L^p ASYMPTOTICS ON THE ZONAL AND SECTORAL HARMONICS

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Abstract. In this paper, we establish asymptotic expressions for the zonal harmonics $P_\ell(\cos\theta)$ and the sectoral harmonics $Y_{\ell\ell}(\theta,\phi)$ in the limit $\ell >> 1$. From doing so, we are able to find L^p growth rates in ℓ for large ℓ. The zonal and sectoral harmonics are specific families of spherical harmonics which are eigenfunctions of the Laplacian on $\mathbb{S}^2$. The asymptotics and L^p estimates tell us about how and where these eigenfunctions "concentrate" on the sphere. The specific concentrations allow us to think of each family of eigenfunctions as defining a particular measure on $\mathbb{S}^2$. This measure will be different in the case of the zonal and sectoral harmonics and we will describe the contrasts below.

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1. Introduction

One very interesting problem in Harmonic Analysis involves taking a compact, Riemannian manifold, and studying the eigenfunctions of the Laplace operator restricted to that manifold. Here, we cannot escape talking about an ambient Euclidean space $\mathbb{R}^k$ at least initially such that our Riemannian manifold is a sub-manifold. It is a trivial matter to describe the Laplacian on $\mathbb{R}^k$ but it is a far less trivial exercise to describe the Laplacian on the compact sub-manifold.

Certain functions on the ambient Euclidean space $\mathbb{R}^k$ restrict to the sub-manifold as eigenfunctions of the Laplace operator on that manifold. We can define these functions as the *"harmonics"* on that manifold, and they will in some sense represent the symmetry of that manifold, if any exists. If the manifold has dimension $d < k$, there will be d parameters labeling distinct harmonics.

It takes a good amount of algebraic and geometric machinery to get to this point but we can then study where and how these harmonics concentrate on the manifold for certain parameter values. Doing this analysis requires using convenient coordinates on the manifold. One of the best ways to study this concentration is by obtaining L^p bounds on the eigenfunctions, for all $p = 2, 3, 4, ...$ where the L^∞ norm is the supremum of the eigenfunction. Sogge established an upper bound for arbitrary manifolds and a lower bound for the case of $\mathbb{S}^2$ as a sub-manifold of $\mathbb{R}^3$. In this paper we will provide rigor in establishing the L^p bounds for certain families of *spherical harmonics*, the eigenfunctions of the Laplace operator on $\mathbb{S}^2$.

There are a couple special families of spherical harmonics that we will concentrate on here. The *Zonal Harmonics* are the collection of spherical harmonics such that $m = 0$ which we will denote by $P_\ell(\cos\theta)$. The *Sectoral Harmonics* are the collection with $m = \ell$ which we will denote by $Y_{\ell\ell}(\theta, \phi)$. Note that ℓ and m are the two parameters resulting from $\mathbb{S}^2$ having dimension 2. We will rigorously derive what ℓ and m represent shortly.

Simply put, in the first part of the paper we will construct the spherical harmonics from first principles and in the last half, we will study how these eigenfunctions concentrate on the sphere for large ℓ.

Before we begin, it will be helpful to consider a more elementary case than the one we will focus on.

Definition. *Let $Q : \mathbb{R}^k \to \mathbb{R}$. We say Q is homogeneous degree-ℓ if for all $t \in \mathbb{R}$, $Q(t\,x) = t^\ell\, Q(x)$, where $x = (x_1, ..., x_k)$.*

Consider now, $\mathbb{R}^2$ with the Laplacian described in polar coordinates as:

$$\Delta_{\mathbb{R}^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

The unit circle $\mathbb{S}^1$, consisting of all unit vectors in $\mathbb{R}^2$, is a compact hypersurface of $\mathbb{R}^2$ much like $\mathbb{S}^2$ is a compact hypersurface of $\mathbb{R}^3$.

Let $f(r, \theta)$ be harmonic on $\mathbb{R}^2$ and homogeneous degree-ω. Homogeneity shows that we can write $f(r, \theta) = r^\omega g(\theta)$ where $g(\theta)$ is the restriction of f to the unit circle. Since f is harmonic

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0$$

Notice that $\frac{\partial f}{\partial r} = \omega r^{\omega-1} g(\theta)$ and $\frac{\partial^2 f}{\partial r^2} = \omega(\omega - 1) r^{\omega-2} g(\theta)$. Therefore,

$$\omega(\omega-1)r^{\omega-2}g(\theta)+\omega r^{\omega-2}g(\theta)+r^{\omega-2}\frac{d^2g}{d\theta^2}=0$$

We may divide through by $r^{\omega-2}$ and simplify to get the final result:

$$\frac{d^2g}{d\theta^2}=-\omega^2 g.$$

The operator $\Delta_{\mathbb{S}^1}=\frac{d^2}{d\theta^2}$ is the Laplacian on the unit circle $\mathbb{S}^1$.

So we've shown that if a function is homogeneous and harmonic on the ambient Euclidean space $\mathbb{R}^2$, then it restricts down the the sub-manifold $\mathbb{S}^1$ to a function $g(\theta)$ such that the differential equation $\frac{d^2}{d\theta^2}g=-\omega^2 g$ is satisfied.

But we know what the solutions to this equation are. The solutions are

$$g(\theta)\backsim e^{\pm i\omega\theta}=\cos\omega\theta\pm i\sin\omega\theta.$$

Therefore, the familiar trigonometric functions come about by restricting harmonic, homogeneous functions on $\mathbb{R}^2$ down to the unit circle. We could, if we liked, define a *"circular harmonic"* as the restriction of harmonic, homogeneous polynomials on $\mathbb{R}^2$ down to $\mathbb{S}^1$ and then prove them to be eigenfunction of $\Delta_{\mathbb{S}^1}$ with eigenvalue $-\omega^2$.

Going into the details would be beyond our scope but it is worth noting that Fourier analysis allows us to decompose a function $f(t)\in L^2(\mathbb{S}^1)$ into:

$$f(t)=\sum_{\omega=-\infty}^{\infty}a_\omega e^{i\omega t},$$

where we interpret a_ω as quantifying the contribution of the ω^{th} mode to $f(t)$.

In this computation, we required that the functions be harmonic and homogeneous on the ambient space. The exact hypotheses needed here will depend, in general, on the manifold in question. The purpose of this computation was to motivate that the familiar trigonometric functions are analogous to the spherical harmonics through the construction described above. Keep these elementary principles in mind throughout the rest of the paper.

2. Notation

We should establish fixed notation for our asymptotic analysis later in the paper.

(i) For f and g functions of a complex variable z which have well defined limits as $z \to z_0$, for some $z_0 \in \mathbb{C}$, we say $f(z) = \mathcal{O}\big(g(z)\big)$ if there exists $C, \delta > 0$ such that $|f| \leqslant C|g|$ whenever $0 < |z - z_0| < \delta$. It is imperative to note that here z can be an independent variable or it can be some sort of parameter. We will mostly use this notation with respect to a parameter that we use to label eigenfunctions.

(ii) If we have that:

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = 1$$

we then say that f and g are *asymptotically equivalent*. We will here use the notation $f(z) \backsim g(z)$ as $z \to z_0$. Once again, z can be either a parameter or an independent variable.

3. Algebraic and Geometric Motivation of Spherical Harmonics

We begin by developing some elementary algebraic and geometric properties of the spherical harmonics. In the process, we uncover crucial characteristics that will help us in the later analysis. These basic properties are motivated by Folland [3].

Definition. *Given Cartesian coordinates (x_1, x_2, x_3) on $\mathbb{R}^3$ we define $\mathbb{S}^2$ as the compact hypersurface consisting of points such that $x_1^2 + x_2^2 + x_3^2 = 1$.*

Definition. *We define $\mathscr{P}_\ell$ to be the space of homogeneous polynomials of degree-ℓ on $\mathbb{R}^3$. Likewise, let*

$$H_\ell = \{P \in \mathscr{P}_\ell \,|\, \Delta P = 0\},$$

$$\mathscr{H}_\ell = \{P\big|_{\mathbb{S}^2} \mid P \in H_\ell\}.$$

So the space H_ℓ is simply the collection of all harmonic, homogeneous polynomials on $\mathbb{R}^3$ and the space $\mathscr{H}_\ell$ is the restriction of each element of H_ℓ to the unit sphere $\mathbb{S}^2$. It is important to note that since polynomials are entire, $\ell \in \mathbb{N} \cup \{0\}$.

Definition. *We define the elements of $\mathscr{H}_\ell$ to be the spherical harmonics of degree ℓ.*

There is an isomorphism between $\mathscr{H}_\ell$ and H_ℓ. Therefore, every element in H_ℓ restricts down to a unique spherical harmonic of degree ℓ, and for every spherical harmonic of degree ℓ, there exists a unique $P_\ell \in H_\ell$ which we will refer to as the *harmonic extension*. This harmonic extension is given by:

$$P_\ell(x) = |x|^\ell Y_\ell\left(\frac{x}{|x|}\right).$$

Let r^2 denote the function from $\mathbb{R}^3$ to $\mathbb{R}$ given by $x \to x_1^2 + x_2^2 + x_3^2$.

We now introduce a notation that makes doing computations with homogeneous polynomials less cumbersome. We define a *multi-index* as an n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$ of non-negative integers such that

$$|\alpha| = \sum_{j=1}^{n} \alpha_j,$$

$$\alpha! = \alpha_1! \cdots \alpha_n!$$

For $x \in \mathbb{R}^n$, we write

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

We employ the above useful notation in the following Proposition.

Proposition. *$\mathscr{P}_\ell = H_\ell \oplus r^2\mathscr{P}_{\ell-2}$, where $r^2\mathscr{P}_{\ell-2} = \{r^2 P \,|\, P \in \mathscr{P}_{\ell-2}\}$.*

Proof. Let $P, Q \in \mathscr{P}_\ell$ and write them explicitly as $P(x) = a_\alpha x^\alpha$ and $Q(x) = b_\beta x^\beta$, employing the summation convention and multi-index notation. We first introduce an inner product on $\mathscr{P}_\ell$. We define $\{P, Q\} = P(\partial)\overline{Q} = a_\alpha \partial^\alpha \overline{Q}$, where $P(\partial)$ is simply the differential operator defined by inputting partial derivatives to the polynomial P. For example, $r^2(\partial) = \frac{\partial}{\partial x_1^2} + \frac{\partial}{\partial x_2^2} + \frac{\partial}{\partial x_3^2} = \Delta$. Clearly the form $\{\cdot,\cdot\}$ is linear in the first entry, conjugate linear in the second, and it maps two elements of $\mathscr{P}_\ell$ to a scalar.

Notice that $\{x^\alpha, x^\beta\} = \alpha!$ whenever $\alpha = \beta$ and $\{x^\alpha, x^\beta\} = 0$ otherwise. Therefore by linearity and conjugate linearity, we see that

$$\{P, Q\} = \{a_\alpha x^\alpha, b_\beta x^\beta\} = \alpha! a_\alpha \overline{b}_\alpha.$$

Therefore, the form $\{\cdot,\cdot\}$ is a scalar product on $\mathscr{P}_\ell$.

