j})/\mathfrak{j}$ by $\gamma(i) = i + \mathfrak{j}$, which is clearly well-defined: If we let μ_J be the Lie operator on $(\mathfrak{i} + \mathfrak{j})/\mathfrak{j}$, then we have that $\mu_J(\gamma(i_1),\gamma(i_2)) = \mu_J(i_1 + \mathfrak{j}, i_2 + \mathfrak{j}) = \mu(i_1,i_2) + \mathfrak{j} = \gamma(\mu(i_1,i_2))$, so γ is a homomorphism.

Now let $x \epsilon (\mathfrak{i} + \mathfrak{j})/\mathfrak{j}$. Then $x = i + j + \mathfrak{j} = i + \mathfrak{j} = \gamma(i)$, so $Im(\gamma) = (\mathfrak{i} + \mathfrak{j})/\mathfrak{j}$. The result will now follow from the First Isomorphism Theorem if we can show that $Ker(\gamma) = \mathfrak{i} \cap \mathfrak{j}$.

Since $\mathfrak{j}$ acts as identity in $(\mathfrak{i} + \mathfrak{j})/\mathfrak{j}$, we see that $Ker(\gamma)$ is the set of all i in $\mathfrak{i}$ such that $\gamma(i) = i + \mathfrak{j} = \mathfrak{j}$. However, this last equality implies that $i \epsilon \mathfrak{j}$, so we have $Ker(\gamma) = \mathfrak{i} \cap \mathfrak{j}$. Then by applying Theorem 1.1 to γ, we see that $(\mathfrak{i} + \mathfrak{j})/\mathfrak{j} \simeq \mathfrak{i}/Ker(\gamma) = \mathfrak{i}/(\mathfrak{i} \cap \mathfrak{j})$. ■

Theorem 1.3 (*Third Isomorphism Theorem*): *If* $\mathfrak{i}$ *and* $\mathfrak{j}$ *are ideals of* $\mathfrak{g}$, *and* $\mathfrak{i} \subset \mathfrak{j}$, *then* $(\mathfrak{g}/\mathfrak{i})/(\mathfrak{j}/\mathfrak{i}) \simeq \mathfrak{g}/\mathfrak{j}$.

Proof: Define γ: $\mathfrak{g}/\mathfrak{i} \longrightarrow \mathfrak{g}/\mathfrak{j}$ by $\gamma(x + \mathfrak{i}) = x + \mathfrak{j}$. Now let μ_I and μ_J be the respective Lie operators, and perform the familiar calculation $\mu_J(\gamma(x_1 + \mathfrak{i}),\gamma(x_2) + \mathfrak{i}) = \mu_J(x_1 + \mathfrak{j},x_2 + \mathfrak{j}) = \mu(x_1,x_2) + \mathfrak{j} = \gamma(\mu(x_1,x_2) + \mathfrak{i}) = \gamma(\mu_I(x_1 + \mathfrak{i},x_2 + \mathfrak{i}))$. So γ is a homomorphism.

Since $\mathfrak{i} \subset \mathfrak{j}$ and $\mathfrak{i}$ is an ideal, we can see that for $y = x + \mathfrak{j} \,\epsilon\, \mathfrak{g}/\mathfrak{j}$, $y = (x - i) + (i + j) \,\epsilon\, \mathfrak{g}/\mathfrak{i}$. Thus, $Im(\gamma) = \mathfrak{g}/\mathfrak{j}$. As before, the result will again follow from the First Isomorphism Theorem if we can show that $Ker(\gamma) = \mathfrak{g}/\mathfrak{i}$.

Let $x = x_0 + \mathfrak{i}$ be in the kernel of γ. Then $\gamma(x) = \gamma(x_0 + \mathfrak{i}) = x_0 + \mathfrak{j} = \mathfrak{j}$, so $x_0 \,\epsilon\, \mathfrak{j}$. Thus, $x = j + \mathfrak{i} \,\epsilon\, \mathfrak{j}/\mathfrak{i}$. Conversely, it is obvious that for $j + \mathfrak{i}$ in the domain, we have $\gamma(j + \mathfrak{i}) = j + \mathfrak{j} = \mathfrak{j}$, so that $Ker(\gamma) = \mathfrak{j}/\mathfrak{i}$. So by Theorem 1.1, we arrive at $(\mathfrak{g}/\mathfrak{i})/(\mathfrak{j}/\mathfrak{i}) \simeq \mathfrak{g}/\mathfrak{j}$. ■

We will use the concept of isomorphisms extensively below, but we should here mention an important type of isomorphism: An ismorphism φ from $\mathfrak{g}$ to $\mathfrak{g}$ is known as an **automorphism**.

1.5 A Familiar Example of a Lie Algebra

An example of a Lie algebra from basic calculus is given by the *cross product* on $\mathbb{R}^3$. Recall that for $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^3$, we have both that

$$\vec{x} \times (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) \times (\vec{x} + \vec{z})$$
$$(\vec{x} + \vec{y}) \times \vec{z} = (\vec{x} + \vec{z}) \times (\vec{y} + \vec{z})$$

and that

$$\vec{x} \times \vec{x} = 0,$$

so the cross product satisfies both (A1) and (A2). To show that it satisfies the Jacobi identity as well, we can calculate directly (the details are not difficult, but they are time-consuming):

$$\begin{aligned}
&\vec{a} \times (\vec{b} \times \vec{c}) + \vec{c} \times (\vec{a} \times \vec{b}) \\
&= (a_1, a_2, a_3) \times [(b_1, b_2, b_3) \times (c_1, c_2, c_3)] + (c_1, c_2, c_3) \times [(a_1, a_2, a_3) \times (b_1, b_2, b_3)] \\
&= (a_1, a_2, a_3) \times [(b_2c_3 - c_2b_3, b_1c_3 - c_1b_3, b_1c_2 - c_1b_2)] \\
&+(c_1, c_2, c_3) \times [(a_2b_3 - b_2a_3, a_1b_3 - b_1a_3, a_1b_2 - b_1a_2)] \\
&= -(b_1, b_2, b_3) \times (a_3c_2 - a_2c_3, a_3c_1 - a_1c_3, a_2c_1 - a_1c_2) \\
&= -(b_1, b_2, b_3) \times [(c_1, c_2, c_3) \times (a_1, a_2, a_3)] \\
&= -\vec{b} \times (\vec{c} \times \vec{a})
\end{aligned}$$

so (A3) is satisfied as well. We will revisit this Lie algebra briefly at the end of Section 2.2.

2 Linear Lie Algebras

2.1 The General Linear Algebra

For a vector space $\mathbf{V}$ of dimension n, denote by $End(\mathbf{V})$ the set of all linear maps φ from $\mathbf{V}$ to itself. Such maps are called *endomorphisms*. It is well-known that this set is a vector space of dimension n^2[8, 5]. Under the operation $\mu(x, y) = xy - yx$, $End(\mathbf{V})$ becomes a Lie algebra.

Let x,y ϵ $End(\mathbf{V})$, and let r,s ϵ $\mathbb{F}$. Then:

(A1)

$$\begin{aligned} \mu(x, ry + sz) &= x(ry + sz) - (ry + sz)x \\ &= x(ry) + x(sz) - (ry)x - (sz)x \\ &= r(xy) + s(xz) - r(yx) - s(zx) \\ &= r(xy - yx) + s(xz - zx) \\ &= r\mu(x, y) + s\mu(x, z) \ ; \end{aligned}$$

(A2)

$$\begin{aligned} \mu(x, x) &= xx - xx \\ &= 0 \ ; \end{aligned}$$

(A3)

$$\begin{aligned} \mu(x, \mu(y, z)) + \mu(y, \mu(z, x)) + \mu(z, \mu(x, y)) &= \mu(x, yz - zy) + \mu(y, zx - xz) + \mu(z, xy - yx) \\ &= x(yz - zy) - (yz - zy)x + y(zx - xz) \\ &\quad -(zx - xz)y + z(xy - yx) - (xy - yx)z \\ &= xyz - xzy - yzx + zyx + yzx - yxz \\ &\quad -zxy + xzy + zxy - zyx - xyz + yxz \\ &= 0. \end{aligned}$$

In this context, μ is called the **commutator** or **bracket** of the Lie algebra and is denoted by $[\cdot,\cdot]$. Furthermore, $End(\mathbf{V})$ is called the **general linear algebra** of dimension n^2 and is denoted by $\mathfrak{gl}(\mathbf{V})$. If $\mathbf{V}$ is a vector space of dimension n, and if we fix a basis $\{x_i\}$ for $\mathbf{V}$, then we can find a basis $\{e_{ij}\}$ $1 \leq i,j \leq n$, for $\mathfrak{gl}(\mathbf{V})$, and identify it with the set of $n \times n$ matrices[11, 9]. We then denote $\mathfrak{gl}(\mathbf{V})$ by $\mathfrak{gl}(n,\mathbb{F})$.

Any subalgebra of $\mathfrak{gl}(\mathbf{V})$ is called a **linear Lie algebra**. A surprising and elegant result from advanced Lie algebra theory is that *any finte-dimensional abstract Lie algebra over a field of characteristic zero is isomorphic to a linear Lie algebra.* This is known as *Ado's Theorem*[1], and it's proof is far beyond the scope of this paper. However, a similar though more restricted version whose proof is not beyond said scope is the following:

Theorem: Every simple lie algebra is isomorphic to a linear Lie algebra.

We will prove this result in Section 2.3. For the rest of this paper, we will be primarily concerned with what are called the *real* linear Lie algebras, where the underlying field $\mathbb{F}$ is $\mathbb{R}$. Whenever the underlying field is not explicitly mentioned, it will be understood that it is the field of real numbers.

