Visualization and Algorithm for Simulations of Electro-Magnetic Field in an Elementary Cell of a Layer of Metamaterials

by

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Contents

Chapter 1

Introduction

1.1 Basic Physics terminology

Physics that we use in the project:

- Visible light is electromagnetic radiation.
- Light have properties of both waves and particles(electromagnetic radiation is emitted and absorbed by photons).
- Electromagnetic radiation has both electric and magnetic field components. They oscillate in phase perpendicular to each other and perpendicular to the direction of energy and wave propagation. Electromagnetic waves are transverse.

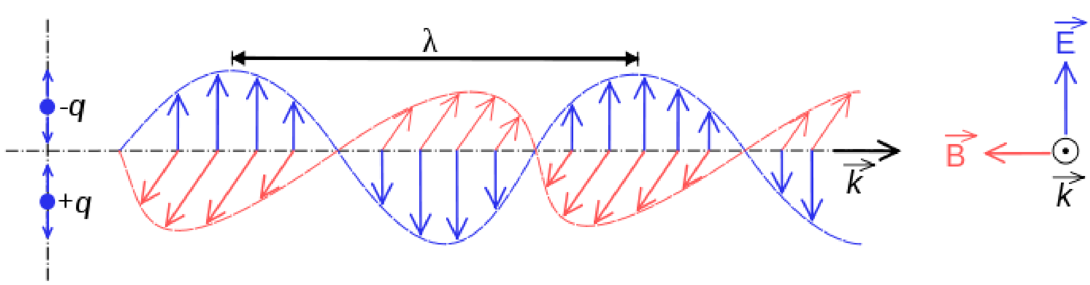
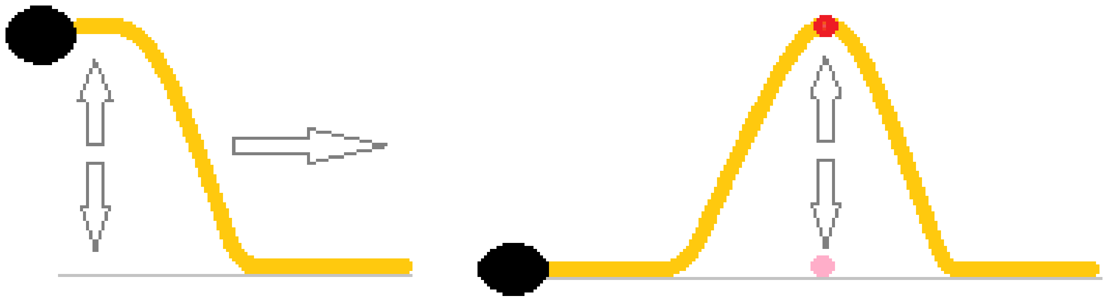
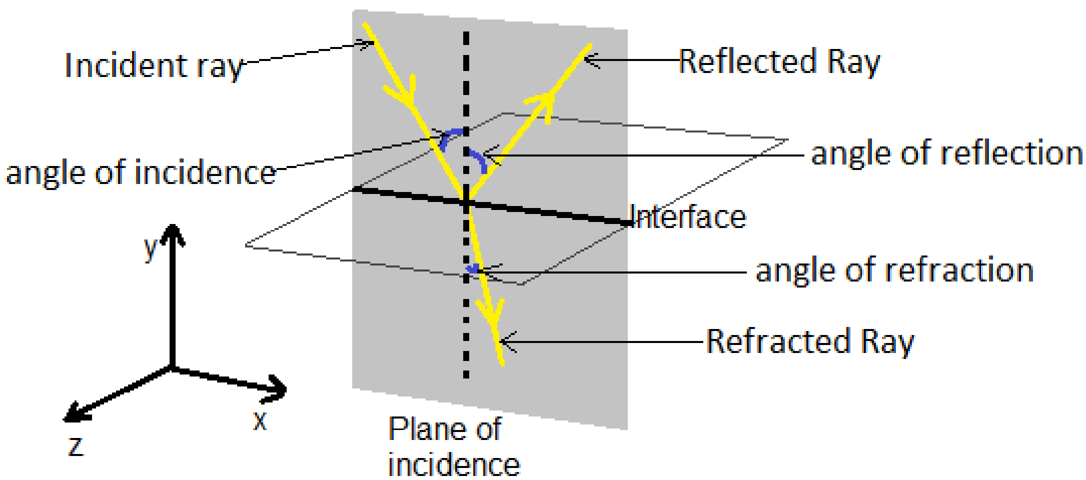


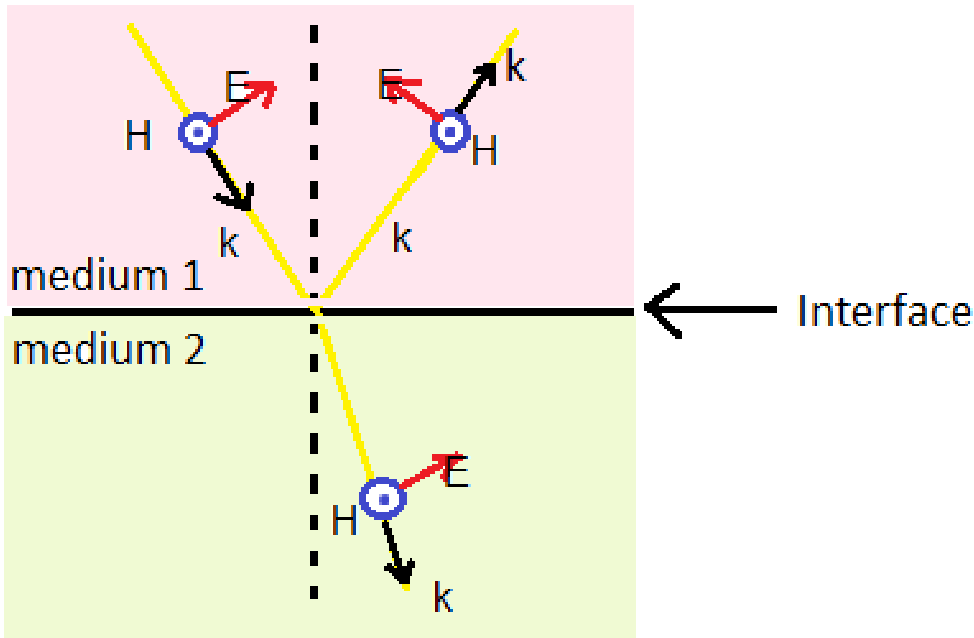
Figure 1.1: Taken from Wikipedia[1].

- Transverse waves the medium is displaced in a direction perpendicular to the motion of the wave.



- Polarization of electromagnetic waves is described by orientation of electric/magnetic field.
- When light/electro-magnetic wave is p-polarized, the only non-zero component of magnetic field is tangential to an interface and the only non-zero components of electric field are parallel to the plane of incidence.





- Boundary conditions - components of electricmagnetic field that are tangent to the interface are continuous across it.

- Electric permittivity of a medium is a measure of how much the medium is permeated by the electric field. We denote it by $\tilde{\epsilon}$. The electric permittivity of free space is always denoted ϵ_0 and equals to $8.8542 * 10^{-12}$.

- Magnetic permeability of a medium is a measure of how capable the medium is to induce/support the magnetic field within it. We it denoted by $\tilde{\mu}$.The permeability of free space is μ_0 and equals to $4\pi * 10^{-7}$.

- Speed of light in a dielectric medium is $v = \sqrt{\tilde{\epsilon}\tilde{\mu}}$.

- Absolute index of refraction is $n = \frac{c}{v} = \sqrt{\frac{\tilde{\epsilon}\tilde{\mu}}{\epsilon_0\mu_0}}$ where c is the speed of light.

- Dielectric constant is $\epsilon = \frac{\tilde{\epsilon}}{\epsilon_0}$ which is dimensionless.

- Relative permeability is $\mu = \frac{\tilde{\mu}}{\mu_0}$ which is dimensionless.

- Maxwell's equations in differential form:

$$\mathrm{curl}\vec{E} = -\mu_0 \partial_t(\mu \vec{H})$$
$$\mathrm{curl}\vec{H} = \epsilon_0 \epsilon \partial_t(\vec{E}).$$

where ϵ and μ are relative permittivity and permeability respectively.

1.2 Introduction to metamaterials

Metamaterials are artificially made materials with nano scale metallic inclusions in a dielectric host medium. Due to this structure, when light, which is considered to be an electro-magnetic wave in this case, interacts with metamaterials, electric and magnetic fields interact resonantly with free electrons of metallic inclusions. One of the results of this electromagnetic interaction of light with metamaterials is negative refraction. The metamaterials with negative refractive index are of special interest because they can be used to create materials with zero refractive index or to create super lens that will resolve objects whose sizes are smaller than the wavelength of light. In this project, we derive the governing equations that describe electric and magnetic fields in metamaterials. Then, we concentrate on numerically solve these equations, so we are able to make numerical simulations of the electric and magnetic fields in metamaterials. Metamaterials are layered structures. Every layer is periodic in its plane, and homogeneous in the vertical direction. We can see it in the following figures.