H_ℓ and $r^2\mathscr{P}_{\ell-2}$ are subspaces of $\mathscr{P}_\ell$. The *direct sum* $\mathscr{P}_\ell = H_\ell \oplus r^2\mathscr{P}_{\ell-2}$ means that every element in $\mathscr{P}_\ell$ can be written as a sum of an element in H_ℓ and an element in $r^2\mathscr{P}_{\ell-2}$, and the two subspaces are orthogonal with respect to the inner product described above.

Now let's take $P \in \mathscr{P}_{\ell-2}$ and $Q \in \mathscr{P}_\ell$ and compute the following

$$\{r^2 P, Q\} = P(\partial) r^2(\partial)\overline{Q} = P(\partial)\overline{\Delta Q} = \{P, \Delta Q\}.$$

(We used in the above line that $r^2(\partial) = \Delta$, the Laplacian on $\mathbb{R}^3$.) It is immediate that if $\Delta Q = 0$, then $\{r^2 P, Q\} = 0$. Conversely, assume now that $\{r^2 P, Q\} = 0$, for all $P \in \mathscr{P}_{\ell-2}$. $\Delta Q \in \mathscr{P}_{\ell-2}$ and we can write:

$$\Delta Q = \sum_{|\alpha|=\ell-2} c_\alpha x^\alpha.$$

We observe that $\partial^\beta(\Delta Q) = \sum c_\alpha \partial^\beta x^\alpha = c_\beta\, \beta!$. Since $P \in \mathscr{P}_{\ell-2}$, and $P(\partial)\overline{\Delta Q} = 0$, it follows that $c_\beta = 0$ for all β. Therefore, $\Delta Q = 0$.

In summary, $\{r^2 P, Q\}$ will vanish for all $P \in \mathscr{P}_{\ell-2}$ if and only if $\Delta Q = 0$ (i.e. $Q \in H_\ell$) from which it follows that H_ℓ is the orthogonal complement to $r^2\mathscr{P}_\ell$ with respect to $\mathscr{P}_\ell$. This completes the proof. □

Corollary. $\mathscr{P}_\ell = H_\ell \oplus r^2 H_{\ell-2} \oplus r^4 H_{\ell-4} \oplus ...$

Proof. This follows trivially by induction on ℓ using the above Proposition. □

Corollary. *Let $P \in \mathscr{P}_\ell$ be arbitrary. $P\big|_{\mathbb{S}^2}$ is then a sum of spherical harmonics of degree at most ℓ.*

Proof. This follows from the above Corollary noting that on $\mathbb{S}^2$, $r^2 = 1$. □

Lemma. *(Euler) Let $Q : \mathbb{R}^k \to \mathbb{R}$ be homogeneous degree-ℓ. Then*

$$\sum_{j=1}^{k} x_j \partial_j Q(x) = \ell Q(x),$$

where $x = (x_1, ..., x_k)$.

Proof. Since $Q \in \mathscr{P}_\ell$ we observe that $Q(tx) = t^\ell Q(x)$ for all $t \in \mathbb{R}$. By chain rule:

$$\frac{d}{dt} Q(tx) = \sum_{j=1}^{k} x_j \partial_j Q(tx) = \frac{d}{dt}\left(t^\ell Q(x)\right) = \ell t^{\ell-1} Q(x).$$

By letting $t = 1$ we observe that:

$$\sum_{j=1}^{k} x_j \partial_j Q(x) = \ell Q(x),$$

which completes the proof. □

Theorem. *(Weierstrass Approximation Theorem) If S is a compact subset of $\mathbb{R}^k$, the restrictions of polynomials to S are dense in $C(S)$ in the L^∞ norm.*

Definition. *For a function f defined on $\mathbb{S}^2$ we define the L^p norm of f as*

$$||f||_p = \left(\int_{\mathbb{S}^2} |f|^p d\Omega \right)^{\frac{1}{p}},$$

where $d\Omega$ is a differential area element on $\mathbb{S}^2$.

Definition. *We define $L^p(\mathbb{S}^2)$ to be the Banach space consisting of functions $f : \mathbb{S}^2 \to \mathbb{C}$ such that:*

$$||f||_p = \left(\int_{\mathbb{S}^2} |f|^p d\Omega \right)^{\frac{1}{p}} < \infty.$$

This space is complete by defining it to be the closure of $C(\mathbb{S}^2)$.

Theorem. $L^2(\mathbb{S}^2) = \bigoplus_{\ell=0}^{\infty} \mathscr{H}_\ell$. *The expression on the right is an orthogonal direct sum with respect to the inner product on $L^2(\mathbb{S}^2)$.*

Proof. Given that $\mathscr{P}_\ell = H_\ell \oplus r^2 H_{\ell-2} \oplus r^4 H_{\ell-4} \oplus ...$ as stated in the above Corollary, the Weierstrass Approximation Theorem tells us that the linear span of $\mathscr{H}_\ell$ for all ℓ is dense in $L^2(\mathbb{S}^2)$.

Let $Y_j \in \mathscr{H}_j$ and $Y_k \in \mathscr{H}_k$. Furthermore, let P_j and P_k be their harmonic extensions in H_j and H_k, respectively. Therefore, since P_j and P_k are harmonic, we may write

$$\int_{\mathbb{B}^2} (P_j \Delta \overline{P_k} - P_k \Delta \overline{P_j}) dx = 0,$$

where $\mathbb{B}^2$ is the unit ball in $\mathbb{R}^3$. Applying the Divergence Theorem to $P_j \nabla P_k$, we observe that

$$\int_{\mathbb{B}^2} (P_j \Delta P_k + \nabla P_j \cdot \nabla P_k) dx = \int_{\mathbb{S}^2} (P_j \partial_\nu P_k) d\Omega,$$

since $\mathbb{S}^2 = \partial \mathbb{B}^2$. $d\Omega$ is a differential area element on $\mathbb{S}^2$; we will define it in coordinates in the next section. Note that $\partial_\nu = \sum_{i=1}^3 x_i \partial_i$ is a radial derivative.

Taking the above expression, interchanging P_j and P_k, and subtracting yields the following useful relation

$$\int_{\mathbb{B}^2} (P_j \Delta P_k - P_k \Delta P_j) dx = \int_{\mathbb{S}^2} (P_j \partial_\nu P_k - P_k \partial_\nu P_j) d\Omega.$$

Therefore,

$$\int_{\mathbb{S}^2} (P_j \partial_\nu \overline{P}_k - \overline{P}_k \partial_\nu P_j) d\Omega = 0.$$

But by Euler's Lemma, $P_j \partial_\nu \overline{P}_k = k P_j \overline{P}_k$ and $\overline{P}_k \partial_\nu P_j = j P_j \overline{P}_k$. Therefore,

$$(k - j) \int_{\mathbb{S}^2} P_j \overline{P}_k d\Omega = (k - j) < Y_j \,|\, Y_k >= 0.$$

In the last line we used that the inner product of the spherical harmonics $< Y_j \,|\, Y_k >$ is equivalent to a surface integral of the harmonic extensions $P_j \overline{P}_k$ over the surface of $\mathbb{S}^2$.

Finally, if $j \neq k$, then we see that $< Y_j \,|\, Y_k >= 0$. Therefore, $\mathscr{H}_j$ is orthogonal to $\mathscr{H}_k$ for all $j \neq k$. This completes the proof. □

The following Proposition determines the dimension of the space $\mathscr{P}_\ell$ of homogeneous degree ℓ polynomials on $\mathbb{R}^3$.

Proposition. $dim(\mathscr{P}_\ell) = \binom{\ell+2}{2}$.

Proof. Employing multi-index notation, we can form a basis of $\mathscr{P}_\ell$ by taking monomials of the form x^α such that $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = \ell$. Therefore, the dimension of this space will be the number of ways we can choose an ordered 3-tuple $(\alpha_1, \alpha_2, \alpha_3)$ of non-negative integers, such that $\alpha_1 + \alpha_2 + \alpha_3 = \ell$.

Therefore, we've reduced this problem to one in combinatorics. Imagine lining up ℓ objects. We want to know the number of ways we can partition the ℓ objects into three groups. We can imagine placing *two* partitions between the ℓ objects, representing the three groups. The number of distinct ways we can do this is simply

$$\binom{\ell+2}{2} = \frac{(\ell+2)!}{2!\,\ell!} = \frac{1}{2}(\ell+1)(\ell+2).$$

This completes the proof. □

Corollary. *$dim(H_\ell) = dim(\mathscr{H}_\ell) = 2\ell+1$.*

Proof. The first equality follows trivially from the isomorphism between H_ℓ and $\mathscr{H}_\ell$, noted earlier. Recall that $\mathscr{P}_\ell = H_\ell \oplus r^2\mathscr{P}_{\ell-2}$. Therefore,

$$dim(\mathscr{H}_\ell) = dim(H_\ell) = \Big\{dim(\mathscr{P}_\ell) - dim(\mathscr{P}_{\ell-2})\Big\}.$$

From this, the second equality follows by applying the above Proposition. □

By this last Corollary, we've shown the following decomposition of $L^2(\mathbb{S}^2)$,

$$L^2(\mathbb{S}^2) = \bigoplus_{\ell=0}^{\infty} \mathscr{H}_\ell,$$

is really an orthogonal direct sum of spaces of increasing dimension $2\ell+1$.

Therefore, given $f \in L^2(\mathbb{S}^2)$, we can define $\Pi_\ell : L^2(\mathbb{S}^2) \to \mathscr{H}_\ell$ as the projection operator onto $\mathscr{H}_\ell$ such that:

$$f = \sum_{\ell=0}^{\infty} \Pi_\ell(f),$$

where the above expression converges in $L^2(\mathbb{S}^2)$.

4. Properties of the Spherical Harmonics as Functions on $\mathbb{S}^2$

In the previous section we established many important algebraic and geometric properties of the spherical harmonics. We demonstrated an isomorphism between spherical harmonics of degree ℓ and harmonic, homogenous degree ℓ polynomials on $\mathbb{R}^3$. We then showed that $L^2(\mathbb{S}^2)$, can be written as an orthogonal direct sum over the spherical harmonics of all degrees, $\ell = 0, 1, ...$ and we were able to do so without worrying about the coordinate representations of the spherical harmonics.

By definition, there exists a unique harmonic, homogenous degree ℓ polynomial on $\mathbb{R}^3$ that restricts to $\mathbb{S}^2$ to give us a spherical harmonic of degree ℓ. Therefore, doing analytical computations with the spherical harmonics will be most natural in a coordinate system on $\mathbb{S}^2$, itself. Let's now describe that system and how it relates to the Cartesian coordinates we had on $\mathbb{R}^3$ previously.

We place a coordinate system on $\mathbb{S}^2$ such that points are labeled by (θ, ϕ) with (x_1, x_2, x_3) and (θ, ϕ) related by

$$x_1 = \cos\phi \sin\theta$$

$$x_2 = \sin\theta \sin\phi$$

$$x_3 = \cos\theta$$

Let $d\Omega = \sin\theta d\theta d\phi$ denote the differential area element or *measure* on $\mathbb{S}^2$ in these coordinates.