2.2 Examples of Linear Lie Algebras

The Special Linear Algebras

For any matrices x and y in $\mathfrak{gl}(n,\mathbb{F})$, we have $Tr(x+y) = Tr(x) + Tr(y)$ and $Tr(xy) = Tr(yx)$, where $Tr(x)$ is the trace of the matrix x. It follows that

$$Tr([x,y]) = Tr(xy - yx) = Tr(xy) - Tr(yx) = 0,$$

so the trace of the bracket is always zero. It is now trivial that the subspace of $\mathfrak{gl}(n,\mathbb{F})$ given by taking all of the $n \times n$ matrices that have trace zero is a subalgebra of $\mathfrak{gl}(n,\mathbb{F})$, denoted by $\mathfrak{sl}(n,\mathbb{F})$. This subalgebra is known as the *special linear algebra.*

Consider the canonical basis for $\mathfrak{sl}(2,\mathbb{R})$:

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since there are three matrices in this basis, there are nine possible brackets: $[e_i,e_j]$ for $1 \leq i,j \leq 3$. However, by (A2) we can eliminate all brackets of the form $[e_i,e_i]$, so this leaves us with six brackets:

$$\begin{array}{ll} [e_1,e_3], & [e_3,e_1], \\ [e_1,e_2], & [e_2,e_1], \\ [e_2,e_3], & [e_3,e_2]. \end{array}$$

Furthermore, by skew-symmetry, we need only specify the brackets for one column—the values for the other column are then determined by these images. We now calculate the brackets in the first column:

$$\begin{aligned} [e_1,e_2] &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e_3 \\ [e_1,e_3] &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -2e_1 \\ [e_2,e_3] &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 2e_2, \end{aligned}$$

and we have the nice result that the special linear algebra is completely determined by the three equations

$$[e_1,e_3] = -2e_1, \ [e_1,e_2] = e_3, \text{ and } [e_2,e_3] = 2e_2.$$

In terms of the structure constants, we see that there are six non-zero entries: $c_{12}^3 = 1$, $c_{21}^3 = -1$, $c_{13}^1 = c_{32}^1 = -2$, and $c_{23}^2 = c_{31}^2 = 2$.

The Symplectic Algebras

To obtain the *symplectic* algebra, let $dim(\mathbf{V}) = 2n$. This condition of even-dimensionality is necessary[8] to define a non-degenerate form f: $\mathbf{V} \times \mathbf{V} \longrightarrow \mathbb{F}$ by the matrix

$$A = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Take the set of all $x \epsilon \mathfrak{gl}(2n,\mathbb{F})$ satisfying $f(xv,w) = -f(v,xw)$ for $v,w \epsilon \mathbf{V}$. These are all of the transformations that are skew-symmetric relative to the form f. This defines a Lie algebra. To see this, we write

$$\begin{aligned} f([x,y]v,w) &= f((xy-yx)v,w) \\ &= f((xy)v-(yx)v,w) \\ &= f((xy)v,w)-f((yx)v,w) \\ &= f(x(yv),w)-f(y(xv),w). \end{aligned}$$

Since x,y satisfy $f((\cdot)v,w) = -f(v,(\cdot)w)$, and $yv,xv \epsilon \mathbf{V}$, we have

$$\begin{aligned} f(x(yv),w)-f(y(xv),w) &= -f(yv,xw)+f(xv,yw) \\ &= f(v,y(xw))-f(v,x(yw)) \\ &= f(v,(yx)w))+f(v,(xy)w)) \\ &= f(v,(yx)w-(xy)w)) \\ &= f(v,(yx-xy)w)) \\ &= -f(v,(xy-yx)w) \\ &= -f(v,[x,y]w)). \end{aligned}$$

We denote the symplectic algebra by $\mathfrak{sp}(2n,\mathbb{F})$

Consider $\mathfrak{sp}(2,\mathbb{R})$. Here f is the matrix $A_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, so any matrix $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathfrak{sp}(2,\mathbb{R})$ satisfies $A_2 x = -x^T A_2$:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a+d \\ -(a+d) & 0 \end{pmatrix} = 0.$$

This implies that b and c are arbitrary, while $d = -a$, so x must be of the form

$$x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

From this, it is clear that $\mathfrak{sp}(2,\mathbb{R})$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

The Orthogonal Algebras

Consider the subset of $\mathfrak{gl}(n,\mathbb{F})$ given by taking all skew-symmetric $n \times n$ matrices. This yields an *orthogonal* algebra. To see that it preseves the bracket, let A and B be skew-symmetric matrices and calculate

$$[A,B]^T = (AB-BA)^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T = BA - AB = -[A,B].$$

We denote this by $\mathfrak{o}(n,\mathbb{F})$. Consider $\mathfrak{o}(3,\mathbb{R})$ with the canonical basis

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The brackets on these basis vectors define the algebra by the relations

$$[e_1,e_2] = e_3, [e_2,e_3] = e_1, [e_3,e_1] = e_2.$$

Recall that the cross-product of vectors in $\mathbb{R}^3$—which defines a Lie algebra, as was seen in Section 1.5—can be defined by the unit vector equations $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, and $\vec{k} \times \vec{i} = \vec{j}$. Define the map from the Lie algebra $(\times, \mathbb{R}^3)$ to $\mathfrak{o}(3,\mathbb{R})$ by

$$\vec{i} \longrightarrow e_1, \vec{j} \longrightarrow e_2, \vec{k} \longrightarrow e_3.$$

This is a Lie algebra isomorphism given by the matrix $A = I_3$, so the cross-product of vectors in $\mathbb{R}^3$ is really nothing more than the orthogonal linear Lie algebra $\mathfrak{o}(3,\mathbb{R})$.

2.3 Representations

We define a *representation* as a homomorphism from an abstract Lie algebra to a linear Lie algebra; *i.e.*, φ: $\mathfrak{g} \longrightarrow gl(n,\mathbb{F}$ where

$$\varphi(\mu(x,y)) = [\varphi(x), \varphi(y)],$$

is a representation of $\mathfrak{g}$. Representations are the key to demonstrating the result mentioned in Section 2.1, and are generally of fundamental importance to the classification and study of Lie algebras, as well as many other algebraic structures[6]. Representation theory is a rich and vitally important area of mathematics, but what is presented here is only what is fundamentally necessary for the purposes of this paper.

The most important representation is the *adjoint representation.* To understand this, we must first define the *adjoint* of an element of $\mathfrak{g}$. Consider the subset of $\mathfrak{gl}(\mathbf{V})$ given by taking all of the endomorphisms δ of $\mathbf{V}$ that satisfy $\delta(xy) = x\delta(y) + \delta(x)y$. Such a mapping is called a **derivation** of $\mathbf{V}$, and the subset containing them is denoted by $Der(\mathbf{V})$ (or $Der(\mathfrak{g})$ if we are viewing $\mathbf{V}$ as a Lie algebra). It is a subalgebra of $\mathfrak{gl}(\mathbf{V})$. To show this, we must demonstrate that it is a linear map, and that it is closed under the bracket.

Let $\delta,\varepsilon \in Der(\mathbf{V})$, $x,y \in \mathbf{V}$, and $c \in \mathbb{F}$. Then

$$\begin{aligned}
(\delta + c\varepsilon)(xy) &= \delta(xy) + c\varepsilon(xy) \\
&= x\delta(y) + \delta(x)y + cx\varepsilon(y) + c\varepsilon(x)y \\
&= (x\delta(y) + cx\varepsilon(y)) + (\delta(x)y + c\varepsilon(x)y) \\
&= x(\delta + c\varepsilon)(y) + (\delta + c\varepsilon)(x)y,
\end{aligned}$$

so $\delta + c\varepsilon \in Der(\mathbf{V})$. Now consider the commutator $[\delta,\varepsilon]$:

$$\begin{aligned}
[\delta,\varepsilon](xy) &= (\delta\varepsilon - \varepsilon\delta)(xy) \\
&= (\delta\varepsilon)(xy) - (\varepsilon\delta)(xy) \\
&= \delta(\varepsilon(xy)) - \varepsilon(\delta(xy)) \\
&= \delta(x\varepsilon(y) + \varepsilon(x)y) - \varepsilon(x\delta(y) + \delta(x)y) \\
&= \delta(x\varepsilon(y)) + \delta(\varepsilon(x)y) - \varepsilon(x\delta(y)) - \varepsilon(\delta(x)y) \\
&= x\delta(\varepsilon(y)) + \delta(x)\varepsilon(y) + \varepsilon(x)\delta(y) + \delta(\varepsilon(x))y) \\
&\quad -x\varepsilon(\delta(y)) - \varepsilon(x)\delta(y) - \delta(x)\varepsilon(y) - \varepsilon(\delta(x))y \\
&= x\delta(\varepsilon(y)) - x\varepsilon(\delta(y)) + \delta(\varepsilon(x))y) - \varepsilon(\delta(x))y \\
&= x(\delta(\varepsilon(y)) - \varepsilon(\delta(y))) + (\delta(\varepsilon(x)) - \varepsilon(\delta(x)))y \\
&= x(\delta\varepsilon - \varepsilon\delta)(y) + (\delta\varepsilon - \varepsilon\delta)(x)y \\
&= x[\delta,\varepsilon](y) + [\varepsilon,\delta](x)y,
\end{aligned}$$

which demonstrates that $Der(\mathfrak{g})$ is a subalgebra of $\mathfrak{gl}(n,\mathbb{F})$.