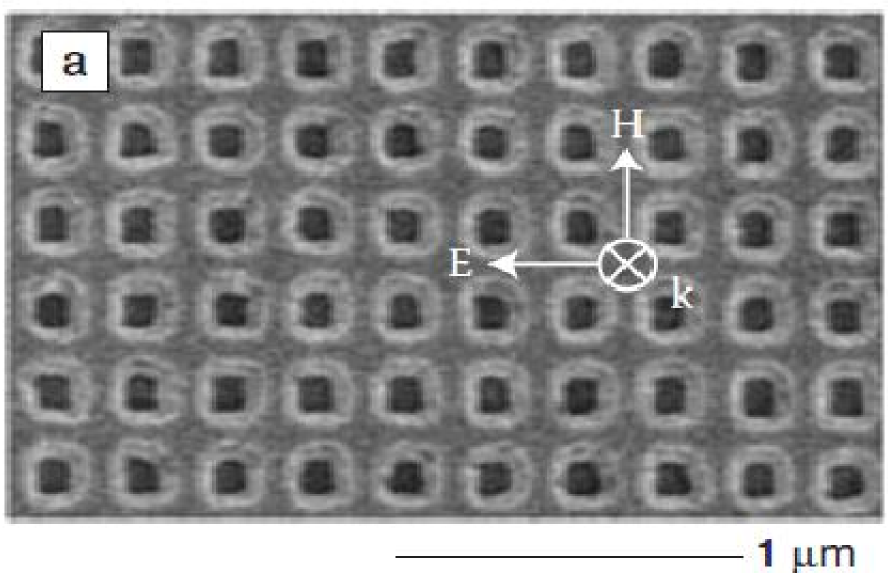


Figure 1.2: A layer of metamaterial. Taken form Chettiar [2].

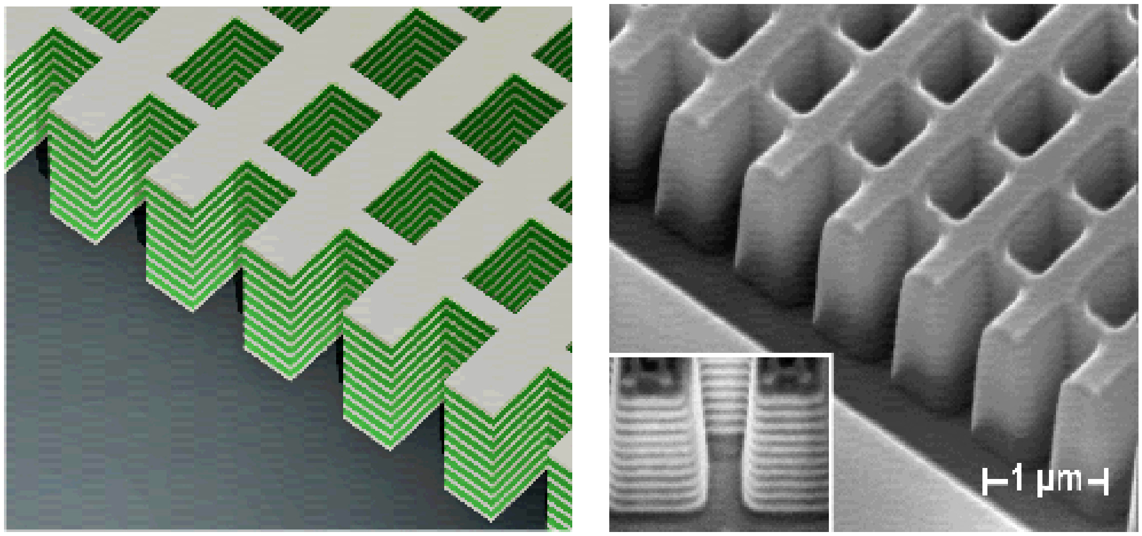


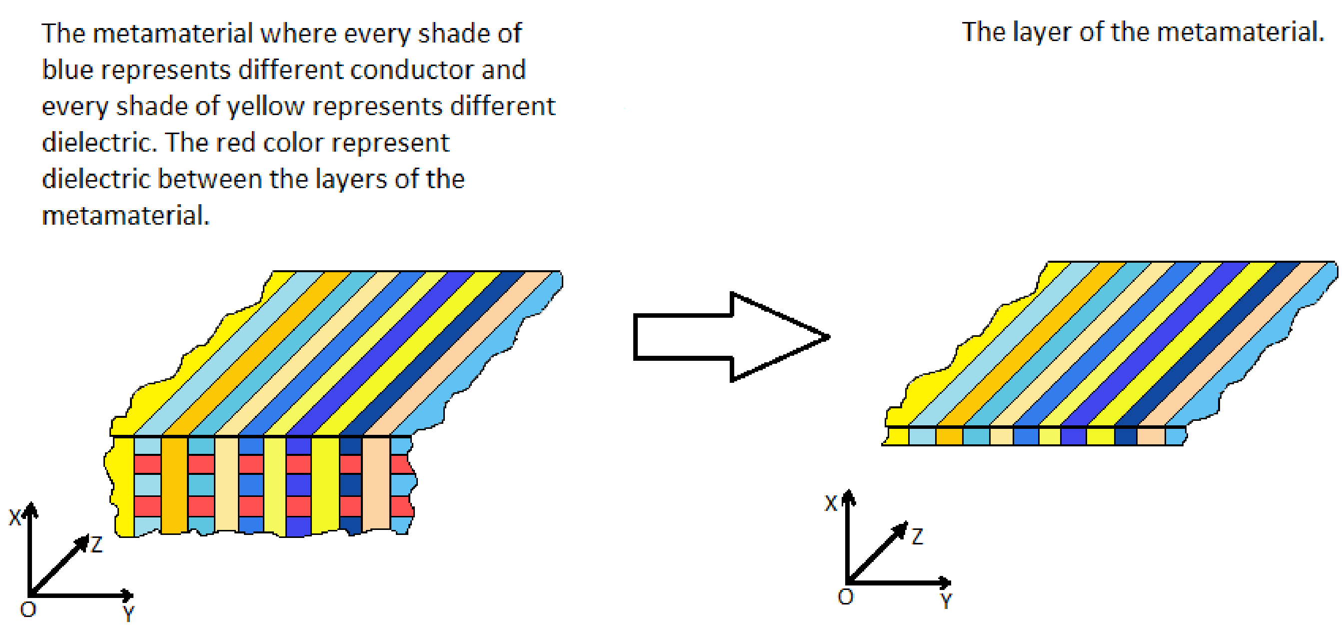
Figure 1.3: Metamaterial with multiple layers. Taken form Valentine [4].

Chapter 2

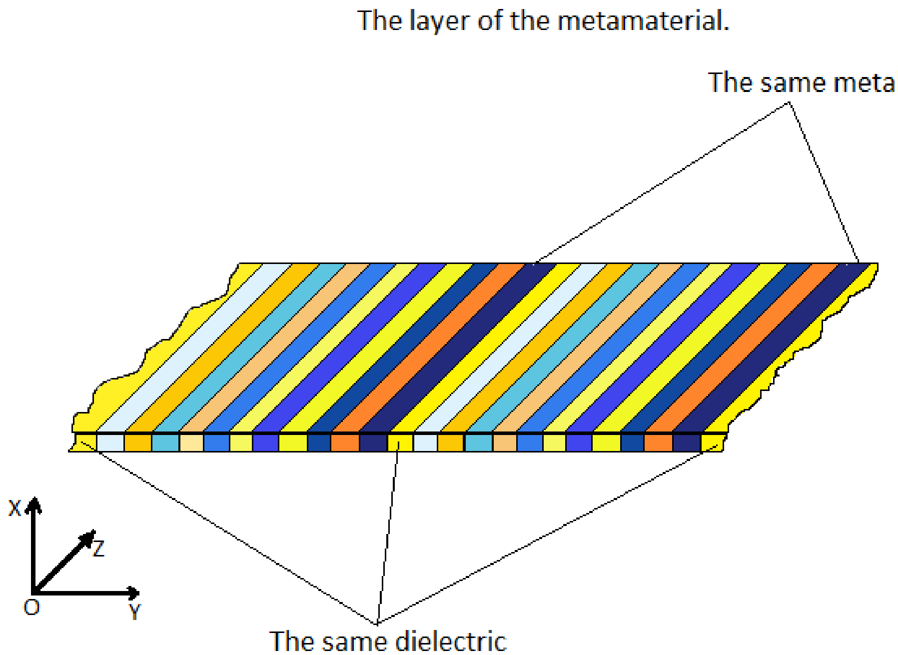
Derivation of governing equations

2.1 Setting Up the Problem

For simplicity, we consider the following metamaterial with the negative index of refraction: every layer is homogeneous in the z and x directions, and has alternation of different conductors with different dielectrics along the y direction.

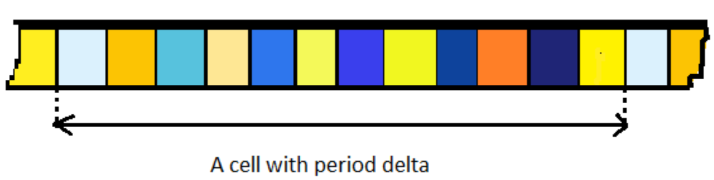


We assume that the metamaterial is periodic in the y direction. This means that every conductor/dielectric that is present in the metamaterial is repeated with some period in the metamaterial.



2.2 The Main Equation

We consider the single layer of the described above metamaterial in $2D$ that is there is no z-direction. A piece of the metamaterial between the same conductors/dielectrics we call the unit cell or structure with period δ.Every conductor/dielectric in the structure is called the element of the structure. We assume that we have s elements of the structure. In other words, s is the number of materials in the cell.Every i-th element of the structure has the boundary y_i with$i+1$-th element where $i = 1, ...s$.