By this convention, $0 \leqslant \phi \leqslant 2\pi$ is called the "azimuthal angle" and $0 \leqslant \theta \leqslant \pi$ is known as the "polar angle."

In these coordinates, we define the Laplacian on $\mathbb{S}^2$ (the spherical Laplacian) as

$$\Delta_{\mathbb{S}^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

This is simply the usual Laplacian on $\mathbb{R}^3$ in spherical coordinates restricted to the sphere. Equivalently, it is $\Delta_{\mathbb{R}^3}$ without the radial variable.

Notice from an earlier definition that we can write $Q(x)$ as $Q\big(\frac{|x|\,x}{|x|}\big) = |x|^\ell Q\big(\frac{x}{|x|}\big)$ by homogeneity. $\frac{x}{|x|}$ is simply an element of the 2-sphere $\mathbb{S}^2$. In particular, if $Q(r, \theta, \phi)$ is a homogeneous degree-ℓ function described in spherical coordinates, it can always be written as

$$Q(r, \theta, \phi) = r^\ell g(\theta, \phi)$$

where $g : \mathbb{S}^2 \to \mathbb{R}$ is the restriction of Q to $\mathbb{S}^2$.

We now prove a theorem that connects the spherical harmonics to the Laplacian on $\mathbb{S}^2$.

Theorem. *Let $P(r, \theta, \phi)$ be a harmonic, homogeneous degree-ℓ function on $\mathbb{R}^3$. Then, the restriction of $P(r, \theta, \phi)$ to $\mathbb{S}^2$ is an eigenfunction of $\Delta_{\mathbb{S}^2}$ with eigenvalue $-\ell(\ell + 1)$.*

Proof. Since P is homogeneous of degree-ℓ we may write $P(r,\theta,\phi) = r^\ell Y(\theta,\phi)$. P being harmonic means the Laplacian of P on $\mathbb{R}^3$ must vanish:

$$\Delta_{\mathbb{R}^3} P = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial r^\ell Y}{\partial r}\right) + \frac{1}{r^2}\Delta_{\mathbb{S}^2}(r^\ell Y) = 0$$

$$\Delta_{\mathbb{R}^3} P = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\ell r^{\ell+1} Y\right) + r^{\ell-2}\Delta_{\mathbb{S}^2} Y = r^{\ell-2}\left(\ell(\ell+1)Y + \Delta_{\mathbb{S}^2} Y\right) = 0$$

It follows clearly that $\Delta_{\mathbb{S}^2} Y = -\ell(\ell+1)Y$ and $Y(\theta,\phi) = P(r,\theta,\phi)\big|_{\mathbb{S}^2}$ which completes the proof. □

Notice in the above proof, exactly where the homogeneity came into play. P being homogeneous of degree-ℓ allowed us to collapse the radial portion of $\Delta_{\mathbb{R}^3}$ into $r^{\ell-2}\ell(\ell+1)Y$ which clearly determined our eigenvalue. Note also that it was *crucial* that the radial direction was everywhere orthogonal to our compact manifold ($\mathbb{S}^2$ in this case). If we had tried restricting down to an ellipsoid $\mathbb{E}^2$, while still employing spherical coordinates on $\mathbb{R}^3$, we could not write $Y(\theta,\phi) = P(r,\theta,\phi)\big|_{\mathbb{E}^2}$ where θ and ϕ label positions on $\mathbb{E}^2$.

Definition. *The Legendre Polynomials or order ℓ are defined by the Rodrigues Formula:*

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell.$$

The Legendre Polynomials are the solutions of Legendre's Differential Equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell}{dx}\right] + \ell(\ell+1)P_\ell(x) = 0,$$

with the initial condition $P_\ell(1) = 1$.

Definition. *The Associated Legendre Polynomials $P_{\ell m}(x)$ are defined by:*

$$P_{\ell m}(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m P_\ell(x)}{dx^m}$$

or equivalently,

$$P_{\ell m}(x) = \frac{(1-x^2)^{\frac{m}{2}}}{2^\ell \ell!}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2+1)^\ell$$

The Associated Legendre Polynomials satisfy the Associated Legendre Differential Equation:

$$(1-x^2)\frac{d^2P_{\ell m}}{dx^2} - 2x\frac{dP_{\ell m}}{dx} + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P_{\ell m} = 0$$

We really only care about these polynomials on the interval $[-1,1]$ and we often change variables from x to θ by letting $x = \cos\theta$. This restricts our attention to the interval $[0,\pi]$ in θ.

We now can think of the space $\mathscr{H}_\ell$ as an *eigenspace* since each element is an eigenfunction of $\Delta_{\mathbb{S}^2}$ with eigenvalue $-\ell(\ell+1)$. However, we also proved that $dim(\mathscr{H}_\ell) = 2\ell+1$, so we need to find an orthonormal basis of the space $\mathscr{H}_\ell$. In the process of finding such a basis, we will discover the coordinate representation of the spherical harmonics.

By an earlier theorem, $Y_\ell \in \mathscr{H}_\ell$ allows us to conclude that $\Delta_{\mathbb{S}^2} Y_\ell = -\ell(\ell+1)Y_\ell$. $\mathbb{S}^2$ is a two dimensional manifold with the two coordinates θ and ϕ as described above. Let's suppose that Y_ℓ can be written as the product

$$Y_\ell = \Theta(\theta)\,\Phi(\phi),$$

which separates the two coordinates, or variables.

Therefore, we can write

$$\Delta_{\mathbb{S}^2} Y_\ell = \Phi\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \Theta\frac{1}{\sin^2\theta}\frac{d}{d\phi^2}\Phi = -\ell(\ell+1)\Theta\Phi = -\ell(\ell+1)Y_\ell.$$

We notice that this equation is separable in the variables θ and ϕ and we can choose the separation constant to be m^2, like so:

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\sin^2\theta = -\frac{1}{\Phi}\frac{d^2}{d\phi^2}\Phi = m^2.$$

When we separate variables in a partial differential equation, we get auxiliary ordinary differential equations which the individual functions Θ and Φ satisfy. The last part of the above expression shows the first of these ordinary differential equations is $\frac{d^2}{d\phi^2}\Phi = -m^2\Phi$, which clearly has the solution:

$$\Phi(\phi) = e^{\pm im\phi}.$$

We insist that Φ be periodic on $\mathbb{S}^1$ which is to say $\Phi(0) = \Phi(2\pi)$. This requires that $m \in \mathbb{Z}$.

The second ordinary differential equation we get out of the separation is:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right)\Theta=0.$$

We can expand the first term to get:

$$\frac{d^2\Theta}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta}+\left(\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right)\Theta=0.$$

If we let $x=\cos\theta$, it is easy to see that the above ordinary differential equation is simply the Associated Legendre Differential Equation in the coordinate $x=\cos\theta$. Therefore:

$$\Theta(\theta)=P_{\ell m}(\cos\theta).$$

So we've shown that $Y_\ell \in \mathscr{H}_\ell$ has the following form:

$$(Y_\ell)_m = Y_{\ell m} = c_{\ell m}P_{\ell m}(\cos\theta)e^{im\phi},$$

where $c_{\ell m}$ is a normalization constant. But we now need to make sense of m. We've already noted that $\Phi(0)=\Phi(2\pi)$ requires that $m\in\mathbb{Z}$. Furthermore, the Associated Legendre functions have the following two properties as noted in the above definition:

$$P_{\ell m}(x)=(1-x^2)^{\frac{m}{2}}\frac{d^m P_\ell(x)}{dx^m}$$

and

$$P_{\ell m}(x)=\frac{(1-x^2)^{\frac{m}{2}}}{2^\ell \ell!}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2+1)^\ell.$$

We can observe a few things from the above equations. First of all, ℓ itself must be integral and non-negative since these function were constructed from homogeneous degree ℓ polynomials. We also note that the Legendre Polynomials of degree ℓ are polynomials of degree ℓ. So if m were to be bigger than ℓ, we would have more than ℓ derivatives, of an ℓ^{th} degree polynomial, which clearly will vanish. Finally, since $\Phi(\phi)=e^{\pm im\phi}$, for every solution labelled by m, there exists another, linearly independent to it, labelled by $-m$. Therefore, we also have that $m>-\ell$. So we must require that $|m|\leqslant\ell$.

Later, we will give asymptotic expressions of the normalization constant $c_{\ell m}$ for certain m and large ℓ.

The above observations enlighten an algebraic property we discovered earlier. Recall that we showed

$$dim(\mathscr{H}_\ell) = 2\ell + 1.$$

But notice that $|m| \leqslant \ell$, with ℓ and m integral, shows that there are $2\ell + 1$ distinct values of m for a given ℓ. So if we can show that the spherical harmonics in each eigenspace $\mathscr{H}_\ell$ form an orthonormal family, we will have shown them to be a basis of the eigenspace.

$$\int_{\mathbb{S}^2} \overline{Y}_{\ell m}(\theta,\phi) Y_{\ell m'}(\theta,\phi) d\Omega = \int_0^{2\pi} \int_0^{\pi} c_{\ell m} c_{\ell m'} P_{\ell m}(\cos\theta) P_{\ell m'}(\cos\theta) e^{i\phi(m'-m)} \sin\theta d\theta d\phi$$

$$= c_{\ell m} c_{\ell m'} \int_0^{\pi} P_{\ell m}(\cos\theta) P_{\ell m'}(\cos\theta) \left(\int_0^{2\pi} e^{i\phi(m'-m)} d\phi \right) \sin\theta d\theta.$$

The ϕ integral clearly vanishes if $m \neq m'$ and it equals 2π when $m = m'$. Therefore, given the normalization constants, $Y_{\ell m}$ for a fixed ℓ and $|m| \leqslant \ell$ is an orthonormal basis of $\mathscr{H}_\ell$.

At this point, it would be helpful to summarize the properties of the spherical harmonics using the coordinates (θ, ϕ) on $\mathbb{S}^2$.

We can define the ℓ^{th} eigenspace $\mathscr{H}_\ell = span\big(\{Y_{\ell m} | -\ell \leqslant m \leqslant \ell\}\big)$ which consists of all spherical harmonics with the eigenvalue $-\ell(\ell+1)$ with respect to the operator $\Delta_{\mathbb{S}^2}$.

The spherical harmonics form a complete basis of square integrable functions on $\mathbb{S}^2$. We denote this space by $L^2(\mathbb{S}^2)$ and we showed earlier that:

$$L^2(\mathbb{S}^2) = \bigoplus_{\ell=0}^{\infty} \mathscr{H}_\ell.$$

The direct sum above, is an orthogonal direct sum, so given the built-in normalization constant, the spherical harmonics are orthonormal on $\mathbb{S}^2$:

$$\int_0^{2\pi} d\phi \int_0^{\pi} \overline{Y}_{\ell m}(\theta,\phi) Y_{\ell' m'}(\theta,\phi) \sin\theta d\theta = \delta_{\ell\ell'} \delta_{mm'}.$$

Therefore, each $Y_{\ell m}$ is normalized to have unit length and the whole family of functions in orthogonal in both ℓ and m.