Now let $x \in \mathfrak{g} = (\mathbf{V},\mu)$. Then the function $ad_x\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ defined by

$$ad_x(y) = \mu(x, y)$$

is an endomorphism known as the **adjoint** of x. To see this, we calculate $ad_x(y + cz) = [x,y+cz] = [x,y] + c[x,z] = ad_x(y) + cad_x(z)$. We can demonstrate that ad_x is a derivation as well by using the Jacobi identity:

$$\begin{aligned}
ad_x([y,z]) &= [x,[y,z]] \\
&= -[y,[z,x]] - [z,[x,y]] \\
&= -[y,-[x,z]] + [[x,y],z] \\
&= [y,[x,z]] + [[x,y],z] \\
&= [y, ad_x(z)] + [ad_x(y), z].
\end{aligned}$$

We are now ready to define the adjoint representation, which will allow us to prove the theorem that all simple Lie algbras are isomorphic to a linear Lie algebra.

The map $\sigma_{\mathfrak{g}}\colon \mathfrak{g} \longrightarrow Der(\mathfrak{g})$ defined by

$$\sigma_{\mathfrak{g}}(x) = ad_x$$

is a representation, called the **adjoint representation**. To demonstrate this, we must show that it satisfies the homomorphism property:

$$\begin{aligned}
[\sigma_{\mathfrak{g}}(x)\sigma_{\mathfrak{g}}(y)](z) &= [ad_x, ad_y](z) \\
&= (ad_x ad_y)(z) - (ad_y ad_x)(z) \\
&= [x,[y,z]] - [y,[x,z]] \\
&= [x,[y,z]] + [y,[z,x]] \\
&= -[z,[x,y]] \\
&= [[x,y],z] \\
&= ad_{[x,y]}(z) \\
&= \sigma_{\mathfrak{g}}([x,y]).
\end{aligned}$$

As a final note, if $Tr(\sigma_{\mathfrak{g}}(x)) = Tr(ad_x)) = 0$ for all x in $\mathfrak{g}$, then we call $\mathfrak{g}$ **unimodular**. As an example, in any abelian algebra, $ad_x \equiv 0$, so all abelian algebras are unimodular. This will become important in Section 2.6. We are now ready to prove the theorem stated in Section 2.1.

Proof of the Theorem

Theorem: Every simple Lie algebra is isomorphic to a linear Lie algebra.

Proof: Let $\mathfrak{g}$ be a simple Lie algebra, and consider the center of $\mathfrak{g}$, $Z(\mathfrak{g})$. There are three options: $Z(\mathfrak{g})$ is trivial, $Z(\mathfrak{g})$ is proper and non-trivial, or $Z(\mathfrak{g}) = \mathfrak{g}$. By the definition of a simple Lie algebra, $\mathfrak{g}$ is non-abelian, so we must have that $Z(\mathfrak{g}) \neq \mathfrak{g}$. Since $\mathfrak{g}$ is simple, $Z(\mathfrak{g})$ cannot be proper. So we must have that $Z(\mathfrak{g}) = 0$ for any simple Lie algebra.

Now consider the adjoint representation $\sigma_\mathfrak{g}$ from $\mathfrak{g}$ to $Der(\mathfrak{g})$. First, we note that by Proposition 2.2, $Im(\sigma_\mathfrak{g})$ is a subalgebra of $Der(\mathfrak{g})$, and so by extension it is a subalgebra $\mathfrak{gl}(n,\mathbb{F})$, and therefore a linear Lie algebra. So we need to show that $\sigma_\mathfrak{g}$ is a monomorphism.

We see that $Ker(\sigma_\mathfrak{g})$ is the set of all x in $\mathfrak{g}$ such that $ad_x \equiv 0$; in other words, we require $[x,y] = 0$ for all y in $\mathfrak{g}$, so that $Ker(\sigma_\mathfrak{g}) = Z(\mathfrak{g}) = 0$ (these are equal as ideals by Proposition 2.1 and the fact that $\mathfrak{g}$ is non-abelian, as noted above). By Proposition 2.3, this implies that $\sigma_\mathfrak{g}$ is a monomorphism, so we conclude that $\mathfrak{g}$ is isomorphic to the linear Lie algebra $Im(\sigma_\mathfrak{g})$. ■

2.4 Ideals Revisited

Let $\mathfrak{g}$ be a Lie algebra, and recall the definition of the ideal $\mathfrak{ij}$ given in Section 1.3. Since we are now dealing solely with linear Lie algebras, we will denote this by $[\mathfrak{ij}]$—the definition is the same. An important ideal of this type is the **derived algebra** of $\mathfrak{g}$, denoted by $[\mathfrak{gg}]$. It follows immediately from the definition that $\mathfrak{g}$ is abelian if and only if $[\mathfrak{gg}] = 0$ —this is simply the statement that the Lie operator maps all pairs to zero. We use this notion of a derived algebra to define two important sequences of ideals:

The **derived series**:

$$\mathfrak{g}^{(0)} = \mathfrak{g},\ \mathfrak{g}^{(1)} = [\mathfrak{gg}],\ \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}\mathfrak{g}^{(1)}],\ \dots,\ \mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}\mathfrak{g}^{(i-1)}],\ \dots$$

We call $\mathfrak{g}$ **solvable** if and only if $\mathfrak{g}^{(i)} = 0$ for some i. No simple Lie algebra is solvable, since $\mathfrak{g}^{(i)} = \mathfrak{g}$ for all i, because a simple Lie algebra has no proper, non-trivial ideals.

The **lower central series**:

$$\mathfrak{g}^0 = \mathfrak{g},\ \mathfrak{g}^1 = [\mathfrak{gg}],\ \mathfrak{g}^2 = [\mathfrak{gg}^1],\ \dots,\ \mathfrak{g}^i = [\mathfrak{gg}^{i-1}],\ \dots$$

We call $\mathfrak{g}$ **nilpotent** if and only if $\mathfrak{g}^i = 0$ for some i.

We now prove a number of propositions about solvable and nilpotent algebras. We begin with two lemmas that will be of use throughout the following proofs:

Lemma 3.1: *If* $\mathfrak{n}_1 \subset \mathfrak{m}_1$ *and* $\mathfrak{n}_2 \subset \mathfrak{m}_2$*, then* $[\mathfrak{n}_1\mathfrak{n}_2] \subset [\mathfrak{m}_1\mathfrak{m}_2]$.

Proof: If $x \,\epsilon\, [\mathfrak{n}_1\mathfrak{n}_2]$, then $x = \lambda_i[x_i,y_i]$ for x_i in $\mathfrak{n}_1$ and y_i in $\mathfrak{n}_2$. But then x_i is in $\mathfrak{m}_1$ and y_i is in $\mathfrak{m}_2$, so $\lambda_i[x_i,y_i]$ is in $[\mathfrak{m}_1\mathfrak{m}_2]$. ■

Lemma 3.2: $(\mathfrak{g}^{(i)})^{(j)} = \mathfrak{g}^{(i+j)}$.

Proof: We must first demonstrate as a sublemma that $[\mathfrak{n}^{(i)}\mathfrak{n}^{(i)}] = [\mathfrak{nn}]^{(i)}$. We proceed by induction. We have as the base case that $[\mathfrak{n}^{(0)}\mathfrak{n}^{(0)}] = [\mathfrak{nn}] = [\mathfrak{nn}]^{(0)}$. Suppose that $[\mathfrak{n}^{(k)}\mathfrak{n}^{(k)}] = [\mathfrak{nn}]^{(k)}$. Then $[\mathfrak{n}^{(k+1)}\mathfrak{n}^{(k+1)}] = [[\mathfrak{n}^{(k)}\mathfrak{n}^{(k)}][\mathfrak{n}^{(k)}\mathfrak{n}^{(k)}]] = [[\mathfrak{nn}]^{(k)}[\mathfrak{nn}]^{(k)}] = [\mathfrak{nn}]^{(k+1)}$.

To prove the lemma, we fix j and induct on i. We have as the base case that $\mathfrak{n}^{(0+j)} = \mathfrak{n}^{(j)} = (\mathfrak{n}^{(0)})^{(j)}$. Now suppose that $\mathfrak{n}^{(k+j)} = (\mathfrak{n}^{(k)})^{(j)}$. Then by the above sublemma, we have that $\mathfrak{n}^{((k+1)+j)} = \mathfrak{n}^{((k+j)+1)} = [\mathfrak{n}^{(k+j)}\mathfrak{n}^{(k+j)}] = [(\mathfrak{n}^{(k)})^{(j)}(\mathfrak{n}^{(k)})^{(j)}] = [(\mathfrak{n}^{(k)}\mathfrak{n}^{(k)}]^{(j)} = (\mathfrak{n}^{(k+1)})^{(j)}$. ■

The previous lemma also holds for the lower central series; the proof is similar.

Proposition 3.1: *Every nilpotent Lie algebra is solvable.*

Proof: We proceed by induction. We have as the base case that $\mathfrak{g}^{(0)} = \mathfrak{g} = \mathfrak{g}^0$. Now suppose $\mathfrak{g}^{(k)} \subset \mathfrak{g}^k$. Then by Lemma 3.1 we have that $\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}\mathfrak{g}^{(k)}] \subset [\mathfrak{g}\mathfrak{g}^k] = \mathfrak{g}^{k+1}$, since $\mathfrak{g}^{(k)} \subset \mathfrak{g}$. Thus, if we suppose that $\mathfrak{g}^i = 0$, then we must have that $\mathfrak{g}^{(i)} = 0$ as well, so every nilpotent algebra must also be solvable. ■

Proposition: *Let* $\mathfrak{g}$ *be a Lie algebra,* $\mathfrak{h}$ *be an arbitrary subalebra of* $\mathfrak{g}$*,* $\mathfrak{i}$ *and* $\mathfrak{j}$ *be arbitrary ideals of* $\mathfrak{g}$*, and let* φ *be a homomorphism from* $\mathfrak{g}$ *to* $\mathfrak{g}_0$*. Then*

(3.2) *If* $\mathfrak{g}$ *is solvable (nilpotent), then so are* $\mathfrak{h}$ *and* $Im(\varphi)$;

Proof: We demonstrate this for solvable algebras; the nilpotent case is similar. We proceed via induction. First, as the base case, we have that $\mathfrak{h} = \mathfrak{h}^{(0)} \subset \mathfrak{g}^{(0)} = \mathfrak{g}$. Then if we suppose $\mathfrak{h}^{(k)} \subset \mathfrak{g}^{(k)}$, we see that $\mathfrak{h}^{(k+1)} = [\mathfrak{h}^{(k)}\mathfrak{h}^{(k)}] \subset [\mathfrak{g}^{(k)}\mathfrak{g}^{(k)}] = \mathfrak{g}^{(k+1)}$, and so every subalgebra of a solvable algebra is solvable.