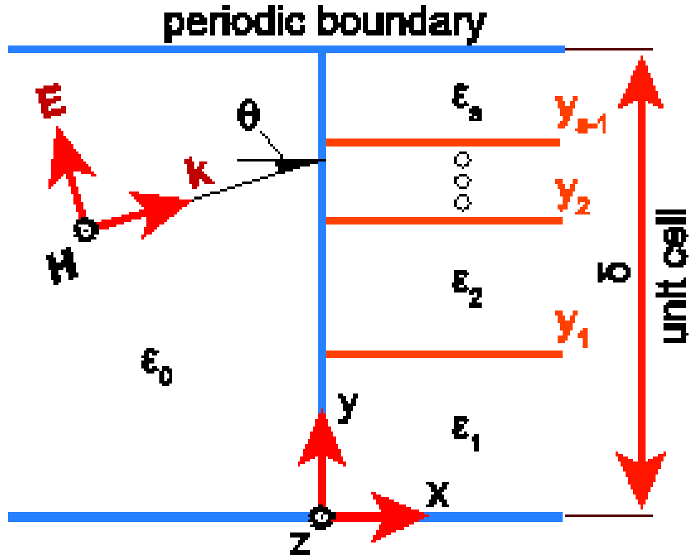


Figure 2.1: $2D$ cross-section of layer of metamaterial, taken from Korotkevich [3].

So, the structure starts at the boundary $y_0 = 0$ and ends at the boundary $y_s = \delta$.Since the structure has different conductors/dielectrics, every i-th element of the structure has dielectric constant ϵ_i where $i = 1, ...s$. The light or electro-magnetic wave is incident to the x-axis at the angle θ.We consider that the light is p-polarized plane wave that is the only non-zero component of magnetic field is perpendicular to the plane of incidence. Then, $\vec{H} = h\hat{z}$ s sufficient to describe the propagation of light through the structure. Thin metal films at optical frequencies can be considered as a material with complex dielectric permittivity and $\mu = 1$. The $2D$ graph of the structure/cell: In every i-th element of the structure where $i = 1, ...s$ we should be looking for a propagating plane wave in the following form:

$$h_i(x, y, t) = a_i^+ e^{(\imath k_x x + \imath k_y^{(i)} y - \imath\omega t)} + a_i^- e^{(-\imath k_x x - \imath k_y^{(i)} y - \imath\omega t)} \tag{2.1}$$

where a_i^+ and a_i^- represent the amplitude of the forward and backward propagating waves/modes in the i-th element, ω is the frequency of oscillations, and $k_y^{(i)}$ is the component of the wave vector in the y direction in the i-th element. Notice that k_x, the component of the wave vector in the x-direction, does not depend on i. The reason is that we are actually trying to find k_x such that it is the same in every element of the structure.

2.3 Normalization of Coordinates

The absolute value of the wave vector in free space is $k_0 = \frac{\omega}{c}$ where c is the speed of light in free space. We see that equation (2.1) can be written as:

$$h_i(x,y,t) = a_i^+ e^{(\imath k_x x \frac{k_0}{k_0} + \imath k_y^{(i)} y \frac{k_0}{k_0} - \imath\omega t)} + a_i^- e^{(-\imath k_x x \frac{k_0}{k_0} - \imath k_y^{(i)} y \frac{k_0}{k_0} - \imath\omega t)}. \tag{2.2}$$

Now, we renormalize δ and coordinates x,y:

$$x \to k_0 x = \tilde{x}$$

$$y \to k_0 y = \tilde{y}$$

$$delta \to k_0 \delta = \tilde{\delta}.$$

We see that new coordinates are dimensionless. Lets drop tilde and use x, y, and δ, but we should remember that now x, y, and δ are renormalized coordinates. As a result we are left with $\frac{k_x}{k_0} := \tilde{k_x}$ and $\frac{k_y^i}{k_0} := \tilde{k_y^i}$.We see that $\tilde{k_x}$ and $\tilde{k_y^i}$ are now dimensionless x and y component of the wave vector respectively. Lets drop tilde and use k_x and k_y^i, but we should remember that k_x and k_y^i are now dimensionless. We remember that the magnitude of the wave vector in the i-th element is:

$$(k^{(i)})^2 = (k_y^{(i)})^2 + (k_x)^2$$

$$k_y^{(i)} = \sqrt{(k^{(i)})^2 - (k_x)^2}$$

$$\frac{k_y^{(i)}}{k_0} = \sqrt{(\frac{k^{(i)}}{k_0})^2 - (\frac{k_x}{k_0})^2}$$

We remember that $\frac{c}{v} = \sqrt{\frac{\tilde{\epsilon}\tilde{\mu}}{\epsilon_0\mu_0}}$ in dielectrics. Since metal inclusions are thin in the metamaterial, then we should consider them as dielectrics. We also remember that $k = \frac{\omega}{v}$ in a medium where v is the speed of light in the medium. Then:

$$\frac{k_y^{(i)}}{k_0} = \sqrt{(\frac{(\frac{\omega}{v_i})^2}{(\frac{\omega}{c})^2} - (\frac{k_x}{k_0})^2}$$

$$\frac{k_y^{(i)}}{k_0} = \sqrt{(\frac{c}{v_i})^2 - (\frac{k_x}{k_0})^2}$$

$$\frac{k_y^{(i)}}{k_0} = \sqrt{(\sqrt{\frac{\tilde{\epsilon}\tilde{\mu}}{\epsilon_0\mu_0}})^2 - (\frac{k_x}{k_0})^2}$$

We remember that: $\mu = \frac{\tilde{\mu_i}}{\mu_0} = 1$ in every element of the structure; $\frac{\tilde{\epsilon_i}}{\epsilon_0} = 1$ is dielectric constant $\epsilon - i$ in the i-th element of the structure where $i = 1, ...s$; $\frac{k_x}{k_0}$ is the dimensionless x component of the wave vector that we called k_x.Then:

$$\frac{k_y^{(i)}}{k_0} = \sqrt{\epsilon_i - (k_x)^2}$$
$$g_i := \imath \frac{k_y^{(i)}}{k_0} = \sqrt{(k_x)^2 - \epsilon_i}$$

We see that g_i is dimensionless. After all of the above manipulations, the equation of the propagating plane wave in every i-th layer where $i = 1, ...s$ is:

$$h_i(x, y, t) = a_i^+ e^{g_i y} e^{\imath k_x x} e^{-\imath\omega t} + a_i^- e^{-g_i y} e^{-\imath k_x x} e^{-\imath\omega t} \tag{2.3}$$

where x, y, k_x, and g_i are dimensionless.

2.4 Coupling Fields of i-th and $i+1$-th elements

For convinience, we rename terms on the right hand side of the equation (2.3): $h_i^+ = a_i^+ e^{g_i y} e^{\imath k_x x} e^{-\imath\omega t}$ and $h_i^- = a_i^- e^{-g_i y} e^{-\imath k_x x} e^{-\imath\omega t}$. Lets derive equations that couple the fields of i-th and $i+1$-th elements of the structure on the i-th boundary. Since the structure is periodic in the y-direction, then i-th boundary occurs at coordinate y_i where $i = 1, ...s$.

2.4.1 Boundary Conditions on the Magnetic Field

We remember that electro-magnetic wave is p-polarized. In other words, the only non-zero component of magnetic field is perpendicular to the plane of incidence that is tangent to the i-th boundary. Boundary conditions imply that magnetic field component that is tangent to the i-th boundary is continuous across it.Then, at the i-th boundary we have the following equation: $(h_i^+ + h_i^-)_{y_i} = (h_{i+1}^+ + h_{i+1}^-)_{y_i}$ where $i = 1, ...s$ and y_i is the point on the i-th boundary at which we evaluate h_i and h_{i+1}.

2.4.2 Maxwells Equations and Boundary Conditions on the Electric Field

ets recall that one of the Maxwells equations is curl$\vec{H} = \epsilon_0\epsilon\partial_t(\vec{E})$. Since electro-magnetic wave is p-polarized, then all non-zero components of electric field are in plane of incidence. This implies that electric filed has non-zero components in the x and y directions respectively. Lets name these components E_x and E_y, then $\vec{E} = E_x\hat{x} + E_y\hat{y}$. We assume that the electric filed is represented by a some kind of a plane wave. Therefore, $\vec{E}$ has the following dependence on t by having $e^{-\imath\omega t}$ term. As a result, the partial derivative of $\vec{E}$ with respect to t is $-\imath\omega\vec{E}$. Since we are going to use curl$\vec{H} = \epsilon_0\epsilon\partial_t(\vec{E})$ in the i-th element of the structure, then this equations is going to look like $\nabla \times h_i\hat{z} = \epsilon_0\epsilon_i\partial_t(\vec{E_i})$ where $\vec{E_i} = E_ix\hat{x} + E_iy\hat{y}$.