For all $f \in L^2(\mathbb{S}^2)$ we can expand f as an arbitrary superposition of spherical harmonics:

$$f(\theta,\phi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}a_{\ell m}Y_{\ell m}(\theta,\phi).$$

The above expansion converges in $L^2(\mathbb{S}^2)$. By using the orthonormality relation we can compute each coefficient $a_{\ell m}$ by projecting f onto the $(\ell,m)^{th}$ basis element:

$$a_{\ell m}=\int_0^{2\pi}d\phi\int_0^{\pi}Y_{\ell m}^*(\theta,\phi)f(\theta,\phi)\sin\theta d\theta.$$

By combining the last two equations we can write the function f as:

$$f(\theta,\phi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}Y_{\ell m}(\theta,\phi)\int_{\mathbb{S}^2}f(\theta',\phi')\overline{Y}_{\ell m}(\theta',\phi')d\Omega'.$$

This integral shows that $\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}Y_{\ell m}^*(\theta',\phi')Y_{\ell m}(\theta,\phi)$ is simply an angular delta function. We can write:

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}Y_{\ell m}^*(\theta',\phi')Y_{\ell m}(\theta,\phi)=\frac{1}{\sin\theta}\delta(\theta-\theta')\delta(\phi-\phi').$$

We can see from the above analysis that the spherical harmonics behave on $\mathbb{S}^2$ like sine and cosine behave on $\mathbb{S}^1$ with respect to Fourier decomposition. For all $f\in L^2(\mathbb{S}^2)$ we can decompose it into the superposition described above. $a_{\ell m}$ quantifies the contribution of the component $Y_{\ell m}$ to the function f.

5. Asymptotic Behavior of the Zonal Harmonics

Darboux gave us an asymptotic expression for the zonal harmonics [Szego 8.21.10] but first we need a couple lemmas.

We first state Watson's Lemma, a proof of which can be found in Murray [2].

Lemma. *(Watson) Let x be real and positive , $T>0$ and $\lambda>-1$. Then as $\ell\to\infty$:*

$$\int_0^T e^{-\ell z} z^{\lambda} g(z)dz \backsim \sum_{n=0}^{\infty} \frac{g^{(n)}(0)\Gamma(\lambda+n+1)}{n!\,\ell^{\lambda+n+1}}.$$

Lemma. *(Schläfli) If $\mathcal{C}$ is any closed contour encircling the origin, we have the following representation of the Legendre Polynomial of order ℓ:*

$$P_\ell(\cos\theta) = \frac{1}{2^{\ell+1}\pi i} \oint_{\mathcal{C}} \frac{e^{-\ell \log(\frac{t-\cos\theta}{t^2-1})}dt}{t-\cos\theta}$$

Proof. We begin with the Rodrigues formula expressed as a function of a complex variable z, $P_\ell(z) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dz^\ell}(z^2-1)^\ell$. Define $f(z) = (z^2-1)^\ell$ and apply Cauchy's Integral Formula to get:

$$\frac{d^\ell f}{dz^\ell} = \frac{\ell!}{2\pi i}\oint_{\mathcal{C}} \frac{(t^2-1)^\ell dt}{(t-z)^{\ell+1}}$$

We plug this into Rodrigues' formula to get:

$$P_\ell(z) = \frac{1}{2^{\ell+1}\pi i}\oint_{\mathcal{C}} \frac{(t^2-1)^\ell dt}{(t-z)^{\ell+1}} \Rightarrow P_\ell(\cos\theta) = \frac{1}{2^{\ell+1}\pi i}\oint_{\mathcal{C}} \frac{(t^2-1)^\ell dt}{(t-\cos\theta)^{\ell+1}}$$

From this it follows trivially that:

$$P_\ell(\cos\theta) = \frac{1}{2^{\ell+1}\pi i} \oint_{\mathcal{C}} \frac{e^{-\ell \log(\frac{t-\cos\theta}{t^2-1})}dt}{t-\cos\theta}$$

which completes the proof. □

Proposition. *(Darboux) As $\ell \to \infty$, there exists $0 < \epsilon < \pi$ such that the asymptotic form for the zonal harmonics given by:*

$$P_\ell(\cos\theta) = \sqrt{\frac{2}{\pi\ell\sin\theta}}\cos\left[(\ell+\frac{1}{2})\theta - \frac{\pi}{4}\right] + \mathcal{O}(\ell^{-\frac{3}{2}})$$

holds uniformly for all $\epsilon < \theta < \pi - \epsilon$.

Proof. We apply the Method of Steepest Descent and express the Zonal Harmonics as:

$$P_\ell(\cos\theta) = \frac{1}{2^{\ell+1}\pi i}\oint_{\mathcal{C}} e^{\ell p(t)} q(t) dt$$

where $p(t) = -\log\left(\frac{t-\cos\theta}{t^2-1}\right)$ and $q(t) = \frac{1}{t-\cos\theta}$. We established this in the above Lemma. Notice that

$$p'(t) = \frac{2t(t-\cos\theta) - (t^2-1)}{(t-\cos\theta)(t^2-1)}$$

which tells us that $p'(t) = 0$ when $t = e^{\pm i\theta}$. These t values we will call the *saddle points*. The primary contributions to the integral representation of P_ℓ comes from values near these saddle points, so by the Method of Steepest Descent, we deform the contour to go through both saddle points. We can take the contour to be the unit circle. But we have two saddle points with opposite imaginary parts, so by symmetry, it is sufficient to simply integrate over the upper half unit circle and multiply by 2. This allows us to concentrate only on the saddle point in the upper half complex plane. Therefore,

$$P_\ell(\cos\theta) = \frac{1}{2^\ell \pi} \mathcal{I}m\left\{ \int_{\mathcal{C}} q(t) e^{\ell p(t)} dt \right\}$$

where $\mathcal{C}$ defines the contour along the unit circle in the upper-half $\mathbb{C}$ plane. Let $t_0 = e^{i\theta}$. We introduce the real variable τ such that $p(t) - p(t_0) = -\tau^2$. This change of variables determines t as a function of τ, call it $t(\tau)$. It is important to note that this is a *local* change of variables; we are mapping a small portion of the contour near the saddle point t_0 to an interval in $\mathbb{R}$. By the change of variables we get:

$$\int_{\mathcal{C}} q(t) e^{\ell p(t)} dt = e^{\ell p(t_0)} \int_{-\infty}^{\infty} e^{-\ell\tau^2} q(t(\tau)) \frac{dt}{d\tau} d\tau + \mathcal{O}(\ell^{-N}),$$

as $\ell \to \infty$ and $N \in \mathbb{N}$. We are able integrate from $-\infty$ to ∞ because the contributions to the integral away from the saddle point are negligible. In summary, a small neighborhood around the saddle point is mapped to a small interval in $\mathbb{R}$, but we can just integrate over all of $\mathbb{R}$ since contributions to the integral from outside of this interval are negligible. We can Taylor expand $p(t) - p(t_0)$ about t_0.

$$\frac{1}{2}(t-t_0)^2 p''(t_0) + \mathcal{O}\big((t-t_0)^3\big) = -\tau^2$$

$$t - t_0 = \left\{ \frac{-2}{p''(t_0)} \right\}^{\frac{1}{2}} \tau + \mathcal{O}(\tau^2)$$

We can also expand $q(t(\tau))$ like so:

$$q(t(\tau)) = q(t_0) + (t-t_0)q'(t_0) + ... = q(t_0) + q'(t_0)\left\{ \frac{-2}{p''(t_0)} \right\}^{\frac{1}{2}} \tau + \mathcal{O}(\tau^2)$$

This yields the following approximation for the zonal harmonics:

$$\int_{\mathcal{C}} q(t)e^{\ell p(t)}dt \backsim \frac{e^{\ell p(t_0)}}{2^\ell \pi i} q(t_0) \left\{ \frac{-2}{p''(t_0)} \right\}^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\ell\tau^2} d\tau = \frac{e^{\ell p(t_0)}}{2^\ell \pi i} q(t_0) \left\{ \frac{-2}{p''(t_0)} \right\}^{\frac{1}{2}} \sqrt{\frac{\pi}{\ell}}$$

By a standard Gaussian integral $\int_{-\infty}^{\infty} e^{-\ell\tau^2} d\tau = \sqrt{\frac{\pi}{\ell}}$. Simple computations show that $q(t_0) = \frac{-i}{\sin\theta}$, $p''(t_0) = \frac{e^{-i\theta}}{i\sin\theta}$, and $p(t_0) = \log 2 + i\theta$. This reduces our approximation to:

$$\mathcal{I}m\left(\int_{\mathcal{C}} q(t)e^{\ell p(t)}dt \right) \backsim 2^\ell \sqrt{\frac{2\pi}{\ell\sin\theta}} (-i) e^{i\left\{(\ell+\frac{1}{2})\theta - \frac{\pi}{4}\right\}}$$

But we know that $P_\ell(\cos\theta) = \frac{1}{2^\ell\pi} \mathcal{I}m \left\{ \int_{\mathcal{C}} q(t)e^{\ell p(t)}dt \right\}$. This allows us to finally conclude that

$$P_\ell(\cos\theta) = \sqrt{\frac{2}{\pi\ell\sin\theta}} \cos\left[(\ell + \frac{1}{2})\theta - \frac{\pi}{4} \right] + E(\ell),$$

where $E(\ell)$ is an error term. We now need to compute the error of our expansion.

We've been studying the following integral:

$$\int_0^\infty e^{-\ell\tau^2} Q(\tau) d\tau,$$

where $Q(\tau) = q(t(\tau))\frac{dt}{d\tau}$. Now we make the change of variables $z = \tau^2$ from which it is easy to see that $d\tau = \frac{1}{2\sqrt{z}}dz$. Therefore:

$$\int_0^\infty e^{-\ell\tau^2} Q(\tau) d\tau = \int_0^\infty \frac{e^{-\ell z}}{2\sqrt{z}} Q(\sqrt{z}) dz.$$

We can now apply Watson's Lemma for $\lambda = -\frac{1}{2}$. The first term of the expansion corresponding to $n = 0$ behaves as $\ell^{-\frac{1}{2}}$, like we saw in our above approximation. The next term corresponding to $n = 0$ behaves as $\ell^{-\frac{3}{2}}$. Therefore,

$$P_\ell(\cos\theta) = \sqrt{\frac{2}{\pi\ell\sin\theta}} \cos\left[(\ell + \frac{1}{2})\theta - \frac{\pi}{4} \right] + \mathcal{O}(\ell^{-\frac{3}{2}}),$$

as desired. Notice that our asymptotics diverge as $\theta \to 0$ and our saddle points $e^{\pm i\theta}$ coalesce. Therefore, there exists $0 < \epsilon < \pi$ such that the above asymptotics hold for all $\epsilon \leqslant \theta \leqslant \pi - \epsilon$.