Now as the base case we see that $(\varphi(\mathfrak{g}))^{(0)} = \varphi(\mathfrak{g}) = \varphi(\mathfrak{g}^{(0)})$. Then if we suppose that $(\varphi(\mathfrak{g}))^{(k)} \subset \varphi(\mathfrak{g}^{(k)})$, we see that since φ is a homomorphism $(\varphi(\mathfrak{g}))^{(k+1)} = [(\varphi(\mathfrak{g}))^{(k)}(\varphi(\mathfrak{g}))^{(k)}]$ $\subset [\varphi(\mathfrak{g}^{(k)})\varphi(\mathfrak{g}^{(k)})] = \varphi([\mathfrak{g}^{(k)}\mathfrak{g}^{(k)}]) = \varphi(\mathfrak{g}^{(k+1)})$, and so every homomorphic image of a solvable algebra is solvable. ■

(3.3) *If* $\mathfrak{i}$ *and* $\mathfrak{g}/\mathfrak{i}$ *are solvable, then so is* $\mathfrak{g}$;

Proof: Suppose that $(\mathfrak{g}/\mathfrak{i})^{(i)} = 0$. If we let π be the canonical homomorphism from $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{i}$, then $\pi(\mathfrak{g}^{(i)}) = (\pi(\mathfrak{g}))^{(i)} = (\mathfrak{g}/\mathfrak{i})^{(i)} = 0$, by Proposition 3.2. Thus $\mathfrak{g}^{(i)} \subset Ker(\pi) = \mathfrak{i}$. So if $\mathfrak{i}^{(j)} = 0$, then by Lemma 3.2 $(\mathfrak{g}^{(i)})^{(j)} = \mathfrak{g}^{(i+j)} = 0$, so $\mathfrak{g}$ is solvable. ■

(3.4) *If* $\mathfrak{g}/Z(\mathfrak{g})$ *is nilpotent, then so is* $\mathfrak{g}$;

Proof: Suppose $\mathfrak{g}/Z(\mathfrak{g})^i = 0$. As in the proof of Proposition 3.3, we have that $\mathfrak{g}^i$ is in $Ker(\pi)$, so $\mathfrak{g}^i \subset Z(\mathfrak{g})$. Then $\mathfrak{g}^{i+1} = [\mathfrak{g}\mathfrak{g}^i] \subset [\mathfrak{g}\ Z(\mathfrak{g})]$. We see that for $x \in [\mathfrak{g}\ Z(\mathfrak{g})]$, $x = \lambda_i[x_i,z_i] = 0$, since z_i is in the center. Thus $[\mathfrak{g}\ Z(\mathfrak{g})] = 0$, so $\mathfrak{g}^{i+1} = 0$, and $\mathfrak{g}$ is nilpotent. ■

(3.5) *If* $\mathfrak{i}$ *and* $\mathfrak{j}$ *are solvable, then so is* $\mathfrak{i} + \mathfrak{j}$;

Proof: By the second isomorphism theorem, $(\mathfrak{i} + \mathfrak{j})/\mathfrak{j} \simeq \mathfrak{i}/(\mathfrak{i} \cap \mathfrak{j})$. Under the canonical homomorphism, we see that $\mathfrak{i}/(\mathfrak{i} \cap \mathfrak{j})$ is a homomorphic image of $\mathfrak{i}$, and so it is also solvable. But then $(\mathfrak{i} + \mathfrak{j})/\mathfrak{j}$ is also solvable, so by Proposition 3.3, if $\mathfrak{j}$ is solvable, then we see that $\mathfrak{i} + \mathfrak{j}$ is solvable. ■

(3.6) *If* $\mathfrak{g}$ *is non-zero and nilpotent, then* $Z(\mathfrak{g}) \neq 0$*.*

Proof: Suppose that $\mathfrak{g}$ is nilpotent. Then we can write $\mathfrak{g}^i = 0$, where i is the smallest such integer. Then $[\mathfrak{g}\mathfrak{g}^{i-1}] = 0$, where $\mathfrak{g}^{i-1} \neq 0$, so we have that $\mathfrak{g}^{i-1} \subset Z(\mathfrak{g})$. ■

There is one last ideal to mention—the **normalizer** of a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, denoted by

$N_{\mathfrak{g}}(\mathfrak{h})$. It is defined as

$$N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid ad_x(h) \in \mathfrak{h},\ h \in \mathfrak{h}\}.$$

To see that it is a subalgebra, we note that if x and y are in $N_{\mathfrak{g}}(\mathfrak{h})$, then so is $ad_{[x,y]}(h) = [[x,y],h]$ by the Jacobi identity, as seen in section 2.3. If $\mathfrak{h}$ is an ideal, then it is clear that $ad_x(h) = [x,h] \in \mathfrak{h}$ for all x, so that $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}$, and conversely. This will be important in the next section.

2.5 Engel's Theorem

There is an important theorem that can be very useful in determining when a Lie algebra is nilpotent, known as *Engel's Theorem.* We call $x \in \mathfrak{gl}(\mathbf{V})$ **ad-nilpotent** if ad_x is a nilpotent endomorphism. This means that there is an i such that $(ad_x)^i = 0$. Then we have:

Theorem 2.3 (Engel's Theorem): *$\mathfrak{g}$ is a nilpotent Lie algebra if and only if all elements of $\mathfrak{g}$ are ad-nilpotent.*

This theorem is also interesting in it's own right, so we here dedicate a section to it's proof. We first demonstrate two essential lemmas.

Lemma 2.3: *If $x \in \mathfrak{gl}(\mathbf{V})$ is nilpotent, then so is ad_x.*

Proof: Define the endomorphisms $\sigma_x(y) = xy$ and $\tau_x(y) = yx$, known respectively as left and right translation. Since x is nilpotent, we can write $x^i = 0$. Now we have that

$$(\sigma_x)^i(y) = x^i y = 0$$

and

$$(\tau_x)^i(y) = yx^i = 0,$$

so the left and right translations are also nilpotent. Furthermore, we have that $(\sigma_x\tau_x)(\mathrm{y}) = xyx = (\tau_x\sigma_x)(\mathrm{y})$, so we can apply the binomial theorem and write:

$$(ad_x(y))^{2i} = [x,y]^{2i} = (xy - yx)^{2i} = (\sigma_x(y) - \tau_x(y))^{2i} = \sum_{k=0}^{2i} \binom{2i}{k} \sigma_x^{2i-k}(-\tau_x)^k.$$

For $k < i$, $2i - k > i$; otherwise, $k \geq i$, so the sum is zero. Therefore, we must have that $(ad_x)^{2i} = 0$, so ad_x is nilpotent. ■

Lemma 2.4: *Let $\mathfrak{g}$ be a linear Lie algebra ($\mathbf{V} \neq 0$) consisting of nilpotent endomorphisms. Then there is a non-zero vector $v \in \mathbf{V}$ such that $xv = 0$ for all $x \in \mathfrak{g}$.*

Proof: We proceed via induction on $dim(\mathfrak{g})$. As the base case, let $dim(\mathfrak{g}) = 1$. Then every x in $\mathfrak{g}$ is a multiple of a single endomorphism e, which is nilpotent. Suppose $e^i = 0$, and let λ be an eigenvalue of e. Then $ev = \lambda v$ for some non-zero v in $\mathbf{V}$. The fact that $e^k v = \lambda^k v$ implies

$$e^{k+1}v = ee^k v = e\lambda^k v = \lambda^k ev = \lambda^{k+1}v$$

shows that $e^i v = \lambda^i v = 0$, so $\lambda = 0$, and we arrive at $ev = 0$.

Now suppose that the conclusion holds for all algebras of dimension less than n, and let $\mathfrak{h}$ be a maximal proper subalgebra of $\mathfrak{g}$, where $dim(\mathfrak{g}) = n$. By Lemma 2.3, every h in $\mathfrak{h}$ is ad-nilpotent, and these are linear maps over the *vector space* $\mathbf{V}/\mathbf{H}$ via the action

$$ad_h(x + \mathbf{H}) = ad_h(x) + \mathbf{H}.$$

We conclude that $\sigma_{\mathfrak{h}}(\mathfrak{h})$ is a subalgebra of $gl(\mathbf{V}/\mathbf{H})$, and since $dim(\sigma_{\mathfrak{h}}) \leq dim(\mathfrak{h}) < dim(\mathfrak{g})$, we can apply the induction hypothesis to conclude that there is a vector $a + \mathbf{H} \neq \mathbf{H}$, such that $ad_h(a) + \mathbf{H} = \mathbf{H}$ for all h in $\mathbf{H}$ $(\mathfrak{h})$. So $[h,a]$ is in $\mathfrak{h}$ for all h, while a is not in $\mathfrak{h}$, so $\mathfrak{h}$ is properly included in $N_{\mathfrak{h}}(\mathfrak{g})$.