(1) $\nabla \times h_i\hat{z} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & h_i \end{pmatrix} = \frac{\partial h_i}{\partial y}\hat{x} - \frac{\partial h_i}{\partial x}\hat{y}$

(2) $\epsilon_0\epsilon_i\frac{\partial\vec{E}}{\partial t} = -\imath\omega\epsilon_0\epsilon_i\vec{E_i} = -\imath\omega\epsilon_0\epsilon_i(E_{ix}\hat{x} + E_{iy}\hat{y})$

(3) $\nabla \times h_i\hat{z} = \epsilon_0\epsilon_i\partial_t(\vec{E_i}) \Rightarrow \frac{\partial h_i}{\partial y}\hat{x} - \frac{\partial h_i}{\partial x}\hat{y} = -\imath\omega\epsilon_0\epsilon_i(E_{ix}\hat{x} + E_{iy}\hat{y})$

(4) $\frac{\partial h_i}{\partial y} = -\imath\omega\epsilon_0\epsilon_iE_{ix}$ and $\frac{\partial h_i}{\partial x} = \imath\omega\epsilon_0\epsilon_iE_{iy}$

(5) Since light is p-polarized, then components of electric field that are parallel to the i-th boundary are continuous across it. We see that only E_x omponent of the electric field is tangent to the i-th boundary then E_x is continuous across it. This implies that $(E_{ix})_{y_i} = (E_{(i+1)x})_{y_i}$ where $i = 1, ...s$.

(6) Since $(E_{ix})_{y_i} = (E_{(i+1)x})_{y_i}$, then $\frac{1}{-\imath\omega\epsilon_0\epsilon_i}(\frac{\partial h_i}{\partial y})_{y_i} = \frac{1}{-\imath\omega\epsilon_0\epsilon_{i+1}}(\frac{\partial h_{i+1}}{\partial y})_{y_i} \Rightarrow \frac{1}{\epsilon_i}(\frac{\partial h_i}{\partial y})_{y_i} = \frac{1}{\epsilon_{i+1}}(\frac{\partial h_{i+1}}{\partial y})_{y_i}$

(7) Let $\gamma_i = \frac{1}{\epsilon_i}$, then $\gamma_i(\frac{\partial h_i}{\partial y})_{y_i} = \gamma_{i+1}(\frac{\partial h_{i+1}}{\partial y})_{y_i}$

(8) $\frac{\partial h_i}{\partial y} = \frac{\partial}{\partial y}(h_i^+ + h_i^-) = g_i(h_i^+ - h_i^-)$

(9) So, we have that: $\gamma_i(\frac{\partial h_i}{\partial y})_{y_i} = \gamma_{i+1}(\frac{\partial h_{i+1}}{\partial y})_{y_i} \Rightarrow \gamma_ig_i(h_i^+ - h_i^-)_{y_i} = \gamma_{i+1}g_{i+1}(h_{i+1}^+ - h_{i+1}^-)_{y_i}$ where $i = 1, ...s$ and y_i is the point on the i-th boundary at which we evaluate h_i and h_i.

From sections 2.4.1 and 2.4.2, we have two equations that couple the fields of i-th and $i+1$-th elements of the structure on the i-th boundary where $i = 1, ...s$:

$$(h_i^+ + h_i^-)_{y_i} = (h_{i+1}^+ + h_{i+1}^-)_{y_i} \tag{2.4}$$

$$\gamma_i g_i (h_i^+ - h_i^-)_{y_i} = \gamma_{i+1} g_{i+1} (h_{i+1}^+ - h_{i+1}^-)_{y_i} \tag{2.5}$$

where $\gamma_i = \frac{1}{\epsilon_i}$ and $i = 1, ...s$.

2.4.3 Bloch Periodic Boundary Conditions

Lets get equations for h_i on the periodic boundaries. We remember that the structure starts at the boundary $y_0 = 0$ and ends at the boundary $y_s = \delta$, , and we have s elements of the structure. Since we have $s+1$ boundaries and general case of incidence with angle θ to the x-direction, then Bloch periodic boundary conditions imply that:

$$h_{i+s}(x, y_{j+s}; t)e^{-\alpha\delta} = h_i(x, y_j; t) \tag{2.6}$$

where $\alpha = \imath sin\theta$ and y_j is some y coordinate within the i-th elemnet. In terms of h_i^+ and h_i^-, the equation (2.6) is represented as:

$$(h_{i+s}^+ + h_{i+s}^-)e^{-\alpha\delta} = h_i^+ + h_i^- \tag{2.7}$$

where $i = 1, ...s$. Since we assumed that $\vec{E}$ is in some form of a plane wave, then it is a periodic function. By applying Bloch periodic boundary conditions, we have that $E_{(i+s)x}(x, y_{j+s}; t)e^{-\alpha\delta} = E_{ix}(x, y_j; t)$ where E_{ix} is the value of the x-component of the electric field at the i-th boundary and y_j is some y coordinate within the i-th element. In section 2.4.2, we have got that $\frac{\partial h_i}{\partial y} = -\imath\omega\epsilon_0\epsilon_i E_{ix}$. We have:

$$\begin{aligned}
E_{(i+s)x}(x, y_{j+s}; t)e^{-\alpha\delta} &= E_{ix}(x, y_j; t) \\
\frac{1}{-\imath\omega\epsilon_0\epsilon_i}\frac{\partial h_i}{\partial y} &= \frac{1}{-\imath\omega\epsilon_0\epsilon_{i+s}}\frac{\partial h_{i+s}}{\partial y}e^{-\alpha\delta} \\
\gamma_i\frac{\partial h_i}{\partial y} &= \gamma_{i+s}\frac{\partial h_{i+s}}{\partial y}e^{-\alpha\delta}
\end{aligned}$$

So, we have that:

$$\gamma_i g_i \frac{\partial h_i}{\partial y} = \gamma_{i+s} g_{i+s} \frac{\partial h_{i+s}}{\partial y} e^{-\alpha\delta} \tag{2.8}$$

where $i = 1, ...s$.

2.5 Matrix-Vector Form

Since factor $e^{-\imath\omega t}$ is present at the left and right hand side of the equations (2.4) and (2.5), we divide the left and right hand side these equations by $e^{-\imath\omega t}$. Now, let's rewrite the equations (c) and (2.5) in matrix-vector form:

$$\mathbf{m}_i \mathbf{d}_{(i,i)} \vec{\mathbf{a}_i} = \mathbf{m}_{i+1} \mathbf{d}_{(i+1,i)} \vec{\mathbf{a}_{i+1}} \tag{2.9}$$

where we introduce

$$\vec{\mathbf{a}_i} = \begin{pmatrix} a_i^+ e^{\imath k_x x} \\ a_i^- e^{-\imath k_x x} \end{pmatrix}, \quad \mathbf{m}_i = \begin{pmatrix} 1 & 1 \\ g_i\gamma_i & -g_i\gamma_i \end{pmatrix}, \quad \mathbf{d}_{(i,j)} = \begin{pmatrix} \mathrm{e}^{g_i y_j} & 0 \\ 0 & \mathrm{e}^{-g_i y_j} \end{pmatrix}. \tag{2.10}$$

for $i, j = 1...s$. Notice that subscript (i, j) of the matrix $\mathbf{d}$ is used to show the dependence of g and y on i,j. Now, we want to get the expression for $\vec{\mathbf{a}_i}$. So, we need to multiply the both side of the equation (2.9) by $\mathbf{d}_{(i,i)}^{-1}\mathbf{m}_i^{-1}$. Then:

Solving (**??**) for $\vec{a}_i$ one can get recurrent equations on $\vec{a}_i$:

$$\vec{\mathbf{a}_i} = \mathbf{d}_{(i,i)}^{-1} \mathbf{m}_i^{-1} \mathbf{m}_{i+1} \mathbf{d}_{(i+1,i)} \vec{\mathbf{a}_{i+1}} \tag{2.11}$$

where $\mathbf{m}_i^{-1} = \frac{1}{2g_i\gamma_i} \begin{pmatrix} g_i\gamma_i & 1 \\ g_i\gamma_i & -1 \end{pmatrix}$ and $\mathbf{d}_{(i,j)}^{-1} = \begin{pmatrix} (\mathrm{e}^{-g_i y_j} & 0 \\ 0 & \mathrm{e}^{g_i y_j} \end{pmatrix}$. We see that for equation (2.11) to exist, $\mathbf{m}_i$ and $\mathbf{d}_{(i,i)}$ should be nonsingular for all i. We notice that $\mathbf{d}_{(i,i)}$ never equals to zero matrix for all i. Since $\mathbf{d}_{(i,i)}$ is diagonal matrix and never equals to zero matrix, then $\mathbf{d}_{(i,i)}$ is nonsingular matrix for all i. We remember that $\gamma_i = \frac{1}{\epsilon_i}$ where ϵ_i is dielectric constant. So, γ_i never equals to zero. Then, for $\mathbf{m}_i$ to be nonsingular, g_i should not be equal to zero for all i. Let $\mathbf{t}_i = \mathbf{d}_{(i,i)}^{-1} \mathbf{m}_i^{-1} \mathbf{m}_{i+1} \mathbf{d}_{(i+1,i)} \vec{\mathbf{a}_{i+1}}$ and by using equation (2.11) we have that:

$$\vec{\mathbf{a}_i} = \mathbf{t}_i \vec{\mathbf{a}_{i+1}} \tag{2.12}$$

2.6 Matrix-Vector Form and Bloch Periodic Boundary Conditions

By applying equations (2.4) and (2.5) from section 2.4.1 and 2.4.2 at the s boundary, $i = s$. We have:

$$(h_s^+ + h_s^-)_{y_s} = (h_{s+1}^+ + h_{s+1}^-)_{y_s} \tag{2.13}$$

$$\gamma_s g_s (h_s^+ - h_s^-)_{y_s} = \gamma_{s+1} g_{s+1} (h_{s+1}^+ - h_{s+1}^-)_{y_s} \tag{2.14}$$

We apply equations (2.7) and (2.8) from section 2.4.3 to the right hand sides of equations (2.13) and (2.14). Since h_{i+s} is evaluated at y_s, then h_1 should be evaluated at y_0. In addition, since the structure is periodic ($\epsilon_i = \epsilon_{i+s}$), then $\gamma_{s+1} = \gamma_1$ and $g_{s+1} = g_1$.