So the Method of Steepest Decent can only be applied if θ is fixed. Our integral estimates converge for ℓ sufficiently large but only if we fix $\epsilon \leqslant \theta \leqslant \pi - \epsilon$ for some $0 < \epsilon < \pi$.

□

There is a slightly stronger statement than the one above. We will state it here and a proof can be found in [5].

Proposition. *Let c be a positive fixed number.*

$$P_\ell(\cos\theta) = \sqrt{\frac{2}{\pi\ell\sin\theta}}\left\{\cos\left((\ell + \frac{1}{2}) - \frac{\pi}{4}\right) + \frac{\mathcal{O}(1)}{\ell\sin\theta}\right\},$$

for all $\frac{c}{\ell} \leqslant \theta \leqslant \pi - \frac{c}{\ell}$.

The advantage of the above statement is clear; the interval over which the asymptotics *do not* hold shrinks to zero as $\ell \to \infty$ whereas in the statement we proved, the asymptotics held only over a fixed interval, independent of ℓ.

Proposition. *Darboux's Asymptotics lead to the following L^p estimates for the zonal harmonics:*

$$||P_\ell||_p \backsim \begin{cases} 1 & p < 4 \\ (\log l)^{\frac{1}{4}} & p = 4 \\ l^{\frac{1}{2}-\frac{2}{p}} & p > 4 \end{cases}$$

Proof. Let $\delta = \frac{\pi}{2}$ and let c be a constant independent of ℓ and p.

$$\int_0^\delta |P_\ell(\cos\theta)|^p \sin\theta d\theta = \int_0^{\frac{c}{\ell}} |P_\ell(\cos\theta)|^p \sin\theta d\theta + \int_{\frac{c}{\ell}}^{\delta} |P_\ell(\cos\theta)|^p \sin\theta d\theta.$$

Notice that it is sufficient to integrate only to $\frac{\pi}{2}$ since $|P_\ell(\cos\theta)|$ is even over the interval $[0, \pi]$.

However, since $P(x) \leqslant 1$ for all $-1 \leqslant x \leqslant 1$, [1, p. 312, (8.53)] we may conclude that

$$\int_0^{\frac{c}{\ell}} |P_\ell(\cos\theta)|^p \sin\theta d\theta \leqslant \int_0^{\frac{c}{\ell}} \sin\theta d\theta \leqslant \int_0^{\frac{c}{\ell}} \theta d\theta = \frac{1}{2}\left(\frac{c}{\ell}\right)^2.$$

And by Darboux's Asymptotics above

$$\int_{\frac{c}{\ell}}^{\delta} |P_\ell(\cos\theta)|^p \sin\theta d\theta = \ell^{-\frac{p}{2}} \int_{\frac{c}{\ell}}^{\delta} (\sin\theta)^{1-\frac{p}{2}} d\theta$$

But we know that $(\sin\theta)^{1-\frac{p}{2}} \leqslant c_p \theta^{1-\frac{p}{2}}$ for a suitable constant c_p. Therefore, so long as $p \neq 4$,

$$\ell^{-\frac{p}{2}} \int_{\frac{c}{\ell}}^{\delta} (\sin\theta)^{1-\frac{p}{2}} d\theta \leqslant c_p \ell^{-\frac{p}{2}} \int_{\frac{c}{\ell}}^{\delta} \theta^{1-\frac{p}{2}} d\theta = C_p \ell^{-\frac{p}{2}} (\delta^{2-\frac{p}{2}} - (\frac{c}{\ell})^{2-\frac{p}{2}}),$$

where C_p is a p dependent constant. In particular, the sign of C_p changes depending on p: when $p < 4$, $C_p > 0$ and when $p > 4$, $C_p < 0$.

(i) If $2 - \frac{p}{2} > 0$ then $p < 4$ and $(\frac{c}{\ell})^{2-\frac{p}{2}}$ tends to zero for large ℓ. So for $p < 4$ we have:

$$\int_0^{\delta} |P_\ell(\cos\theta)|^p \sin\theta d\theta \backsim \left(\frac{c}{\ell}\right)^2 + \ell^{-\frac{p}{2}} \backsim \ell^{-\frac{p}{2}}.$$

But for L^p estimates we still need to take p^{th} roots. This shows that $||P_\ell||_p \backsim \ell^{-\frac{1}{2}}$.

(ii) If $2 - \frac{p}{2} < 0$ then $p > 4$. This implies that $||P_\ell||_p^p \backsim \left(\frac{c}{\ell}\right)^2 + \ell^{-\frac{p}{2}}((\frac{\ell}{c})^{\frac{p}{2}-2} - \delta^{2-\frac{p}{2}}) \backsim \ell^{-2}$. This shows that for $p > 4$ $||P_\ell||_p \backsim \ell^{-\frac{2}{p}}$.

(iii) When $p = 4$ we can't apply product rule and instead integrate to get a logarithm. $\ell^{-2} \int_{\frac{c}{\ell}}^{\delta} (\sin\theta)^{-1} d\theta$ and $(\sin\theta)^{-1} \leqslant c_4 \theta^{-1}$. This shows that $\ell^{-2} \int_{\frac{c}{\ell}}^{\delta} \theta^{-1} d\theta$ is an upper bound on the L^4 norm.

$$\ell^{-2} \int_{\frac{c}{\ell}}^{\delta} \theta^{-1} d\theta = \ell^{-2}(\log\delta - \log(\frac{c}{\ell})) \backsim \ell^{-2} \log\ell$$

$$||P_\ell||_4 \backsim \ell^{-\frac{1}{2}} (\log\ell)^{\frac{1}{4}}$$

What we've shown so far is:

$$||P_\ell||_p < \begin{cases} \ell^{-\frac{1}{2}} & p < 4 \\ \ell^{-\frac{1}{2}} (\log\ell)^{\frac{1}{4}} & p = 4 \\ \ell^{-\frac{2}{p}} & p > 4 \end{cases}$$

However, we would like to be L^2 normalized which means that $||P_\ell||_2 = 1$. We can achieve this by taking $c_\ell P_\ell(\cos\theta)$ to be the normalized zonal harmonics for a suitable normalization constant c_ℓ. Notice that for $\ell >> 1$, $c_\ell \backsim \ell^{\frac{1}{2}}$. This yields the normalized growth rates:

$$c_\ell ||P_\ell||_p \backsim \ell^{\frac{1}{2}} ||P_\ell||_p < \begin{cases} 1 & p < 4 \\ (\log l)^{\frac{1}{4}} & p = 4 \\ l^{\frac{1}{2}-\frac{2}{p}} & p > 4 \end{cases}$$

So we've computed upper bounds on the L^p estimates desired. We need to now compute lower bounds to complete the proof.

We want to find lower bounds on the following integral:

$$\ell^{-\frac{p}{2}} \int_{\frac{1}{\ell}}^{\frac{\pi}{2}} \theta^{1-\frac{p}{2}} \left| \cos((\ell + \frac{1}{2})\theta - \frac{\pi}{4}) \right|^p d\theta$$

We change variables such that $t = (\ell + \frac{1}{2})\theta - \frac{\pi}{4}$ and $dt = (\ell + \frac{1}{2})d\theta$, and the integral becomes

$$\ell^{-\frac{p}{2}} \int_{\alpha_\ell}^{\beta_\ell} (\ell + \frac{1}{2})^{\frac{p}{2}-2} (t + \frac{\pi}{4})^{1-\frac{p}{2}} |\cos t|^p dt$$

where $\alpha_\ell = 1 - \frac{\pi}{4} + \frac{1}{2\ell}$ and $\beta_\ell = \frac{\ell\pi}{2}$. Notice that

$$\ell^{-\frac{p}{2}} (\ell + \frac{1}{2})^{\frac{p}{2}-2} \geqslant (\ell + \frac{1}{2})^{-2} \backsim \ell^{-2}.$$

The above expression is valid for all p. Therefore, our entire integral is bounded below by

$$\ell^{-2} \int_{\alpha_\ell}^{\beta_\ell} (t + \frac{\pi}{4})^{1-\frac{p}{2}} |\cos t|^p dt \geqslant \ell^{-2} \int_{\frac{3\pi}{2}}^{\frac{\ell\pi}{2}} (t + \frac{\pi}{4})^{1-\frac{p}{2}} |\cos t|^p dt$$

The final inequality holds because we increase the lower limit of integration, and therefore integrate over a smaller interval to get a lower bound. We can clearly write

$$\int_{\frac{3\pi}{2}}^{\frac{\ell\pi}{2}} (t + \frac{\pi}{4})^{1-\frac{p}{2}} |\cos t|^p dt = \sum_{k=3}^{\ell-1} \int_{\frac{k\pi}{2}}^{\frac{(k+1)\pi}{2}} (t + \frac{\pi}{4})^{1-\frac{p}{2}} |\cos t|^p dt$$

and define $f(t) = (1 + \frac{\pi}{2})^{1-\frac{p}{2}}$. By the Mean Value Theorem there exists $t_k \in [\frac{k\pi}{2}, \frac{(k+1)\pi}{2}]$, for all k such that

$$\sum_{k=3}^{\ell-1} \int_{\frac{k\pi}{2}}^{\frac{(k+1)\pi}{2}} f(t)|\cos t|^p dt = \sum_{k=3}^{\ell-1} \int_{\frac{k\pi}{2}}^{\frac{(k+1)\pi}{2}} f(t_k)|\cos t|^p dt.$$

So long as $p > 2$, $f(t)$ is decreasing: $f(t_k) \geqslant \left(\frac{(k+1)\pi}{2} + \frac{\pi}{4}\right)^{1-\frac{p}{2}}$. So we can bound the integral even further from below by

$$\sum_{k=3}^{\ell-1} \left(\frac{(k+1)\pi}{2} + \frac{\pi}{4}\right)^{1-\frac{p}{2}} \int_{\frac{k\pi}{2}}^{\frac{(k+1)\pi}{2}} |\cos t|^p dt$$

We now notice that

$$C = \int_{\frac{k\pi}{2}}^{\frac{(k+1)\pi}{2}} |\cos t|^p dt$$

is simply a constant. We now want to find even further lower bounds on

$$\sum_{k=3}^{\ell-1} \left(\frac{k\pi}{2} + \frac{3\pi}{4}\right)^{1-\frac{p}{2}}.$$

Notice that so long as $p > 2$ and $t \in \left[\frac{k\pi}{2}, \frac{(k+1)\pi}{2}\right]$, we may conclude

$$\left(\frac{k\pi}{2} + \frac{3\pi}{4}\right)^{1-\frac{p}{2}} \geqslant \left(\frac{t\pi}{2} + \frac{3\pi}{4}\right)^{1-\frac{p}{2}}$$

Therefore,

$$\sum_{k=3}^{\ell-1} \left(\frac{k\pi}{2} + \frac{3\pi}{4}\right)^{1-\frac{p}{2}} \geqslant \sum_{k=3}^{\ell-1} \int_{\frac{k\pi}{2}}^{\frac{(k+1)\pi}{2}} \left(\frac{t\pi}{2} + \frac{3\pi}{4}\right)^{1-\frac{p}{2}} dt$$

$$= \int_{\frac{3\pi}{2}}^{\frac{\ell\pi}{2}} \left(\frac{t\pi}{2} + \frac{3\pi}{4}\right)^{1-\frac{p}{2}} dt.$$

Now we have three different cases to worry about:

(i) First let $p > 4$. In this case we can simply integrate to show that:

$$\ell^{-2}\int_{\frac{3\pi}{2}}^{\frac{\ell\pi}{2}}\left(\frac{t\pi}{2}+\frac{3\pi}{4}\right)^{1-\frac{p}{2}}dt=\ell^{-2}\Bigg\{\frac{1}{(\frac{9\pi}{4})^{\frac{p}{2}-2}}-\frac{1}{(\frac{\ell\pi}{2}+\frac{3\pi}{4})^{\frac{p}{2}-2}}\Bigg\}\backsim\frac{\ell^{-2}}{(\frac{9\pi}{4})^{\frac{p}{2}-2}}.$$

In the last line, we can neglect the second term because it goes to zero as $\ell\to\infty$. Therefore, we've shown that the summation simply goes like a constant, dependent on p, but independent of ℓ. Let's absorb all of the constants into the single p-dependent constant, call it C_p.