However, since $\mathfrak{h}$ was supposed to be a *maximal* proper subalgebra, this forces $N_{\mathfrak{h}}(\mathfrak{g})$ $= \mathfrak{g}$ (otherwise, $\mathfrak{h}$ would not be maximal). Therefore, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Furthermore, if $dim(\mathbf{V}/\mathbf{H}) > 1$, then $dim(\mathfrak{h}) > 2$, which implies that there is a proper subalgebra of $\mathfrak{g}$ properly containing $\mathfrak{h}$. This, however, contradicts the maximality of $\mathfrak{h}$, so we conclude that $\mathfrak{h}$ has co-dimension one, which allows us to write $\mathfrak{g} = \mathfrak{h} + span(g)$, where g is in $\mathfrak{g} - \mathfrak{h}$.

By induction, $K = \{v \in \mathbf{V} \mid hv = 0,\ h \in \mathfrak{h}\}$ is non-empty. We need to show that for x $\in span(g)$, $xk = 0$ for some $k \in K$. Since x is nilpotent, we must have that its eigenvalues are all zero. Since $\mathfrak{h}$ is an ideal, K is invariant under x:

$$(hx)k = h(xk) = x(hk) - [x, h]k = x0 - 0 = 0,$$

for $x \in span(g)$, $h \in \mathfrak{h}$, and $k \in K$. Then x restricted to K has an eigenvector k in K, so that $xk = 0$. We conclude that there exists a v in $\mathbf{V}$ such that $xv = 0$, for all x in $\mathfrak{g}$. ■

Note that absolutely no assumptions about the underlying field of the vector space were made in this proof. This level of generality holds in the proof of Engel's Theorem.

Proof of Engel's Theorem: If we assume that $\mathfrak{g}$ is nilpotent—say $\mathfrak{g}^n$—then

$$[x_1, [x_2, [...[x_n, y]...]]] = 0, \text{ or } ad_{x_1}(ad_{x_2}(...(ad_{x_n}(y))...) = 0,$$

for any x_i,$y \in \mathfrak{g}$. In particular, if we choose $x_i = x_{i+1}$ for $i = 1$,...,$n-1$, and $y \neq 0$, then we have that $(ad_x)^n = 0$, for all x in $\mathfrak{g}$, so every element of $\mathfrak{g}$ is ad-nilpotent.

Now we assume that all of the elements of $\mathfrak{g}$ are ad-nilpotent, and we induct on the dimension of $\mathfrak{g}$ (assume without loss of generality that $\mathfrak{g} \neq 0$). If $dim(\mathfrak{g}) = 1$, then the result is trivial, since the abelian algebra is trivially nilpotent (see Section 2.6 below). Now assume that the result holds for $dim(\mathfrak{g}) < n$, and let $dim(\mathfrak{g}) = n$. We saw in Section 2.3 that the algebra $\sigma_{\mathfrak{g}}(\mathfrak{g})$ is a subalgebra of $\mathfrak{gl}(\mathfrak{g})$, so we can apply Lemma 2.4 to conclude that there exists a non-zero x in $\mathfrak{g}$ such that $[y,x] = 0$ for all y in $\mathfrak{g}$. This implies that $x \in Z(\mathfrak{g})$ so we must have $Z(\mathfrak{g}) \neq 0$. The quotient algebra $\mathfrak{g}/Z(\mathfrak{g})$ must have a strictly smaller dimension than $\mathfrak{g}$, because otherwise we would have $\mathfrak{g} = 0$. Since all of the elements of $\mathfrak{g}$ and $Z(\mathfrak{g})$ are ad-nilpotent, and since the sum of ad-nilpotent maps is also ad-nilpotent, $\mathfrak{g}/Z(\mathfrak{g})$ consists of ad-nilpotent endomorphisms, and so we can apply the induction hypothesis to conclude that $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent. Then by Proposition 3.4, $\mathfrak{g}$ must also be nilpotent. ■

2.6 Classifications of Lie Algebras

In this section we will discuss the classification of all 1-, 2-, 3-dimensional real Lie algebras. All 3- and 4- dimensional real (and complex) Lie algebras have been classified up to isomorphism with a linear Lie algebra[12], but we here are only concerned with those algebras of dimension less than or equal to 3.

The 1- and 2-Dimensional Lie Algebras

Start with a one-dimensional vector space $\mathbf{V}$, with a basis $\{e\}$. If we now define a Lie operator μ on this space, we see that for all x,y in $\mathbf{V}$

$$\mu(x, y) = \mu(re, se) = rs\mu(e, e) = 0,$$

for some r,s in $\mathbb{R}$. In terms of the structure constants, c_{ij}^k is always zero, since there is only one basis vector. Thus, any Lie algebra defined on a one-dimensional vector space is necessarily isomorphic to the abelian algebra. This result can clearly be generalized to cover all underlying fields since (A2) holds regardless of $char(\mathbb{F})$. Note that we can always define $\mu(x, y) = 0$ regardless of the dimension of $\mathbf{V}$, so in the following examples, we are interested only in the *non-abelian* 2- and 3-dimensional algebras.

For a more interesting case, we turn to the two-dimensional real Lie algebras, so let dim($\mathbf{V}$) = 2 and fix a basis $\{e_1, e_2\}$ for $\mathbf{V}$. It should be clear that there is only one non-trivial product for these basis vectors: $\mu(e_1, e_2) = -\mu(e_2, e_1)$. Thus, for any x,y in $\mathbf{V}$, we have that $\mu(x, y)$ is a scalar multiple of $\mu(e_1, e_2)$, so μ must be of the form $\mu(e_1, e_2) = re_1 + se_2$, for some scalars r,s. In order to get a non-abelian algebra, we impose without loss of generality that $s \neq 0$. We claim that this algebra is isomorphic to the one given by

$$\mu_0(\alpha, \beta) = \beta.$$

Define the linear map φ by the matrix:

$$A = \begin{pmatrix} s & -r \\ 0 & \frac{1}{s} \end{pmatrix}.$$

We can clearly see that

$$A^{-1} = \begin{pmatrix} \frac{1}{s} & r \\ 0 & s \end{pmatrix},$$

so φ is invertible and the two algebras are isomorphic as vector spaces. Now we show that it satisfies the homomorphism property, *i.e.*, that it preserves the Lie operator:

$$\begin{aligned} A^{-1}\mu_0(Ae_1, Ae_2) &= A^{-1}\mu_0(s\alpha, -r\alpha + \frac{1}{s}\beta) \\ &= -rsA^{-1}\mu_0(\alpha, \alpha) + A^{-1}\mu_0(\alpha, \beta) \\ &= A^{-1}\beta \\ &= re_1 + se_2 \\ &= \mu(e_1, e_2) \end{aligned}$$

Thus there is one non-abelian Lie algebra of dimension two up to isomorphism. The linear Lie algebra that exemplifies this abstract algebra is the set ξ of matrices of the following form with the basis B:

$$\xi = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}, \qquad B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

which is nothing more than the subalgebra of $\mathfrak{gl}(2,\mathbb{R})$ consisting of all upper triangular matrices with trace 0 (a subalgebra of $\mathfrak{sl}(2,\mathbb{R})$).

The Bianchi Classification

The classification of the 3-dimensional real Lie algebras is of direct relevance to cosmological models because of the connection between real linear Lie algebras and the various groups of geometrical physical operators[4]. It can be achieved via the *Bianchi classification*. In this section, we will use the bracket notation for clarity. We begin by recalling that any 3-dimensional algebra is determined by the three brackets $[e_2,e_3]$, $[e_3,e_1]$, and $[e_1,e_2]$ of basis vectors, the others being given by anti-symmetry. For each bracket, we write out the structure coefficients:

$$[e_2, e_3] = c^k_{23}e_k, \quad [e_3, e_1] = c^k_{31}e_k, \quad [e_1, e_2] = c^k_{12}e_k.$$

Given this, we can describe the algebra by the matrix equation

$$([e_2, e_3], [e_3, e_1], [e_1, e_2]) = (e_1, e_2, e_3)C$$

where C is the matrix of structure coefficients

$$C = \begin{pmatrix} c^1_{23} & c^1_{31} & c^1_{12} \\ c^2_{23} & c^2_{31} & c^2_{12} \\ c^3_{23} & c^3_{31} & c^3_{12} \end{pmatrix}.$$

The strategy now is to find a condition on C that is equivalent to the Jacobi identity. This will tell us which structure coefficients are permissible. Thus, we can classify all 3-dimensional real Lie algebras in terms of these constraints. First, we write $C = S + T$, where S is a symmetric matrix and

$$T = \begin{pmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{pmatrix} \quad \text{where} \quad t = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}.$$

We must now prove the following: *the Jacobi identity holds if and only if $St = 0$.* To demonstrate this, we first compute:

$$\begin{aligned}[][[e_2, e_3], e_1] + [[e_3, e_1], e_2] + [[e_1, e_2], e_3] &= [c^k_{23}e_k, e_1] + [c^k_{31}e_k, e_2] + [c^k_{12}e_k, e_3] \\ &= (c^2_{12} - c^3_{31})[e_2, e_3] + (c^3_{23} - c^1_{12})[e_3, e_1] + (c^1_{31} - c^2_{23})[e_1, e_2].\end{aligned}$$

Since $C = S + T$, we see that $c^2_{12} - c^3_{31} = s_{32} + t_{32} - s_{23} - t_{23} = 2t_1$, from the fact that S is symmetric and T is skew-symmetric. The second and third terms are computed similarly. So we have

$$\begin{aligned}[][[e_2, e_3], e_1] + [[e_3, e_1], e_2] + [[e_1, e_2], e_3] &= 2t_1[e_2, e_3] + 2t_2[e_3, e_1] + 2t_3[e_1, e_2] \\ &= 2([e_2, e_3], [e_3, e_1], [e_1, e_2])t \\ &= 2(e_1, e_2, e_3)Ct.\end{aligned}$$

We see that since $Tt = 0$, $Ct = (S + T)t = St + Tt = St$. Thus, if the Jacoby identity holds, then $2(e_1,e_2,e_3)Ct = 0$. Multiplying on either side by $(e_1,e_2,e_3)^T$, we arrive at $St = 0$. Clearly, if $Ct = 0$, the Jacobi identity for the given basis holds as well. To see that this result is invariant under a change of basis, we let B be a change of basis matrix and write $y_i = b_{1i}x_1 + b_{2i}x_2 + b_{3i}x_3$. One can then use Matlab (the computation is very lengthy) to show that

$$[[f_2, f_3], f_1] + [[f_3, f_1], f_2] + [[f_1, f_2], f_3] = det(B)([[e_2, e_3], e_1] + [[e_3, e_1], e_2] + [[e_1, e_2], e_3]),$$

so that

$$[[f_2, f_3], f_1] + [[f_3, f_1], f_2] + [[f_1, f_2], f_3] = 2det(B)(e_1, e_2, e_3)Ct.$$

Thus, we conclude that the result holds regardless of basis, and the equivalency is proven.