$$(h_{s+1}^+ + h_{s+1}^-)_{y_s} e^{-\alpha\delta} = (h_1^+ + h_1^-)_{y_0}$$

$$\gamma_{s+1} g_{s+1} (h_{s+1}^+ - h_{s+1}^-)_{y_s} e^{-\alpha\delta} = \gamma_1 g_1 (h_1^+ - h_1^-)_{y_0}$$

The two above equations in the matrix-vector form:

$$\begin{pmatrix} 1 & 1 \\ \gamma_1 g_1 & -\gamma_1 g_1 \end{pmatrix} \begin{pmatrix} e^{g_1 y_0} & 0 \\ 0 & e^{-g_1 y_0} \end{pmatrix} \begin{pmatrix} a_i^+ e^{\imath k_x x} \\ a_i^- e^{-\imath k_x x} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \gamma_s g_s & -\gamma_s g_s \end{pmatrix} \begin{pmatrix} e^{g_s y_s} & 0 \\ 0 & e^{-g_s y_s} \end{pmatrix} \begin{pmatrix} a_s^+ e^{\imath k_x x} \\ a_s^- e^{-\imath k_x x} \end{pmatrix} e^{-\alpha\delta}$$

$$\mathbf{m}_1 \mathbf{d}_{(1,0)} \vec{\mathbf{a}_1} = \mathbf{m}_s \mathbf{d}_{(s,s)} e^{-\alpha\delta} \vec{\mathbf{a}_s}$$

$$\vec{\mathbf{a}_s} = \mathbf{d}_{(s,s)}^{-1} \mathbf{m}_s^{-1} \mathbf{m}_1 \mathbf{d}_{(1,0)} e^{\alpha\delta} \vec{\mathbf{a}_1}$$

Recall that $y_0 = 0$, then $\mathbf{d}_{(1,0)} = \mathbf{I}$. So, we have that

$$\vec{\mathbf{a}_s} = \mathbf{t}_s \vec{\mathbf{a}_1} \tag{2.15}$$

where $\mathbf{t}_s = \mathbf{d}_{(s,s)}^{-1} \mathbf{m}_s^{-1} \mathbf{m}_1 e^{\alpha\delta}$

2.7 Eigenvalue Problem

Now, we can use equations (2.12) and (2.15) to derive equations for $\vec{\mathbf{a}}_1$,$\vec{\mathbf{a}}_2$,$\vec{\mathbf{a}}_3$,...,$\vec{\mathbf{a}}_s$:

(1) $\vec{\mathbf{a}}_1 = \mathbf{t}_1 \vec{\mathbf{a}}_2$

(2) $\vec{\mathbf{a}}_2 = \mathbf{t}_2 \vec{\mathbf{a}}_3 \Rightarrow \vec{\mathbf{a}}_1 = \mathbf{t}_1 \vec{\mathbf{a}}_2 = \mathbf{t}_1 \mathbf{t}_2 \vec{\mathbf{a}}_3$

(3) $\vec{\mathbf{a}}_3 = \mathbf{t}_3\vec{\mathbf{a}}_4 \Rightarrow \vec{\mathbf{a}}_1 = \mathbf{t}_1\vec{\mathbf{a}}_2 = \mathbf{t}_1\mathbf{t}_2\vec{\mathbf{a}}_3 = \mathbf{t}_1\mathbf{t}_2\mathbf{t}_3\vec{\mathbf{a}}_4$

$\vdots$

(i) $\vec{\mathbf{a}}_i = \mathbf{t}_i\vec{\mathbf{a}}_{i+1} \Rightarrow \mathbf{t}_{i-1}\vec{\mathbf{a}}_i = \mathbf{t}_{i-1}\mathbf{t}_i\vec{\mathbf{a}}_{i+1} \Rightarrow ... \Rightarrow \vec{\mathbf{a}}_1 = \mathbf{t}_1\vec{\mathbf{a}}_2 = \mathbf{t}_1\mathbf{t}_2\vec{\mathbf{a}}_3 = ... = \mathbf{t}_1\mathbf{t}_2...\mathbf{t}_i\vec{\mathbf{a}}_{i+1}$ $\vdots$

(s) $\vec{\mathbf{a}}_s = \mathbf{t}_s\vec{\mathbf{a}}_1 \Rightarrow \vec{\mathbf{a}}_1 = \mathbf{t}_1\vec{\mathbf{a}}_2 = \mathbf{t}_1\mathbf{t}_2\vec{\mathbf{a}}_3 = ... = \mathbf{t}_1\mathbf{t}_2...\mathbf{t}_s\vec{\mathbf{a}}_1$

The last equation from the part s can be rewriten as:

$$\vec{\mathbf{a}}_1 = \mathbf{t}\vec{\mathbf{a}}_1 \tag{2.16}$$

where $\mathbf{t} = \prod_{i=1}^{s} \mathbf{t}_i$ with $\mathbf{t}_i = \mathbf{d}_{(i,i)}^{-1}\mathbf{m}_i^{-1}\mathbf{m}_{i+1}\mathbf{d}_{(i+1,i)}\vec{\mathbf{a}_{i+1}}$ for $i = 1, ..., s-1$ and $\mathbf{t}_s = \mathbf{d}_{(s,s)}^{-1}\mathbf{m}_s^{-1}\mathbf{m}_1 e^{\alpha\delta}$. We see that equation (2.16) is the eigenvalue problem for $\vec{\mathbf{a}}_1$ with eigenvalue 1: $(\mathbf{I} - \mathbf{t})\vec{\mathbf{a}}_1 = \vec{0}$. For the above system to have non-trivial solution, the determinant of $(\mathbf{I} - \mathbf{t})$ should be equal to 0. So, by using the formula for the characteristic equation for determinants of $2x2$ matrices and $\lambda = 1$, we have that:

$$1 - Tr(\mathbf{t}) + det(\mathbf{t}) = 0 \tag{2.17}$$

2.8 Properties of Determinant

By product property of determinants, we have that:

$$det(\mathbf{t}) = det(\mathbf{t}_1)det(\mathbf{t}_2)det(\mathbf{t}_3)...det(\mathbf{t}_s).$$

Let's calculate the dtereminant of matrix $det(\mathbf{t}_i)$:

$$det(\mathbf{t}_i) = det(\mathbf{d}_{(i,i)}^{-1})det(\mathbf{m}_i^{-1})det(\mathbf{m}_{i+1})det(\mathbf{d}_{(i+1,i)}) = \frac{1}{\gamma_i g_i}\gamma_{i+1}g_{i+1}.$$

The determinant of matrix $\mathbf{t}_s$:

$$det(\mathbf{t}_s) = e^{2\alpha\delta}det(\mathbf{d}_{(s,s)}^{-1})det(\mathbf{m}_s^{-1})det(\mathbf{m}_1) = e^{2\alpha\delta}\frac{\gamma_1 g_1}{\gamma_s g_s}.$$

By using the above results, we derive that $det(\mathbf{t}) = e^{2\alpha\delta}$ and substitute it into the equation (2.17):

$$1 - Tr(\mathbf{t}) = 0$$

$$Tr(\mathbf{t}) = 1 + e^{2\alpha\delta} = \frac{2e^{\alpha\delta}(e^{-\alpha\delta}+e^{\alpha\delta})}{2} = 2e^{\alpha\delta}cosh\alpha\delta$$

$$Tr(\mathbf{t}) = 2e^{\alpha\delta}cosh\alpha\delta \tag{2.18}$$

Let's recall the following facts:

(1) $\mathbf{t} = \prod_{i=1}^{s} \mathbf{t}_i$ with $\mathbf{t}_i = \mathbf{d}^{-1}_{(i,i)}\mathbf{m}_i^{-1}\mathbf{m}_{i+1}\mathbf{d}_{(i+1,i)}\mathbf{a}_{\vec{i+1}}$ for $i = 1, ..., s-1$ and $\mathbf{t}_s = \mathbf{d}^{-1}_{(s,s)}\mathbf{m}_s^{-1}\mathbf{m}_1 e^{\alpha\delta}$.

(2) $\mathbf{m}_i = \begin{pmatrix} 1 & 1 \\ g_i\gamma_i & -g_i\gamma_i \end{pmatrix}$, and $\mathbf{d}_{(i,j)} = \begin{pmatrix} \mathrm{e}^{g_i y_j} & 0 \\ 0 & \mathrm{e}^{-g_i y_j} \end{pmatrix}$, with $h\gamma_i = \frac{1}{\epsilon_i}$ and $g_i = \sqrt{(k_x)^2 - \epsilon_i}$

(3) $\alpha = sin\theta$ where θ is the incidence angle to the x-direction of light/electro-magnetic wave. This angle is given to us.

(4) δ is the period that is given to us.

So, we see that only unknown variable in the equation (2.18) is k_x. After we solve the non-linear equation (2.18) for k_x, we should evaluate $g_i = \sqrt{(k_x)^2 - \epsilon_i}$ in every element of the structure. However, the main task is to evaluate the amplitudes of the forward and backward propagating waves/modes in the i-th element. So, the values of k_x has been evaluated by Professor Alexander O. Korotkevich by using Lehmer-Schur algorithm based on the Argument Principle.