As an overall lower bound, we've found that for $p>4$:

$$||P_\ell||_p\geqslant(C_p\ell^{-2})^{\frac{1}{p}}\backsim\ell^{-\frac{2}{p}}.$$

However, recall from above that the normalization constant for the zonal harmonics behaves as $\ell^{\frac{1}{2}}$ for large ℓ. Accounting for this, we find for a lower bound $\ell^{\frac{1}{2}-\frac{2}{p}}$. Therefore, in the case $p>4$, we have shown that we can find p-dependent constants C_{p_1} and C_{p_2} such that:

$$C_{p_1}\ell^{\frac{1}{2}-\frac{2}{p}}\leqslant\ell^{\frac{1}{2}}||P_\ell||_p\leqslant C_{p_2}\ell^{\frac{1}{2}-\frac{2}{p}}$$

Therefore, $\ell^{\frac{1}{2}}||P_\ell||_p\backsim\ell^{\frac{1}{2}-\frac{2}{p}}$.

(ii) Consider now the case $p=4$.

$$\ell^{-2}\int_{\frac{3\pi}{2}}^{\frac{\ell\pi}{2}}\left(\frac{t\pi}{2}+\frac{3\pi}{4}\right)^{-1}dt=\frac{2\ell^{-2}}{\pi}\int_{\frac{3\pi}{2}}^{\frac{\ell\pi}{2}}\left(t+\frac{3}{2}\right)^{-1}dt$$

$$=\frac{2\ell^{-2}}{\pi}\Bigg(\log\big(\frac{\ell\pi}{2}+\frac{3}{2}\big)-\log\big(\frac{3\pi}{2}+\frac{3}{2}\big)\Bigg)\backsim\ell^{-2}\log\ell.$$

As dictated by the p-norm, we take this expression to the power $\frac{1}{4}$. This yields the unnormalized growth rates to be:

$$||P_\ell||_4\backsim\ell^{-\frac{1}{2}}(\log\ell)^{\frac{1}{4}},$$

and with the asymptotic expression of $c_\ell\backsim\ell^{\frac{1}{2}}$, we get:

$$\ell^{\frac{1}{2}}||P_\ell||_4\backsim(\log\ell)^{\frac{1}{4}}.$$

(iii) Consider, finally, the case $2<p<4$. Here we must compute the integral:

$$\ell^{-2}\int_{\frac{3\pi}{2}}^{\frac{\ell\pi}{2}}\left(\frac{t\pi}{2}+\frac{3\pi}{4}\right)^{1-\frac{p}{2}}dt \backsim \ell^{-2}(\ell+\frac{3}{2})^{2-\frac{p}{2}} \backsim \ell^{-\frac{p}{2}}.$$

When we take the expression to the power $\frac{1}{p}$ we get when $2 < p < 4$:

$$||P_\ell||_p \backsim \ell^{-\frac{1}{2}},$$

and clearly, when we multiple by $c_\ell \backsim \ell^{\frac{1}{2}}$ we get:

$$\ell^{\frac{1}{2}}||P_\ell||_p \backsim 1.$$

This completes the proof. □

6. The Zonal Harmonics as a Measure on $\mathbb{S}^2$

Notice that as a consequence of the above Proposition,

$$||P_\ell||_\infty = \sup_{x\in\mathbb{S}^2} P_\ell(x) \backsim \ell^{\frac{1}{2}}$$

which tells us that as $\ell \to \infty$ the zonal harmonics have a supremum which also tends to infinity as $\ell^{\frac{1}{2}}$. However, to retain L^2-normalization, the zonal harmonics must become more concentrated over some region.

We conclude this from the above Proposition, but to see exactly *where* this concentration is occurring we must refer back to Darboux's Asymptotics:

$$P_\ell(\cos\theta) = \sqrt{\frac{2}{\pi\ell\sin\theta}}\cos\left[(\ell+\frac{1}{2})\theta-\frac{\pi}{4}\right]+\mathcal{O}(\ell^{-\frac{3}{2}})$$

This tells us that the peaks, and corresponding regions of concentration on $\mathbb{S}^2$ are the north and south poles corresponding to $\sin\theta = 0$.

It may appear that as $\ell \to \infty$ the zonal harmonics converge to some sort of Dirac delta function at the two poles, but not every sharply peaked, highly concentrated function behaves mathematically as a delta function. The determining characteristic of a delta function $\delta(x - x_0)$ on $\mathbb{R}^k$ is that for all well-behaved functions $f : \mathbb{R}^k \to \mathbb{R}$,

$$\int_{\mathbb{R}^k} \delta(x - x_0) f(x) dx = f(x_0).$$

We can think of $\delta(x - x_0)dx$ as being a *measure* on $\mathbb{R}^k$ in the following sense: if we define $d\Omega(x_0) = \delta(x - x_0)dx$, we have that:

$$\int_{\mathbb{R}^k} f(x) d\Omega(x_0) = f(x_0).$$

In an analogous way, we can imagine $c_\ell |P_\ell(\cos\theta)|^2$ as defining a measure on $\mathbb{S}^2$. The following proposition establishes this measure. But first, we need a lemma:

Lemma. *(Riemann-Lebesgue) Let $f(t)$ be piecewise continuous on $[0, 2\pi]$. Then*

$$\lim_{\ell\to\infty} \int_0^{2\pi} f(t) e^{i\ell t} dt = 0.$$

Proposition. *Let $f(\theta, \phi) \in L^2(\mathbb{S}^2)$ be continuous and let $c_\ell P_\ell(\cos\theta)$ be the normalized zonal harmonics. Then the following limit holds:*

$$\lim_{\ell\to\infty} \int_{\mathbb{S}^2} c_\ell^2 |P_\ell(\cos\theta)|^2 f(\theta, \phi) d\Omega = \frac{1}{\pi} \int_0^{\pi} \frac{1}{2\pi} \int_0^{2\pi} f(\theta, \phi) d\theta d\phi.$$

Proof. Recall from Darboux's asymptotics, we need to multiply his expression by c_ℓ to obtain the normalized zonal harmonics. From what we've done above, we expect c_ℓ to behave as $\ell^{\frac{1}{2}}$ but we care about a numerical factor multiplying c_ℓ in this case as well. The normalization condition is:

$$2\pi \int_0^{\pi} c_\ell^2 |P_\ell(\cos\theta)|^2 \sin\theta d\theta = 1.$$

By Darboux we know $|P_\ell(\cos\theta)|^2 \sin\theta = 2(\ell\pi)^{-1} \cos^2\left((\ell + \frac{1}{2})\theta - \frac{\pi}{4}\right) + \mathcal{O}(\ell^{-\frac{3}{2}})$. Therefore, we have the following approximate integral which we can use to determine c_ℓ:

$$\int_{\frac{c}{\ell}}^{\pi - \frac{c}{\ell}} \cos^2\left((\ell + \frac{1}{2})\theta - \frac{\pi}{4}\right) d\theta + \mathcal{O}(\ell^{-\frac{1}{2}}) = \frac{\ell}{4c_\ell^2}.$$

As we saw in an earlier proof, contributions to the integral from the region $[0, \frac{c}{\ell}] \cup [\pi - \frac{c}{\ell}, \pi]$ go to zero as ℓ^{-2} so we may neglect them.

We can now write the cosine term as:

$$\cos^2\left((\ell+\frac{1}{2})\theta-\frac{\pi}{4}\right)=\left(\frac{e^{i\left((\ell+\frac{1}{2})\theta-\frac{\pi}{4}\right)}+e^{-i\left((\ell+\frac{1}{2})\theta-\frac{\pi}{4}\right)}}{2}\right)^2=\frac{1}{2}+\frac{1}{4}e^{i((2\ell+1)\theta-\frac{\pi}{2})}+\frac{1}{4}e^{-i((2\ell+1)\theta-\frac{\pi}{2})}$$

So we can write the integral over the cosine as:

$$\int_0^\pi \frac{1}{2}d\theta+\frac{1}{4}\int_0^\pi e^{i((2\ell+1)\theta-\frac{\pi}{2})}d\theta+\frac{1}{4}\int_0^\pi e^{-i((2\ell+1)\theta-\frac{\pi}{2})}d\theta.$$

We extend the region of integration to all of $[0,\pi]$ since the contributions near 0 and π are negligible as we mentioned above. Trivially, the first integral is $\frac{\pi}{2}$. The second and third integrals go to zero as $\ell\to\infty$ by direct computation.

We see, therefore, that $\frac{\ell}{4c_\ell^2}=\frac{\pi}{2}+\mathcal{O}(\ell^{-\frac{1}{2}})$ which yields the final expression for our normalization constant:

$$c_\ell \backsim \sqrt{\frac{\ell}{2\pi}}.$$

Now that we have the normalization constant, we want to study the following integral:

$$\int_0^{2\pi}\int_0^\pi c_\ell^2|P_\ell(\cos\theta)|^2 f(\theta,\phi)\sin\theta d\theta d\phi=\int_0^\pi \frac{2c_\ell^2}{\pi\ell}\cos^2\left((\ell+\frac{1}{2})\theta-\frac{\pi}{4}\right)\int_0^{2\pi} f(\theta,\phi)d\phi d\theta.$$

$$=\frac{2c_\ell^2}{\pi\ell}\int_0^\pi\left(\frac{1}{2}+\frac{1}{4}e^{i((2\ell+1)\theta-\frac{\pi}{2})}+\frac{1}{4}e^{-i((2\ell+1)\theta-\frac{\pi}{2})}\right)\frac{1}{2\pi}\int_0^{2\pi} f(\theta,\phi)d\phi d\theta.$$

Let's define $\int_0^{2\pi} f(\theta,\phi)d\phi=g(\theta)$. Clearly, if f is continuous, then g is continuous. Notice that the second two terms in parentheses contribute to:

$$\frac{e^{i\frac{\pi}{2}}}{4}\int_0^\pi e^{i(2\ell+1)\theta}g(\theta)d\theta=\frac{1}{2}\int_0^{2\pi} e^{i(\ell+\frac{1}{2})\theta}g(\frac{\theta}{2})d\theta=\frac{1}{2}\int_0^{2\pi} e^{i\ell\theta}\big(e^{\frac{i\theta}{2}}g(\frac{\theta}{2})\big)d\theta.$$

We again realize these integrals as Fourier coefficients, and use Riemann-Lebesgue to conclude they go to zero as $\ell\to\infty$.