Using this equivalency, it can be shown[2, 6] that every 3-dimensional real Lie algebra is equivalent to either $so(3)$, $sl(2,\mathbb{R})$, or an algebra of the form

$$\begin{aligned} [e_2, e_3] &= ae_1 + be_2 \\ [e_3, e_1] &= ce_1 + de_2 \\ [e_1, e_2] &= 0 \end{aligned}$$

where the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is of one of the following forms:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \quad \begin{pmatrix} \alpha & 1 \\ -1 & -\alpha \end{pmatrix}$$

where α is a positive real number. The previous classification scheme is known as the **Bianchi classification** of the Lie algebras. Each of these algebras can be classified according to whether or not it is simple, solvable, nilpotent, or unimodular[11, 7]. We show this for two particular algebras.

The 3-dimensional real *Heisenberg* algebra, denoted by $\mathfrak{H}$ is both nilpotent (and therefore also solvable) and unimodular. It is defined by the single bracket $[e_2,e_3] = e_1$. To see that it is nilpotent, we note that $[\mathfrak{H},\mathfrak{H}]$ is just a scalar multiple the basis vector e_1:

$$[\mathfrak{H}, \mathfrak{H}] = [a_i e_i, b_j e_j] = a_i b_j [e_i, e_j] = (a_2 b_3 - a_3 b_2) e_1.$$

However, all brackets involving this vector are zero, so that

$$[\mathfrak{H}, [\mathfrak{H}, \mathfrak{H}]] = [\mathfrak{H}, ke_1] = [a_i e_i, ke_1] = 0,$$

which is equivalent to $\mathfrak{H}^2 = 0$, so the Heisenberg algebra is nilpotent.

We can also use Engel's Theorem to establish that the Heisenberg algebra is nilpotent. Clearly, ad_{e_1} is nilpotent, and it is easy to very that

$$ad_{e_2}^2 = ad_{e_3}^2 = 0,$$

so that all elements of $\mathfrak{H}$ are ad-nilpotent. Then by Engel's Theorem, $\mathfrak{H}$ is nilpotent.

To see that it is unimodular, we must show that $Tr(ad_{e_i})) = 0$ for each basis element e_i in $\mathfrak{H}$ (this is sufficient since $ad_x = ad_{\sum a_i e_i} = \sum a_i ad_{e_i}$, so $Tr(ad_x) = \sum a_i Tr(ad_i)$). We can calculate

$$\begin{aligned} &ad_{e_1}(e_1) = 0, \quad ad_{e_1}(e_2) = 0, \quad ad_{e_1}(e_3) = 0 \\ &ad_{e_2}(e_1) = 0, \quad ad_{e_2}(e_2) = 0, \quad ad_{e_2}(e_3) = e_1 \\ &ad_{e_3}(e_1) = 0, \quad ad_{e_3}(e_2) = -e_1, \quad ad_{e_3}(e_3) = 0 \end{aligned}$$

so the matrices of ad_{e_i} are

$$ad_{e_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ad_{e_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ad_{e_3} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

which are clearly all of trace zero, so the Heisenberg algebra is unimodular.

The special linear algebra is also unimodular. It has the basis $[e_1,e_3]$ = -$2e_1$, $[e_2,e_3] = 2e_2$, and $[e_1,e_2] = e_3$, and we can see that

$$\begin{aligned} ad_{e_1}(e_1) = 0, \quad & ad_{e_1}(e_2) = e_3, \quad & ad_{e_1}(e_3) = -2e_1, \\ ad_{e_2}(e_1) = -e_3, \quad & ad_{e_2}(e_2) = 0, \quad & ad_{e_2}(e_3) = 2e_2, \\ ad_{e_3}(e_1) = 2e_1, \quad & ad_{e_3}(e_2) = -2e_2, \quad & ad_{e_3}(e_3) = 0. \end{aligned}$$

So the matrices of ad_{e_i} are

$$ad_{e_1} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad ad_{e_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix} \quad ad_{e_3} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These are all of trace zero, so $\mathfrak{sl}(2,\mathbb{R})$ is unimodular. It is also simple[9], and therefore neither solvable or nilpotent.

3 Contractions of Lie Algebras

3.1 The Definition of a Contraction

The Automorphism Group $GL(\mathbf{V})$

Consider the subset of $End(\mathbf{V})$ consisting of all *invertible* endomorphisms of $\mathbf{V}$, where $dim(\mathbf{V}) = n$. In order for a linear map γ from $\mathbf{V}$ to $\mathbf{V}$ to have an inverse γ^{-1}, γ must be a bijective map, so this subset consists of all isomorphisms from $\mathbf{V}$ to itself, which are known as **automorphisms**, denoted by $GL(\mathbf{V})$, or in matrix terms, $GL(n,\mathbb{R})$. For the purposes of this section, we will identify $GL(\mathbf{V})$ with $GL(n,\mathbb{R})$, and identify an invertible endomorphism γ with its associated non-singular matrix $[T]_\gamma$. For the sake of brevity, we will simply write T for $[T]_\gamma$.

Under the operation of matrix multiplication (also to be understood as function composition), $GL(n,\mathbb{R})$ is a group: Associativity is given by the fact that matrix multiplication is always associative. It should be clear that the identity matrix I_n serves as the identity element. It is non-singular with the inverse $(I_n)^{-1} = I_n$, and for any $T \in GL(n,\mathbb{R})$, we have

$$(TI_n)x = T(I_nx) = Tx = I_n(Tx) = (I_nT)x,$$

for any $x \in \mathbf{V}$. By definition, every T has an inverse T^{-1}, with $T^{-1}T = TT^{-1} = I_n$.

The Definition of a Contraction

Consider now a function T: $(0,1] \longrightarrow GL(n,\mathbb{R})$. For $\varepsilon \in (0,1]$, $T(\varepsilon)$—or more simply, T_ε—is by definition an isomorphism of the vector space $\mathbf{V}$. Given a Lie algebra $\mathfrak{g}$ with a Lie bracket $[\cdot,\cdot]$, we can define a new Lie bracket, $[\cdot,\cdot]_\varepsilon$, in the following way:

$$[x, y]_\varepsilon = T_\varepsilon^{-1}[T_\varepsilon x, T_\varepsilon y],$$

for all x,y in $\mathbf{V}$. Since $T_\varepsilon \,\epsilon\, GL(n,\mathbb{R})$, then we have that $\mathfrak{g}_\varepsilon = (\mathbf{V},[\cdot,\cdot]_\varepsilon)$ is isomorphic to $\mathfrak{g}$. If

$$\lim_{\varepsilon\to 0^+}[x,y]_\varepsilon = \lim_{\varepsilon\to 0^+} T_\varepsilon^{-1}[T_\varepsilon x, T_\varepsilon y] = [x,y]_0$$

exists for all x,y in $\mathbf{V}$, then the Lie algebra $\mathfrak{g}_0 = (\mathbf{V},[\cdot,\cdot]_0)$ is called a **contraction** of $\mathfrak{g}$, and we write $\mathfrak{g} \longrightarrow \mathfrak{g}_0$, or if the specific matrix-family T_ε is important, $\mathfrak{g} \xrightarrow{T_\varepsilon} \mathfrak{g}_0$ (we also refer to the limiting process itself as a contraction). If T is continuous for all ε, then $\mathfrak{g}_0$ is called a *continuous* contraction. In this paper, we are concerned only with continuous contractions, so we will use the term contraction to mean a continuous contraction, unless noted otherwise. We often refer to ε as the *contraction parameter* and to the matrix family T as the *contraction matrix*.

For any Lie algebra $\mathfrak{g}$, there exist at least two contractions. If we define the contraction matrix by $T_\varepsilon = \varepsilon I_n$, then we have that

$$\begin{aligned}
[x,y]_\varepsilon &= (\varepsilon I_n)^{-1}[(\varepsilon I_n)x,(\varepsilon I_n)y] \\
&= (\varepsilon^{-1} I_n)[\varepsilon(I_n x),\varepsilon(I_n y)] \\
&= \varepsilon^{-1}[\varepsilon x,\varepsilon y] \\
&= \varepsilon[x,y],
\end{aligned}$$

so we see that

$$\lim_{\varepsilon\to 0^+}[x,y]_\varepsilon = \lim_{\varepsilon\to 0^+}\varepsilon[x,y] = 0.$$

Thus, every Lie algebra can be contracted to the abelian one. This is called the **trivial** contraction of $\mathfrak{g}$.