2.9 Argument Principle

If a function $f(z)$ is meromorphic (analytic except for poles) in the domain interior to a positively oriented contour Γ; analytic and nonzero on Γ; then

$$\frac{1}{2\pi\imath}\int\limits_{\Gamma}\frac{f'(z)}{f(z)}\mathrm{d}z = \frac{1}{2\pi\imath}\int\limits_{\Gamma}\ln'(f(z))\mathrm{d}z = (Z - P).$$

where Z – is number of zeros of the function in the domain and P – number of poles.

2.10 Amplitudes

After we found k_x, we need to calculate a_i^+ and a_i^- that is the amplitudes of the forward and backward propagating waves/modes in the i-th element.

2.10.1 Singularity Condition

We found k_x from the condition that the matrix $(\mathbf{I}-\mathbf{t})$ is singular. This implies that $\vec{\mathbf{a}}_1$ where $(\mathbf{I}-\mathbf{t})\vec{\mathbf{a}}_1 = \vec{0}$ has infinitely many non-trivial solutions. Let $(\mathbf{I}-\mathbf{t}) =: \mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. Let's use the following method. Since $\mathbf{A}$ is singular, then rows of the matrix $\mathbf{A}$ are linearly dependent. This implies that $\exists \quad c \in \mathbb{C}$ such that $A_{21} = cA_{11}$ and $A_{22} = cA_{12} \Longrightarrow \mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ cA_{11} & cA_{12} \end{pmatrix}$.

Lets multiply the first row by $-\alpha$ and add it to the second row.Then,$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix} \Rightarrow$

$\begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix} \vec{\mathbf{a}}_1 = 0 \Rightarrow \vec{\mathbf{a}}_1 = \begin{pmatrix} A_{12} \\ -A_{11} \end{pmatrix}$

We apply the same method to the first row. Then, by applying the method to the second row the solutions $\vec{\mathbf{a}}_1 = c_1 \begin{pmatrix} A_{12} \\ -A_{11} \end{pmatrix}$ and $\vec{\mathbf{a}}_1 = c_2 \begin{pmatrix} A_{22} \\ -A_{21} \end{pmatrix}$ where $c_1, c_2 \in \mathbb{C}$

2.10.2 Matrix Manipulations

Lets recall that $\vec{\mathbf{a}_i} = \begin{pmatrix} a_i^+ e^{\imath k_x x} \\ a_i^- e^{-\imath k_x x} \end{pmatrix}$, then $\vec{\mathbf{a}_1} = \begin{pmatrix} a_1^+ e^{\imath k_x x} \\ a_1^- e^{-\imath k_x x} \end{pmatrix}$. We see that we can rewrite $\vec{\mathbf{a}_1}$ in the following form: $\vec{\mathbf{a}_1} = \begin{pmatrix} a_1^+ e^{\imath k_x x} \\ a_1^- e^{-\imath k_x x} \end{pmatrix} = \begin{pmatrix} e^{\imath k_x x} & 0 \\ 0 & e^{-\imath k_x x} \end{pmatrix} \begin{pmatrix} a_1^+ \\ a_1^- \end{pmatrix}$. Let's substitute $\vec{\mathbf{a}_1} = \begin{pmatrix} e^{\imath k_x x} & 0 \\ 0 & e^{-\imath k_x x} \end{pmatrix} \begin{pmatrix} a_1^+ \\ a_1^- \end{pmatrix}$ for example into $\vec{\mathbf{a}}_1 = c_1 \begin{pmatrix} A_{12} \\ -A_{11} \end{pmatrix}$:

$$\begin{pmatrix} e^{\imath k_x x} & 0 \\ 0 & e^{-\imath k_x x} \end{pmatrix} \begin{pmatrix} a_1^+ \\ a_1^- \end{pmatrix} = c_1 \begin{pmatrix} A_{12} \\ -A_{11} \end{pmatrix}$$

$$\begin{pmatrix} a_1^+ \\ a_1^- \end{pmatrix} = c_1 \begin{pmatrix} e^{-\imath k_x x} & 0 \\ 0 & e^{\imath k_x x} \end{pmatrix} \begin{pmatrix} A_{12} \\ -A_{11} \end{pmatrix}$$

We choose $c_1 = e^{-ik_x}$.

After we calculate $\vec{\mathbf{a}}_1$, we are going to use $\vec{\mathbf{a}}_s = \mathbf{t}_s \vec{\mathbf{a}}_1$ to find $\vec{\mathbf{a}}_s$ and $\vec{\mathbf{a}}_i = \mathbf{t}_i \vec{\mathbf{a}}_{i+1}$ to calculate all other amplitudes $i = 2, ..., s-1$. After we found k_x, g_i, and $\vec{\mathbf{a}}_i$, we are finally able to reconstruct the magnetic filed at the every point of the cell. Since we know $\vec{H}$ at every point of the cell. Then by using Maxwell's equation $\nabla \times \vec{H} = \epsilon_0 \epsilon \partial_t(\vec{E})$, we are able to find the electric field at every point in the structure.

Chapter 3

Numerical Simulations.

We are using MATLAB to calculate the values of the forward and backward propagating waves/modes in the i-th element. After we evalute a_i^+ and a_i^- in every i-th element, we use Maxwell's equation $\nabla \times \vec{H} = \epsilon_0 \epsilon \partial_t(\vec{E})$ to calculate the magnetic and electric fields.

3.1 Initial Data for the Case of Two Elements

We consider the structure with two elements that is s=2. We are given the following tables:

	The Width of the Element (nm)	Dielectric Constant
Element 1	5.0	-26.0790 + 0.8819i
Element 2	45.0	2.7224 - 0.0296i

Number of the k_x	Dimensionless Value
1	-1.695679238387135e-01 -2.956935340797049e+00
2	4.421583361978319e-03 -1.648402109808226e+01
3	3.508479947541616e-03 -9.057013389637862e+01
4	2.239931603186373e-02 -5.022221535165307e+01
5	1.052240428479498e-02 -3.326157067505196e+01
6	-2.239931603186461e-02 5.022221535165307e+01

Number of the k_x	Dimensionless Value
7	2.431188818640963e+00 -7.432525696627290e+01
8	-3.012814624814569e+00 -1.106218640105945e-01
9	2.174406378391880e-02 9.771154602270143e+01
10	-4.421583361978352e-03 1.648402109808226e+01
11	-1.052240428479628e-02 3.326157067505196e+01
12	1.516021336386181e-01 7.702337370307291e+01
13	9.556865697890394e-03 4.064610767458256e+01
14	-2.431188818640970e+00 7.432525696627295e+01
15	3.035552044654712e-03 5.738486460259395e+01
16	-9.556865697890387e-03 -4.064610767458256e+0
17	-1.461185840105554e-01 7.017733929509339e+01
18	3.012814624814563e+00 1.106218640105944e-01
19	2.132085694979038e-02 2.366851262791773e+01
20	1.461185840105556e-01 -7.017733929509339e+01
21	-3.035552044656346e-03 -5.738486460259395e+01
22	1.695679238387134e-01 2.956935340797049e+00
23	2.431295481128431e+00 7.462001939515700e+01
24	-1.516021336386175e-01 -7.702337370307293e+01
25	-2.174406378391873e-02 -9.771154602270143e+01
26	-2.132085694979038e-02 -2.366851262791773e+01
27	-2.431295481128424e+00 -7.462001939515702e+01
28	-3.508479947541663e-03 9.057013389637862e+01

3.2 Graphs

The following graphs represent the magnetic and electric fields for the firts 4 values of k_x (If reader is interested in the rest of the graphs, we attach them in the Appendix A.). We observe that the condition of the tangetntial component of the magnetic field being continuous over the boundary is satisfied. Since only the x component of the electric fields is conserved over the boundary, we cannot excpect any particular behaviour from the electric field over the boundary. We are attaching the first 4 x components of the electric fields to show that they are conserved over the boundary. We see that the Bloch periodic boundary conditions are satisfied by both fields.