Therefore, as $\ell\to\infty$ we have the following approximation:

$$\int_0^{2\pi}\int_0^\pi c_\ell^2|P_\ell(\cos\theta)|^2 f(\theta,\phi)\sin\theta d\theta d\phi=\left(\frac{2c_\ell^2}{\pi\ell}\right)\frac{1}{2}\int_0^\pi\int_0^{2\pi} f(\theta,\phi)d\theta d\phi+\mathcal{O}(\ell^{-2})$$

$$= \frac{1}{\pi}\int_0^{\pi}\frac{1}{2\pi}\int_0^{2\pi} f(\theta,\phi)d\theta d\phi + \mathcal{O}(\ell^{-2}),$$

which completes the proof. □

There are a couple points to make from the above proposition. Firstly, notice that we can trivially write:

$$\frac{1}{\pi}\int_0^{\pi}\frac{1}{2\pi}\int_0^{2\pi} f(\theta,\phi)d\theta d\phi = \frac{1}{\pi}\int_0^{\pi}\frac{1}{2\pi}\int_0^{2\pi} \frac{f(\theta,\phi)}{\sin\theta}\sin\theta d\theta d\phi = \frac{1}{\pi}\int_0^{\pi}\frac{1}{2\pi}\int_0^{2\pi} \frac{f(\theta,\phi)}{\sin\theta}d\Omega.$$

This is simply the average value of the function $\frac{f(\theta,\phi)}{\sin\theta}$ on $\mathbb{S}^2$. More importantly, we can realize $c_\ell^2|P_\ell(\cos\theta)|^2$ as giving us a measure on $\mathbb{S}^2$ for large ℓ, in the following sense:

$$\lim_{\ell\to\infty} c_\ell^2|P_\ell(\cos\theta)|^2 d\Omega = \frac{1}{\sin\theta}d\Omega.$$

As mentioned earlier, a Dirac Delta Function $\delta(x-x_0)$, thought of as a measure on $\mathbb{R}^k$, can be paired in an integral with a continuous function f to output $f(x_0)$. We now see the analogy to what we did above: $c_\ell^2|P_\ell(\cos\theta)|^2$, thought of as a measure on $\mathbb{S}^2$ for $\ell\to\infty$, can be paired in an integral with a continuous function f to output the average value of $\frac{f}{\sin\theta}$ on $\mathbb{S}^2$.

Therefore, the above Proposition shows that the normalized zonal harmonics *do not* converge to a Delta Function at the north and south poles.

7. Asymptotic Behavior of the Sectoral Harmonics

Proposition. *Let $Y_{\ell\ell}$ be the secotoral harmonic of order ℓ. Then $Y_{\ell\ell} = c_{\ell\ell}(x_1+ix_2)^\ell$, where $c_{\ell\ell}$ is an ℓ dependent constant.*

Proof. Recall that $x_1 = \cos\phi\sin\theta$ and $x_2 = \sin\theta\sin\phi$ by the coordinates we've fixed. $(x_1+ix_2)^\ell = \sin^\ell\theta(\cos\phi + i\sin\phi)^\ell = (\sin^\ell\theta)e^{i\ell\phi}$. But we know that:

$$P_{\ell\ell}(x) = (1-x^2)^{\frac{\ell}{2}}\frac{d^\ell P_\ell(x)}{dx^\ell}$$

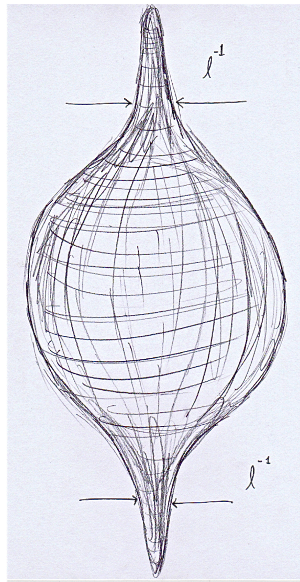

FIGURE 1. $c_\ell|P_\ell(\cos\theta)|$ shown for large l

But ℓ derivatives of an ℓ^{th} order polynomial simply yields a constant and we change variables by letting $x = \cos\theta$ to get $P_{\ell\ell}(\cos\theta) = c_{\ell\ell}\sin^\ell\theta$. Therefore, $c_{\ell\ell}(x_1 + ix_2)^\ell = c_{\ell\ell}(\sin^\ell\theta)e^{i\ell\phi} = c_{\ell\ell}P_{\ell\ell}(\cos\theta)e^{i\ell\phi} = Y_{\ell\ell}$. □

Remark. $\log(1-\xi) = -\xi + \mathcal{O}(\xi^2)$. *In particular, the bounds* $-2\xi \leqslant \log(1-\xi) \leqslant -\xi$ *hold over the interval* $0 \leqslant \xi \leqslant \frac{1}{2}$.

Proof. To see this, Taylor expand $\log(1-\xi)$ about $\xi = 0$. We first note that $\frac{d}{d\xi}\big(\log(1-\xi)\big) = -\frac{1}{1-\xi}$ and $\frac{d^2}{d\xi^2}\big(\log(1-\xi)\big) = \frac{1}{(1-\xi)^2}$. Using the Taylor Expansion with Remainder, this shows that:

$$\log(1-\xi) = \log(1-0) + (\xi - 0)\big(\frac{-1}{1-0}\big) - \frac{1}{2}\left(\frac{1}{(1-c)^2}\right)\xi^2 = -\xi - \frac{1}{2}\left(\frac{1}{(1-c)^2}\right)\xi^2,$$

for some $c \in [0, \xi]$.

We write $\log(1-\xi) = -\xi + E(\xi)$ where $E(\xi)$ is an error term from the Taylor expansion. Written out explicitly as above, we see that:

$$E(\xi) = -\frac{1}{2}\left(\frac{1}{(1-c)^2}\right)\xi^2,$$

for some $c \in [0, \xi]$.

Since we're assuming $\xi \in [0, \frac{1}{2}]$, we observe the following upper bound on the absolute value of the error:

$$|E(\xi)| \leqslant \frac{1}{2} \max_{0 \leqslant c \leqslant \frac{1}{2}} \left(\frac{1}{(1-c)^2}\right)\xi^2.$$

Over the given interval, the function $\frac{1}{(1-\xi)^2}$ is an increasing function so that we may conclude:

$$\frac{1}{2} \max_{0 \leqslant c \leqslant \frac{1}{2}} \left(\frac{1}{(1-c)^2}\right)\xi^2 \leqslant \frac{1}{2}(4\xi^2) = 2\xi^2,$$

for $\xi \in [0, \frac{1}{2}]$. Therefore, for all ξ in the same interval:

$$\frac{|E(\xi)|}{\xi} \leqslant 2\xi \leqslant 1.$$

This is equivalent to stating $-\xi < E(\xi) < \xi$. But we can already see from above that so long as $\xi \in [0, \frac{1}{2}]$, $E(\xi) < 0$. Therefore:

$$-\xi < E(\xi) < 0,$$

from which it follows that:

$$-2\xi < -\xi + E(\xi) = \log(1-\xi) < -\xi.$$

This completes the proof. □

Proposition. *For all $|x_3|^2 < \frac{1}{2}$, we have the following bounds:*

$$e^{-\ell x_3^2} \leqslant |x_1 + i x_2|^\ell \leqslant e^{-\frac{\ell}{2} x_3^2}.$$

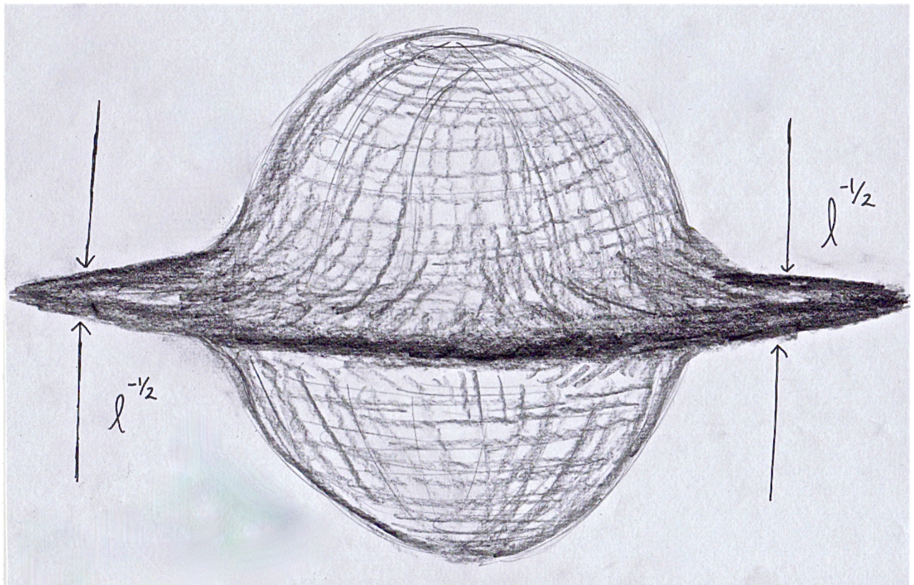

FIGURE 2. $c_{\ell\ell}|Y_{\ell\ell}|$ shown for large l

Proof. We've shown above that $c_{\ell\ell}Y_{\ell\ell} = c_{\ell\ell}(x_1 + ix_2)^\ell$ but let's take the absolute value of $Y_{\ell\ell}$ to get rid of the exponential phase. $|Y_{\ell\ell}| = |x_1 + ix_2|^\ell = (x_1^2 + x_2^2)^{\frac{\ell}{2}} = (1 - x_3^2)^{\frac{\ell}{2}} = \left(e^{\log(1-x_3^2)}\right)^{\frac{\ell}{2}}$. We can now apply the results of the above Proposition to $\xi = x_3^2$. Namely:

$$-2x_3^2 < \log(1 - x_3^2) < -x_3^2.$$

From this, the result follows trivially. □

Proposition. *The $L^p(\mathbb{S}^2)$ estimates for the normalized sectoral harmonics $Y_{\ell\ell}$ are given for all $p \in \mathbb{N}$ by:*

$$||Y_{\ell\ell}||_p \backsim \ell^{\frac{1}{4}-\frac{1}{2p}}.$$

Proof. We want to get upper and lower bounds on the integral:

$$c_{\ell\ell}\left(\int_0^\pi |Y_{\ell\ell}|^p \sin\theta d\theta\right)^{\frac{1}{p}}.$$