Now define T by $T_\varepsilon = I_n$. Then we have that

$$\lim_{\varepsilon\to 0^+}(I_n)^{-1}[I_n x, I_n y]_\varepsilon = [x,y],$$

so $\mathfrak{g} \longrightarrow \mathfrak{g}$. Thus, every Lie algebra can be contracted to itself. More generally, if $\mathfrak{g} \longrightarrow \mathfrak{g}_0$, where $\mathfrak{g}_0$ is isomorphic to $\mathfrak{g}$, then this is called an **improper** contraction. Otherwise, the contraction is **proper**. Improper contractions amount to little more than a change of basis, since the algebras are isomorphic. In order for a contraction to be proper, either $\lim_{\varepsilon\to 0^+} T_\varepsilon$ fails to exist for at least one component of T_ε or

$$\lim_{\varepsilon\to 0^+} T_\varepsilon = T_0$$

exists, but T_0 is singular.

It is often useful to rewrite the definition of a contraction in terms of the structure coefficients. First note that since T_ε is linear, if we fix a basis $\{e_i\}$, $1 \le i \le n$, then $T_\varepsilon e_i = \sum (T_\varepsilon)_{il} e_l = (T_\varepsilon)_i^l e_l$, where summation over l is understood. This allows us to write

$$\begin{aligned}
T_\varepsilon^{-1}[T_\varepsilon e_i, T_\varepsilon e_j] &= T_\varepsilon^{-1}[(T_\varepsilon)_i^l e_l, (T_\varepsilon)_j^m e_m] \\
&= (T_\varepsilon)_i^l (T_\varepsilon)_j^m T_\varepsilon^{-1}[e_l, e_m] \\
&= (T_\varepsilon)_i^l (T_\varepsilon)_j^m T_\varepsilon^{-1}(c_{lm}^k e_k) \\
&= (T_\varepsilon)_i^l (T_\varepsilon)_j^m c_{lm}^k T_\varepsilon^{-1} e_k \\
&= (T_\varepsilon)_i^l (T_\varepsilon)_j^m c_{lm}^k T_\varepsilon^{-1} e_k \\
&= (T_\varepsilon)_i^l (T_\varepsilon)_j^m (T_\varepsilon^{-1})_k^r c_{lm}^k e_r.
\end{aligned}$$

We can now define the new structure coefficients in the limit as ε goes to zero:

$$\lim_{\varepsilon \to 0^+} (T_\varepsilon)^l_i (T_\varepsilon)^m_j (T^{-1}_\varepsilon)^r_k c^k_{lm} = \overset{\frown}{c}{}^r_{ij},$$

where the $\overset{\smile}{c}{}^r_{ij}$ are the components of the new structure coefficient tensor. This is well-defined since the new coefficients will satisfy

$$\begin{gathered} \overset{\frown}{c}{}^k_{ij} + \overset{\frown}{c}{}^k_{ji} = \overset{\frown}{c}{}^k_{ii} = 0 \\ \textstyle\sum_{k=1}^n (\overset{\frown}{c}{}^k_{ij} \overset{\frown}{c}{}^q_{kp} + \overset{\frown}{c}{}^k_{jp} \overset{\frown}{c}{}^q_{kp} + \overset{\frown}{c}{}^k_{pi} \overset{\frown}{c}{}^q_{kj}) = 0. \end{gathered}$$

whenever the c^k_{lm} do[12].

Subclasses of Contractions

Contractions first arose in the study of the limits of bases defining groups of operators in mechanical systems[12]. Often in physical applications, it is useful and even necessary to restrict the form that the contraction matrix T_ε is allowed to take. The *Saletan contractions* are given by matrices of the form

$$T_\varepsilon = T_0 + \varepsilon T_1,$$

where T_0 and T_1 are constant matrices, so the individual components of T_ε are linear with repsect to the contraction parameter.

Another class of contractions that are useful in physics are the *generalized Inönü-Wigner-contractions*, or *generalized IW-contractions* for short[10]. The linearity condition is replaced with the condition

$$(T_\varepsilon)_{ij} = \varepsilon^p \delta_{ij},$$

where δ_{ij} is the Kronecker delta and $p \in \mathbb{Z}$. In the case of the 3-dimensional Lie algebras, all contractions are known and can be realized as generalized IW-contractions with non-negative integer exponents. This is not in general true for algebras of dimension 4 or higher.

Typically, there will exist many different specific contractions from a given Lie algebra to another, so it is useful to define the notion of *equivalent* contractions. Let $\mathfrak{g}$ be isomorphic to $\mathfrak{g}_0$, let $\mathfrak{g}'$ be isomorphic to $\mathfrak{g}'_0$, and let

$$\mathfrak{g} \xrightarrow{T_\varepsilon} \mathfrak{g}_0 \text{ and } \mathfrak{g}' \xrightarrow{T'_\epsilon} \mathfrak{g}'_0$$

be contractions. Then $\xrightarrow{T_\varepsilon}$ and $\xrightarrow{T'_\varepsilon}$ are **loosely equivalent** contractions. This definition is perfectly adequate in the sense that in order to demonstrate that $\mathfrak{g}$ contracts to $\mathfrak{g}_0$, we need only find *one* contraction between algebras that are isomorphic to $\mathfrak{g}$ and $\mathfrak{g}_0$ respectively. Given a contraction $\mathfrak{g} \xrightarrow{T_\varepsilon} \mathfrak{g}_0$ and $A, B \in Aut(\mathbf{V})$ (the set of automorphisms of $\mathbf{V}$), suppose there exists another contraction matrix given by

$$S_\varepsilon = A T_\varepsilon B^{-1}.$$

Then the contractions $\mathfrak{g} \xrightarrow{T_\varepsilon} \mathfrak{g}_0$ and $\mathfrak{g} \xrightarrow{S_\varepsilon} \mathfrak{g}_0$ are called **strictly equivalent** contractions. For the purposes of this paper, we will be concerned only with loosely equivalent contractions, and we will use the term equivalent to mean loosely equivalent.

Though we shall not use it, there is a general class of contractions that is widely known, and we mention it here for the sake of completeness. Let δ: $\mathbb{N} \longrightarrow$ (0,1] be a strictly decreasing function. Then given a continuous family of non-singular matrices T: (0,1] $\longrightarrow$ $GL(n,\mathbb{R})$, we have that $T_{\delta(n)}$ is a function from $\mathbb{N}$ to $GL(n,\mathbb{R})$, where

$$\lim_{n\to\infty} T_{\delta(n)}^{-1}[T_{\delta(n)}y, T_{\delta(n)}y] = [x,y]_0$$

defines a contraction whenever T does. More generally, let $n = 0,1,2,...$ and define a function U: $\mathbb{N} \longrightarrow GL(n,\mathbb{R})$. If

$$\lim_{n\to\infty}[x,y]_n = \lim_{n\to\infty} U_n^{-1}[U_n x, U_n y] = [x,y]_0$$

exists, then we call $\mathfrak{g}_0$ the *sequential contraction* of $\mathfrak{g}$.

3.2 Some Non-Trivial Examples of Contractions

In general, every abelian algebra can be contracted to itself. This is the special case where a contraction is both trivial and improper. Since every 1-dimensional Lie algebra is isomorphic to the abelian algebra, this is the only contraction that exists up to equivalence. In 2 dimensions, we know that there is only one non-abelian algebra. It can of course be contracted both to itself and to the abelian algebra, and it is obvious that up to equivalence, these are the only 2-dimensional contractions.

For some i,j, note that unless T_ε is diagonal, we cannot in general infer $[T_\varepsilon e_i, T_\varepsilon e_j] = 0$ from the fact that $[e_i,e_j] = 0$, for any ε. So in general, we must compute the limit of each bracket $[e_1,e_2]$, $[e_1,e_3]$, $[e_2,e_3]$ of basis vectors in $\mathfrak{g}$ to obtain the brackets in the contraction $\mathfrak{g}_0$, though recall from Section 2.2 that these brackets suffice to determine *any* 3-dimensional Lie algebra.

Examples in 3 Dimensions: Contractions to $\mathfrak{H}$

Recall that the algebra $\mathfrak{sl}(2,\mathbb{R})$ with the canonical basis is determined by the brackets $[e_1,e_2] = e_3$, $[e_1,e_3] = -2e_1$, and $[e_2,e_3] = 2e_2$. Define the contraction matrix by

$$T_\varepsilon = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \varepsilon \\ \varepsilon & 0 & 0 \end{pmatrix}$$

with the inverse

$$T_\varepsilon^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{\varepsilon} \\ -1 & 0 & 0 \\ 0 & \frac{1}{\varepsilon} & 0. \end{pmatrix}$$

We can now compute

$$\begin{aligned} T_\varepsilon^{-1}[T_\varepsilon e_1, T_\varepsilon e_2] &= T_\varepsilon^{-1}[\varepsilon e_3, e_1] \\ &= -\varepsilon T_\varepsilon^{-1}[e_1, e_3] \\ &= 2\varepsilon T_\varepsilon^{-1} e_1 \\ &= -2\varepsilon e_2 \end{aligned}$$

so that

$$\lim_{\varepsilon\to 0^+} T_\varepsilon^{-1}[T_\varepsilon e_1, T_\varepsilon e_2] = \lim_{\varepsilon\to 0^+} -2\varepsilon e_2 = 0.$$

Similarly, we can compute that $\lim_{\varepsilon\to 0^+} T_\varepsilon^{-1}[T_\varepsilon e_1, T_\varepsilon e_3]$ = -$2\varepsilon e_3 = 0$. Lastly, we see that

$$\begin{aligned} T_\varepsilon^{-1}[T_\varepsilon e_2, T_\varepsilon e_3] &= T_\varepsilon^{-1}[e_1, \varepsilon e_2] \\ &= \varepsilon T_\varepsilon^{-1}[e_1, e_2] \\ &= \varepsilon T_\varepsilon^{-1} e_3 \\ &= \varepsilon(\frac{1}{\varepsilon} e_1) \\ &= e_1, \end{aligned}$$

so that $\lim_{\varepsilon\to 0^+} T_\varepsilon^{-1}[T_\varepsilon e_2, T_\varepsilon e_3] = e_1$. This demonstrates that $\mathfrak{sl}(2,\mathbb{R})$ can be contracted to the algebra given by the single bracket $[e_2, e_3] = e_1$, which is the Heisenberg algebra $\mathfrak{H}$ encountered in Section 2.6.