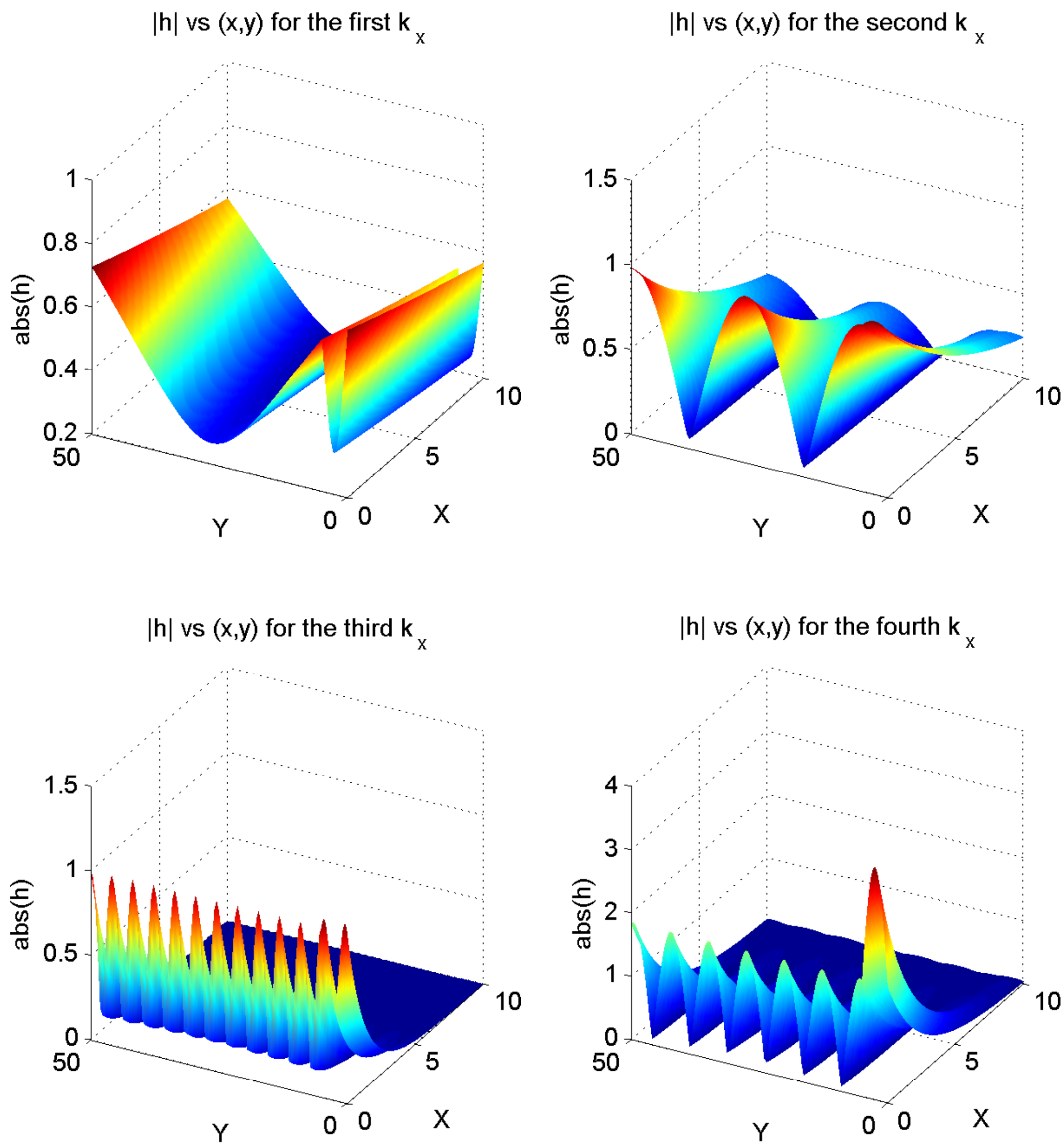


Figure 3.1: Magnetic Fields for the first 4 k_x

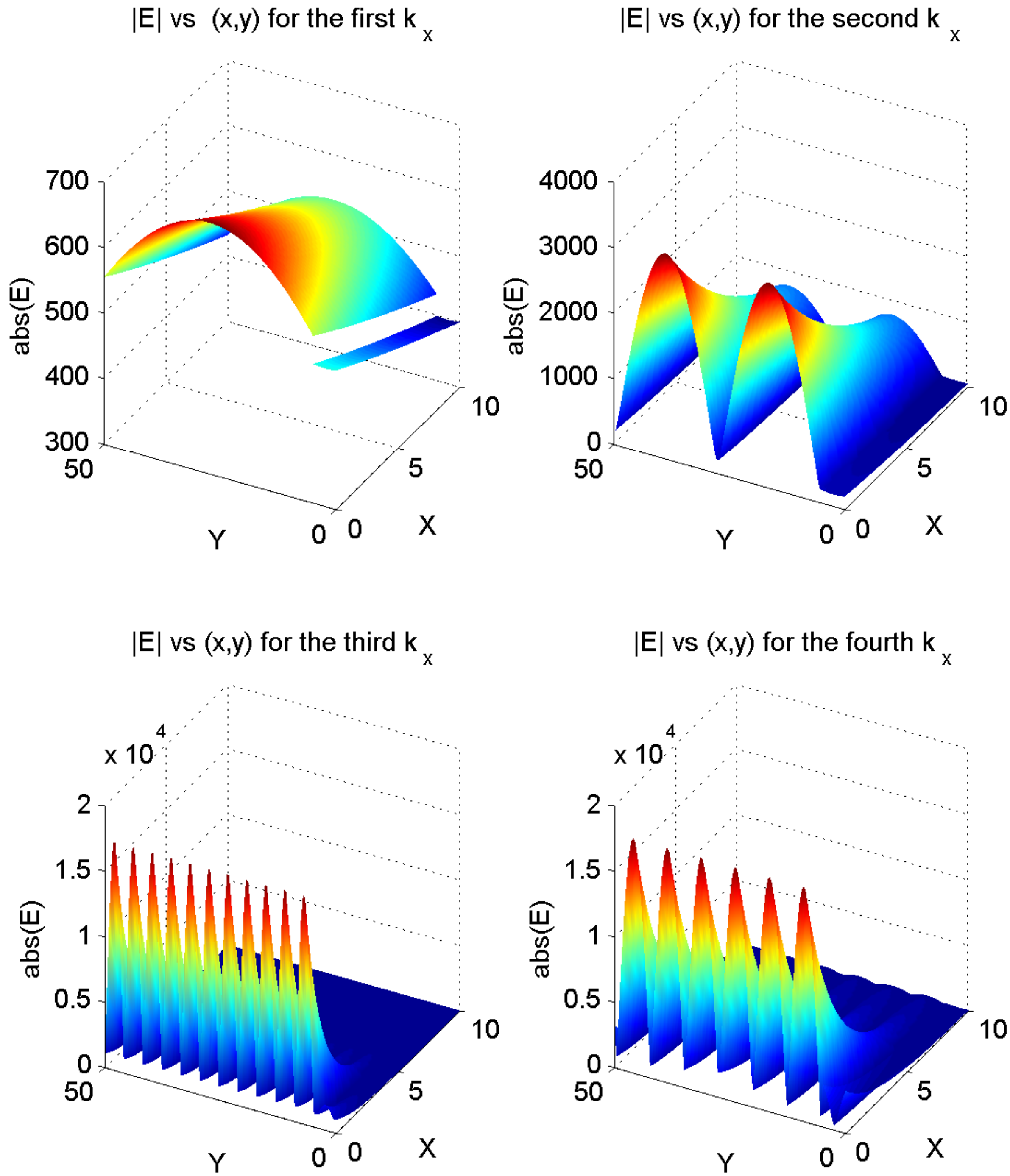


Figure 3.2: Electric Fields for the first 4 k_x

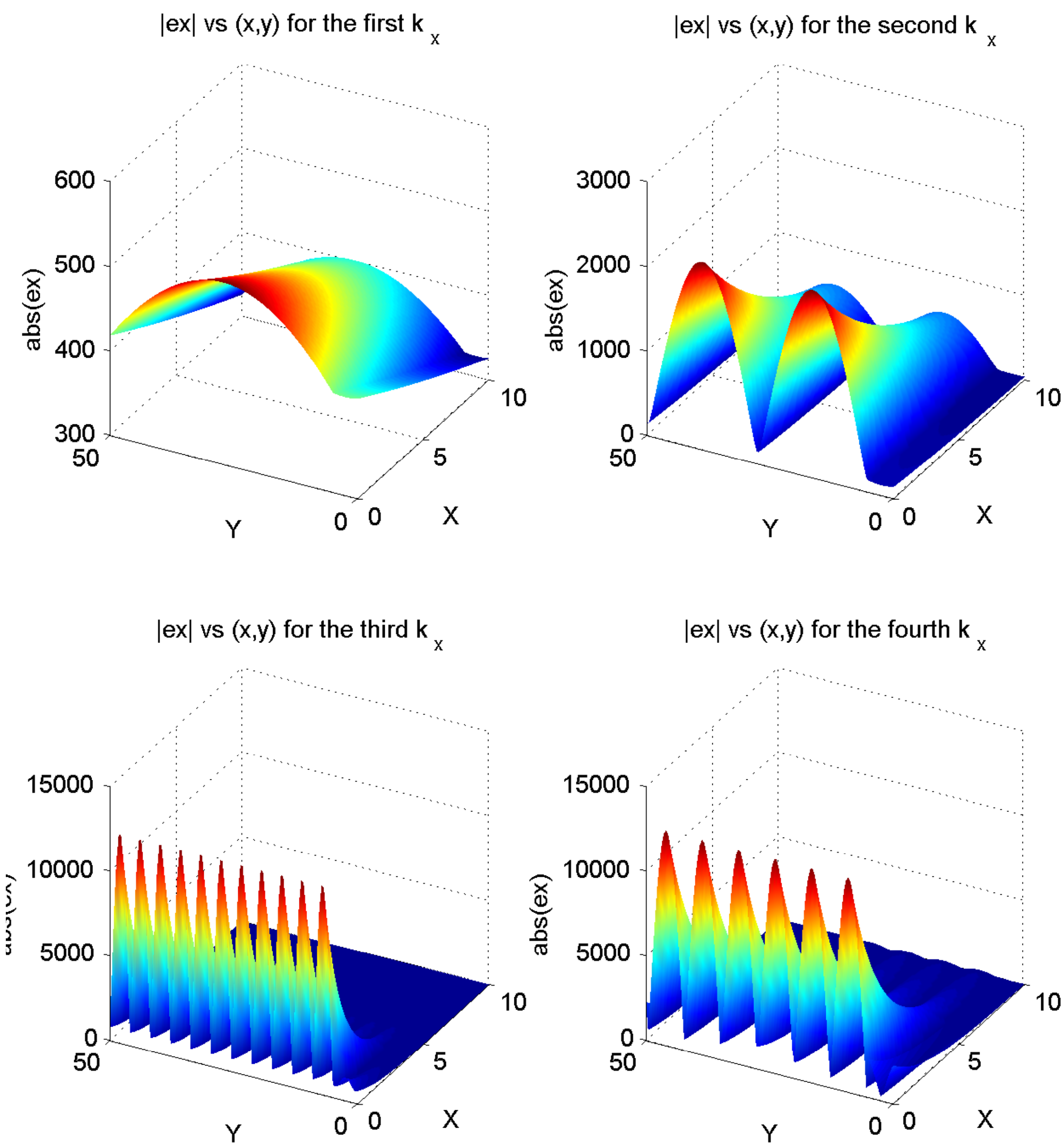


Figure 3.3: x components of electric Fields for the first 4 k_x

Chapter 4

Conclusion

The numerical simulations show that our method of setting up and deriving the governing equations for the magnetic and electric fields indeed works. As a result, we belive that this method can be applyed to any type of periodic metamaterials with negative index of refraction. The use of this method produces non-experimental approach to finding and studying the behaviour of magnetic and electric fields in metamaterials with negative index of refraction. In future works, we would like to consider the $3D$ case of the layer of the metamaterial with negative index of refraction. In addition, we want to expand the amount layers from 1 to n.