However, $x = \cos\theta$ which implies that $dx_3 = -\sin\theta d\theta$ and we can re-write the integral, using the above proposition, as:

$$c_{\ell\ell}\int_{-1}^{1}|x_1+ix_2|^{\ell p}dx_3 \leqslant c_{\ell\ell}\int_{-1}^{1}e^{-\frac{\ell p}{2}x_3^2}dx_3 \leqslant c_{\ell\ell}\int_{-\infty}^{\infty}e^{-\frac{\ell p}{2}x_3^2}dx_3.$$

But by a very standard Gaussian integration:

$$\int_{-\infty}^{\infty}e^{-\frac{p\ell}{2}x_3^2}dx_3 = \sqrt{\frac{2\pi}{\ell p}} \backsim \ell^{-\frac{1}{2}}.$$

Therefore, we have the following result for the upper bounds on the L^p norms:

$$c_{\ell\ell}||Y_{\ell\ell}||_p \leqslant c_{\ell\ell}\ell^{-\frac{1}{2p}}.$$

To compute the lower bounds, we use the result that $|x_1+ix_2|^\ell \geqslant e^{-\ell x_3^2}$ so long as $|x_3|^2 \leqslant \frac{1}{2}$. Therefore:

$$\int_{-1}^{1}|x_1+ix_2|^{\ell p}dx_3 \geqslant \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}e^{-p\ell x_3^2}\frac{1}{\sqrt{2}}dx_3.$$

By a change of variables we can re-write this integral as:

$$\frac{1}{\sqrt{2\ell}}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}}e^{-px_3^2}dx_3 = \frac{1}{\sqrt{2\ell}}\left(\int_{-\infty}^{\infty}e^{-px_3^2}dx_3 - 2\int_{\frac{\ell}{\sqrt{2}}}^{\infty}e^{-px_3^2}dx_3\right).$$

The second term decays exponentially as $\ell \to \infty$ and we've established the following lower bounds:

$$c_{\ell\ell}||Y_{\ell\ell}||_p \geqslant c_{\ell\ell}\ell^{-\frac{1}{2p}}.$$

The normalization constant $c_{\ell\ell}$ yields L^2 normalization. We observe that the asymptotic expression for large ℓ is $c_{\ell\ell} = \ell^{\frac{1}{4}}$.

Therefore, the normalized family of sectoral harmonics is given by $\ell^{\frac{1}{4}}Y_{\ell\ell}$ and the L^p growth rates are given by:

$$\ell^{\frac{1}{4}}||Y_{\ell\ell}||_p \backsim \ell^{\frac{1}{4}-\frac{1}{2p}}.$$

□

8. The Sectoral Harmonics as Point Mass Measure on $\mathbb{S}^2$

Recall from an earlier section that the zonal harmonics converge to a measure $\frac{1}{\sin\theta}d\Omega$ on $\mathbb{S}^2$. We now want to study the measure on $\mathbb{S}^2$ defined by the sectoral harmonics for large ℓ.

Proposition. *Let $f \in L^2(S^2)$ be continuous and take $\psi_\ell(\theta,\phi) = Y_{\ell\ell}(\theta,\phi)$ as the normalized Sectoral Harmonics. Then the following limit holds:*

$$\lim_{l\to\infty}\int_0^{2\pi}\int_0^{\pi}|\psi_\ell|^2 f(\theta,\phi)\sin\theta d\theta d\phi = \frac{1}{2\pi}\int_0^{2\pi} f(\frac{\pi}{2},\phi)d\phi$$

Proof. It is enough to show:

$$\int_0^{2\pi}\int_0^{\pi}|\psi_\ell|^2\Big\{f(\theta,\phi)-f(\frac{\pi}{2},\phi)\Big\}\sin\theta d\theta d\phi = \int_0^{\pi}|\psi_\ell|^2\bigg(\int_0^{2\pi}\Big\{f(\theta,\phi)-f(\frac{\pi}{2},\phi)\Big\}d\phi\bigg)\sin\theta d\theta \to 0,$$

as $\ell \to \infty$.

By assumption $f(\theta,\phi)$ is uniformly continuous at $\theta = \frac{\pi}{2}$. Therefore, given $\epsilon > 0$ there exists $\delta > 0$ such that $|f(\theta,\phi) - f(\frac{\pi}{2},\phi)| < \frac{\epsilon}{2}$ for all $|\theta - \frac{\pi}{2}| < \delta$ and any $\phi \in [0,2\pi]$. This gives the following bounds:

$$\left|\int_0^{2\pi} f(\theta,\phi) - f(\frac{\pi}{2},\phi)d\phi\right| < \epsilon\pi.$$

Therefore,

$$\left|\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta}|\psi_\ell|^2\bigg(\int_0^{2\pi}\Big\{f(\theta,\phi)-f(\frac{\pi}{2},\phi)\Big\}d\phi\bigg)\sin\theta d\theta\right|$$

$$< \int_0^{\pi}|\psi_\ell|^2\left|\int_0^{2\pi}\Big\{f(\theta,\phi)-f(\frac{\pi}{2},\phi)\Big\}d\phi\right|\sin\theta d\theta$$

$$< \epsilon\pi \int_0^\pi |\psi_\ell|^2 \sin\theta d\theta$$

But by normalization of the Sectoral Harmonics we know that:

$$\int_0^\pi |\psi_\ell|^2 \sin\theta d\theta = \frac{1}{2\pi}.$$

This yields the following upper bound:

$$\left| \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} |\psi_\ell|^2 \Bigg(\int_0^{2\pi} \left\{ f(\theta,\phi) - f(\frac{\pi}{2},\phi) \right\} d\phi \Bigg) \sin\theta d\theta \right| < \frac{\epsilon}{2}.$$

So we let $\epsilon > 0$ be arbitrary, we fix the δ that comes naturally out of the continuity of f at $\theta = \frac{\pi}{2}$ and break up the interval $[0,\pi]$ like so:
$[0,\pi] \to [0, \frac{\pi}{2} - \delta], [\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta], [\frac{\pi}{2} + \delta, \pi]$.

This allows us to integrate over the region $[0,\pi]$ piece by piece, like so:

$$\int_0^\pi dz = \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} dz + \int_0^{\frac{\pi}{2}-\delta} dz + \int_{\frac{\pi}{2}+\delta}^{\pi} dz.$$

We have estimates on the integral over the region $[\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta]$ and now we simply need to do the same for the intervals $[0, \frac{\pi}{2} - \delta]$ and $[\frac{\pi}{2} + \delta, \pi]$.

Since $f \in L^2(S^2)$ and continuous, there exists an $M > 0$ such that $|f(\theta,\phi) - f(\frac{\pi}{2},\phi)| < 2M$. We know such an M exists because since $\mathbb{S}^2$ is compact and f is continuous, we can take M as the global supremum.

Note that δ is still fixed from the continuity of f. Therefore:

$$\left| \int_0^{2\pi} \big| f(\theta,\phi) - f(\frac{\pi}{2},\phi) \big| d\phi \right| < 4M\pi.$$

Since the Sectoral Harmonics decay about the equator of $\mathbb{S}^2$ like $\ell^{-\frac{1}{2}}$ we can sufficiently extend our window by taking $\frac{1}{\ell} < \delta < |x_3|$.

This shows that $|\psi_\ell|^2 \sim \ell^{\frac{1}{2}} e^{\ell \log(1-x_3^2)} < \ell^{\frac{1}{2}} e^{\ell \log(1-\delta^2)} = \ell^{\frac{1}{2}} e^{-\ell \log(\frac{1}{1-\delta^2})} < \frac{\epsilon}{8M\pi}$ provided $\ell > N$ for N sufficiently large. Therefore:

$$\Big|\int_0^{\frac{\pi}{2}-\delta} |\psi_\ell|^2 \Big(\int_0^{2\pi} \Big\{f(\theta,\phi) - f(\frac{\pi}{2},\phi)\Big\}d\phi\Big) \sin\theta d\theta\Big|$$

$$+\Big|\int_{\frac{\pi}{2}+\delta}^{\pi} |\psi_\ell|^2 \Big(\int_0^{2\pi} \Big\{f(\theta,\phi) - f(\frac{\pi}{2},\phi)\Big\}d\phi\Big) \sin\theta d\theta\Big|$$

$$< \frac{4M\pi\epsilon}{8M\pi} = \frac{\epsilon}{2}$$

Finally, this yields the overall upper bound:

$$\Big|\int_0^{\pi} |\psi_\ell|^2 \Big(\int_0^{2\pi} \Big\{f(\theta,\phi) - f(\frac{\pi}{2},\phi)\Big\}d\phi\Big) \sin\theta d\theta\Big| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Which shows that:

$$\lim_{\ell\to\infty}\int_0^{\pi} |\psi_\ell|^2 \Big(\int_0^{2\pi} \Big\{f(\theta,\phi) - f(\frac{\pi}{2},\phi)\Big\}d\phi\Big) \sin\theta d\theta = 0$$

And as we noted at the beginning of the proof, from this, it follows that:

$$\lim_{l\to\infty}\int_0^{2\pi}\int_0^{\pi} |\psi_\ell|^2 f(\theta,\phi)\sin\theta d\theta d\phi = \frac{1}{2\pi}\int_0^{2\pi} f(\frac{\pi}{2},\phi)d\phi$$

□

Note that the right hand side is simply an average value of the function f at the equator of $\mathbb{S}^2$ (defined by $\theta = \frac{\pi}{2}$). Notice that:

$$\int_0^{\pi} \lim_{\ell\to\infty} |\psi_\ell|^2 f(\theta,\phi)\sin\theta d\theta = \frac{1}{2\pi} f(\frac{\pi}{2},\phi).$$

This behavior is reminiscent of a Dirac Delta Function. However, instead of being concentrated at a point, as in $\delta(x-x_0)$, the sectoral harmonics concentrate at the equatorial line $\theta = \frac{\pi}{2}$. For all continuous $f \in L^2(\mathbb{S}^2)$, define the function $\delta(\theta - \frac{\pi}{2})$ by:

$$\int_0^{\pi} \delta(\theta - \frac{\pi}{2}) f(\theta,\phi)\sin\theta d\theta = f(\frac{\pi}{2},\phi),$$

for all $\phi \in [0, 2\pi]$. Such a function can be considered as a point mass at the equatorial line. Then the measure determined by $|Y_{\ell\ell}|^2$ for large ℓ is:

$$\lim_{\ell\to\infty} ||Y_{\ell\ell}|^2 = \frac{1}{2\pi}\delta(\theta - \frac{\pi}{2}).$$

References

[1] Cahill, Kevin, *Physical Mathematics.* Cambridge University Press, 1st Edition, 2013.
[2] Murray, J.D., *Asymptotic Analysis.* Clarendon Press, 1974.
[3] Folland, Gerald B., *Introduction to Partial Differential Equations.* Princeton University Press, 2nd Edition, 1995.
[4] Szego, Gabor, *Orthogonal Polynomials.* American Mathematical Society, Colloquium Publications, Volume 23, 1939.
[5] Szego, Gabor, *Asymptotische Entwicklungen der Jacobischen Polynome.* 1933.