The characteristic polynomial for T_ε is $T - \lambda I_n = \lambda^3 - \varepsilon^2$, which does not split over $\mathbb{R}$ for any ε, so T_ε is not diagonalizable. Thus the contraction just given is *not* a generalized IW-contraction—however, since each entry is linear with respect to the contraction parameter ε, we can write $T_\varepsilon = T_0 + \varepsilon T_1$ where

$$T_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

which shows that $\mathfrak{sl}(2,\mathbb{R}) \xrightarrow{T_\varepsilon} \mathfrak{H}$ is a Saletan contraction.

Now consider $\mathfrak{o}(3,\mathbb{R})$, and define the contraction matrix S_ε by

$$S_\varepsilon = \begin{pmatrix} \varepsilon^2 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}.$$

The inverse is clearly the diagonal matrix A with $(A)_{ii} = ((S_\varepsilon)_{ii})^{-1}$. This will give a generalized IW-contraction if we can demonstrate that the limit exists. Consider the structure coefficients of $\mathfrak{o}(3,\mathbb{R})$:

$$c_{12}^3 = c_{23}^1 = c_{31}^2 = 1.$$

We can now write

$$\begin{aligned} \widehat{c}_{12}^3 &= (T_\varepsilon)_1^1 (T_\varepsilon)_2^2 (T_\varepsilon^{-1})_3^3 c_{12}^3 \\ &= \varepsilon^2 \cdot \varepsilon \cdot \frac{1}{\varepsilon} \\ &= \varepsilon^2, \end{aligned}$$

so in the limit as ε goes to zero, $\overset{\frown}{c}_{12}^3 = 0$. Similarly, we find that

$$\begin{aligned} \widehat{c}_{31}^2 &= (T_\varepsilon)_3^3 (T_\varepsilon)_1^1 (T_\varepsilon^{-1})_2^2 c_{31}^2 \\ &= \varepsilon \cdot \varepsilon^2 \cdot \frac{1}{\varepsilon} \\ &= \varepsilon^2 \end{aligned}$$

and

$$\begin{aligned}\widehat{c}^{1}_{23} &= (T_\varepsilon)^2_2 (T_\varepsilon)^3_3 (T_\varepsilon^{-1})^1_1 c^1_{23} \\ &= \varepsilon \cdot \varepsilon \cdot \frac{1}{\varepsilon^2} \\ &= 1\end{aligned}$$

so that in the limit as ε goes to zero, we have $\breve{c}^{2}_{31} = 0$ and $\breve{c}^{1}_{23} = 1$, which is the only non-zero structure coefficient in the contraction. Thus, the only remaining non-zero bracket is $[e_2,e_3] = e_1$, and the contraction is once again the Heisenberg algebra $\mathfrak{H}$. Therefore, we have that $\mathfrak{o}(3,\mathbb{R}) \xrightarrow{S_\varepsilon} \mathfrak{H}$ is a generalized IW-contraction.

3.3 Necessary Contraction Criteria and Deformations

Examples of Necessary Conditions for Contractions

As noted in Section 3.2, every Lie algebra can be contracted both to the abelian algebra of the same dimension and to an algebra isomorphic to itself. These cases are uninteresing. In light of the Bianchi classification (Section 2.6), what we want to know is which 3-dimensional algebras are contractions of one or more of the others. Sufficient conditions for a contraction to exist are difficult to obtain, but since there is a managable number of 3-dimensional algebras, we can find all contractions up to equivalency by the following procedure[12]: (1) generate a list of *necessary conditions* that concern *invariant* and *semi-invariant* quantites of Lie algebras; (2) test all pairs of 3-dimensional Lie algebras that could provide a proper contraction against these conditions; (3) for any pair that satisfy every condition on the list, explicitly construct a contraction matrix or demonstrate that no such matrix can exist.

This procedure is much more efficient than simply attempting to apply (3) across all cases, but the number of necessary conditions is very large[12], and it is impractical here to attempt to demonstrate every one. We list several here, but omit the proofs. Suppose that $\mathfrak{g} \longrightarrow \mathfrak{g}_0$ is a proper contraction. Then we have:

(1) $dim(\mathfrak{g}_0^k) \leq dim(\mathfrak{g}^{(k)})$;
(2) $dim(\mathfrak{g}_0^k) \leq dim(\mathfrak{g}^k)$;
(3) $tr(ad_x) = 0$ for all x in $\mathfrak{g}$ implies $tr(ad_{x_0}) = 0$ for all x_0 in $\mathfrak{g}_0$.

These conditions imply that if $\mathfrak{g}$ is solvable, nilpotent, or unimodular, then so is $\mathfrak{g}_0$. As an example of the above procedure, note that since $\mathfrak{H}$ is the only nilpotent 3-dimensional algebra, it has no proper contractions at all.

Deformations

We end with some brief comments on deformations of Lie algebras. Note that is presented here is generalized to n dimensions. Denote the set of all real Lie algebras of dimension n by $\mathfrak{L}_n$. Define the group action Φ: $GL(n,\mathbb{R}) \times \mathfrak{L}_n \longrightarrow \mathfrak{L}_n$ by

$$\Phi(T, \mathfrak{g}) = \mathfrak{g}_0$$

where $\mathfrak{g}_0$ is the Lie algebra given by

$$T^{-1}[Tx, Ty].$$

This is well-defined, since we have that $\Phi(I_n,\mathfrak{g})$ gives back $\mathfrak{g}$, and that $\Phi(T_1T_2,\mathfrak{g})$ and $\Phi(T_1,\Phi(T_2,\mathfrak{g}))$ both yield the same algebra by the relation $(T_1T_2)^{-1} = T_2^{-1}T_1^{-1}$. The orbits of Φ are nothing more than the isomorphism classes of $\mathfrak{L}_n$, which we shall denote by $\mathcal{O}(\mathfrak{g})$, where $\mathfrak{g}$ is any representative of the isomorphism class.

We can identify an element $\mathfrak{g}$ of $\mathfrak{L}_n$ via its structure constants c_{ij}^k with respect to a given basis $\{e_i\}$ with an ordered n^3-tuple in $\mathbb{R}^{n^3}$. A change in basis will clearly give an algebra isomorphic to $\mathfrak{g}$. This corresponds to the contraction matrix yielding an improper contraction[13].

We can now write $\mathfrak{L}_n \subset \mathbb{R}^{n^3}$, impose the standard Euclidean (or Zariski—[3] for this similar, but different, approach) topology on $\mathfrak{L}_n$, and speak of the closure of $\mathcal{O}(\mathfrak{g})$, denoted by $\overline{\mathcal{O}(\mathfrak{g})}$ of $\mathfrak{g}$. It can be shown that a contraction of $\mathfrak{g}$ is proper if and only if it lies in $\overline{\mathcal{O}(\mathfrak{g})}$ of $\mathfrak{g}$.

Broadly speaking, a *deformation* of a Lie algebra is a continous path $\varepsilon \ \epsilon\ [0,1] \longmapsto \mathfrak{L}_n$. However, for our purposes, we are concerned only with the following restriction: A **deformation of plateau type** is a deformation defined by

$$\zeta_\varepsilon : [0,1] \longrightarrow \mathcal{O}(\mathfrak{g})$$

where ζ_0 is not isomorphic to ζ_1, but $\zeta_\varepsilon \simeq \zeta_1$ for all ε in [0,1]. In other words, a deformation of plateau type is restricted to the orbit of $\mathfrak{g}$. We can assume without loss of generality that the path ζ_ε begins on the vector of structure constants defining $\mathfrak{g}$ the canonical basis, so long as it ends in the closure of the orbit, yielding a non-isomorphic algebra. We denote a deformation by $\mathfrak{g}_0 \xrightarrow{\zeta_\varepsilon} \mathfrak{g}$.

The motivation for the restricted definition in the previous paragraph is given by the following remarks. Suppose that $\mathfrak{g} \xrightarrow{T_\varepsilon} \mathfrak{g}_0$ is a contraction and $\mathfrak{g}' \xrightarrow{\zeta_\varepsilon} \mathfrak{g}'_0$ is a deformation of plateau type. If $\mathfrak{g} \simeq \mathfrak{g}'_0$, $\mathfrak{g}_0 \simeq \mathfrak{g}'$, and

$$\zeta_\varepsilon = \zeta_{T_\varepsilon}$$

for $0 < \varepsilon \leq 1$, then we call $\mathfrak{g} \xrightarrow{T_\varepsilon} \mathfrak{g}_0$ the **inverse** of $\mathfrak{g}' \xrightarrow{\zeta_\varepsilon} \mathfrak{g}'_0$, and vice versa. In terms of the necessary contraction criteria seen in the previous section, we can now give a much stronger equivalence between contractions and deformations of plateau type. We conclude by presenting the following theorem:

Given any continuous contraction $\mathfrak{g} \xrightarrow{T_\varepsilon} \mathfrak{g}$, *there exists a deformation of plateau type* $\mathfrak{g}' \xrightarrow{\zeta_\varepsilon} \mathfrak{g}'_0$ *that is inverse to it, and vice versa.*[13].

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