Chapter 5

Appendix A

The magnetic fields for different k_x:

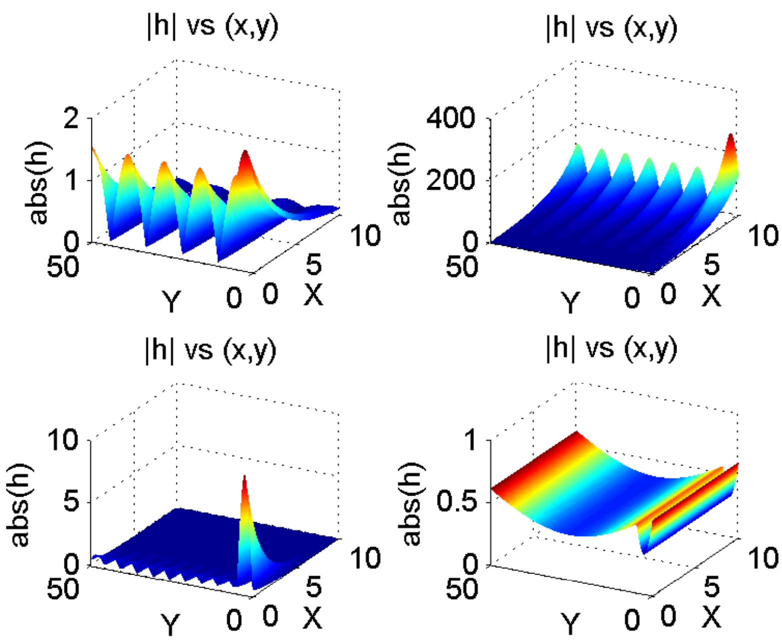


Figure 5.1: Magnetic Fields for the second 4 k_x's

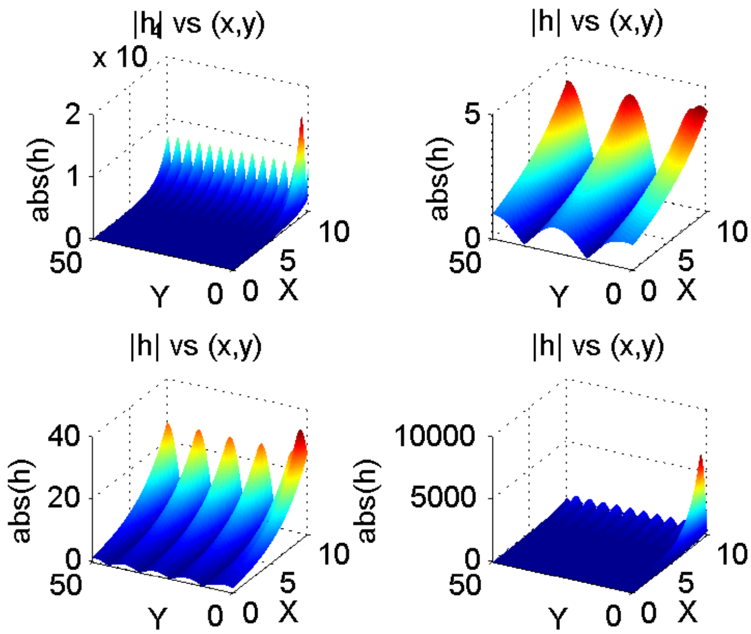


Figure 5.2: Magnetic Fields for the third 4 k_x's

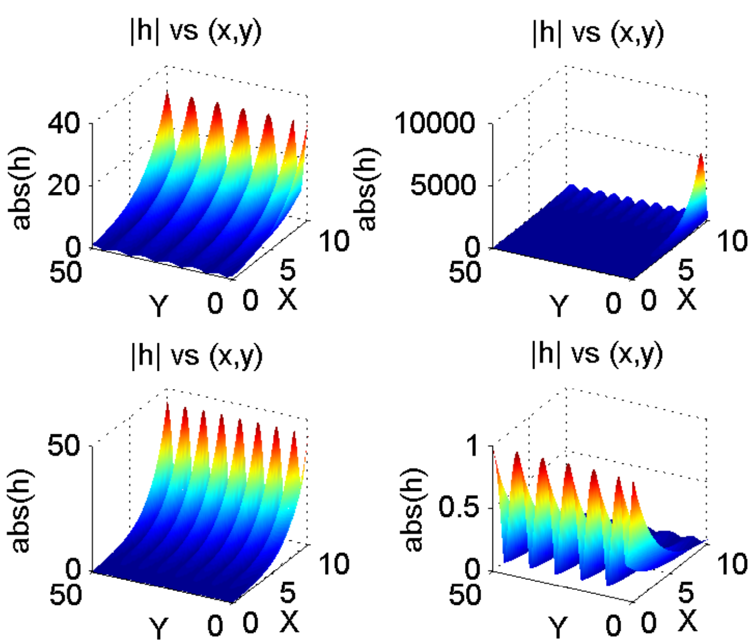


Figure 5.3: Magnetic Fields for the fourth 4 k_x's

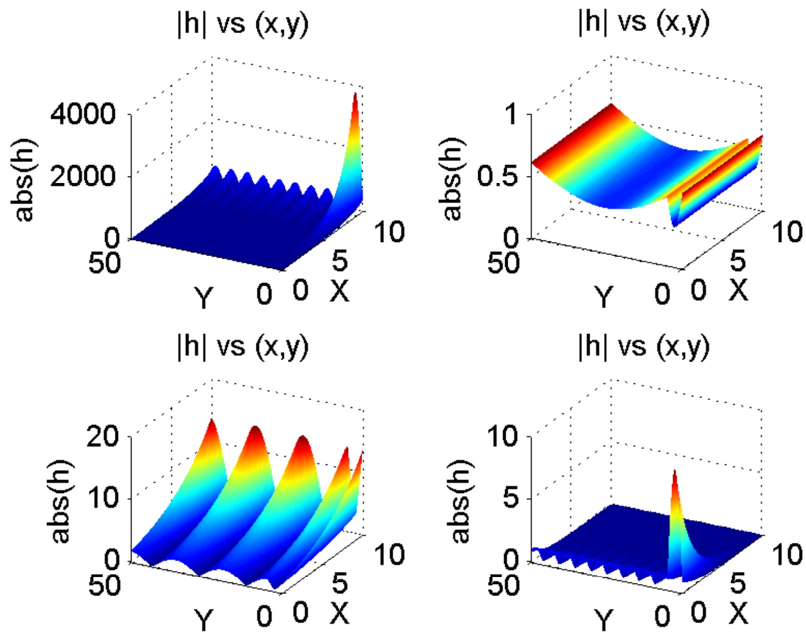


Figure 5.4: Magnetic Fields for the fifth 4 k_x's

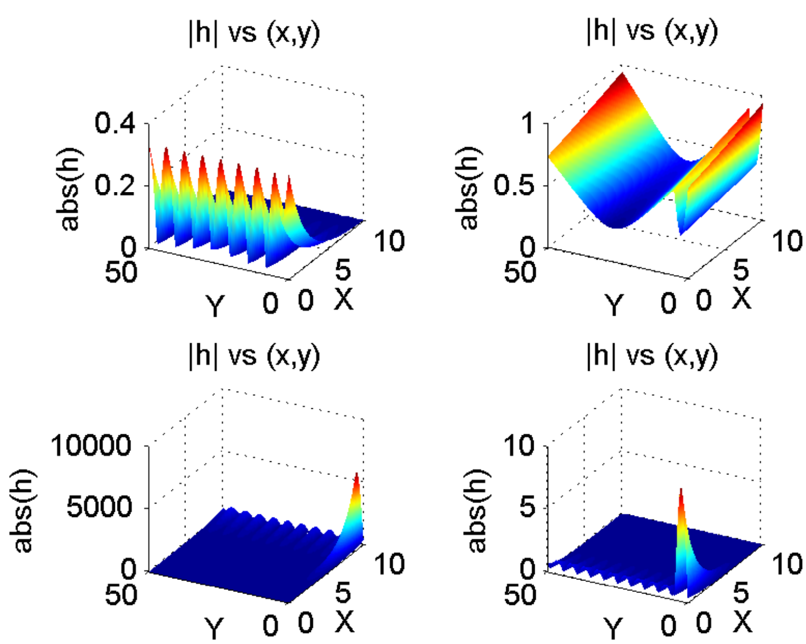


Figure 5.5: Magnetic Fields for the sixth 4 k_x's

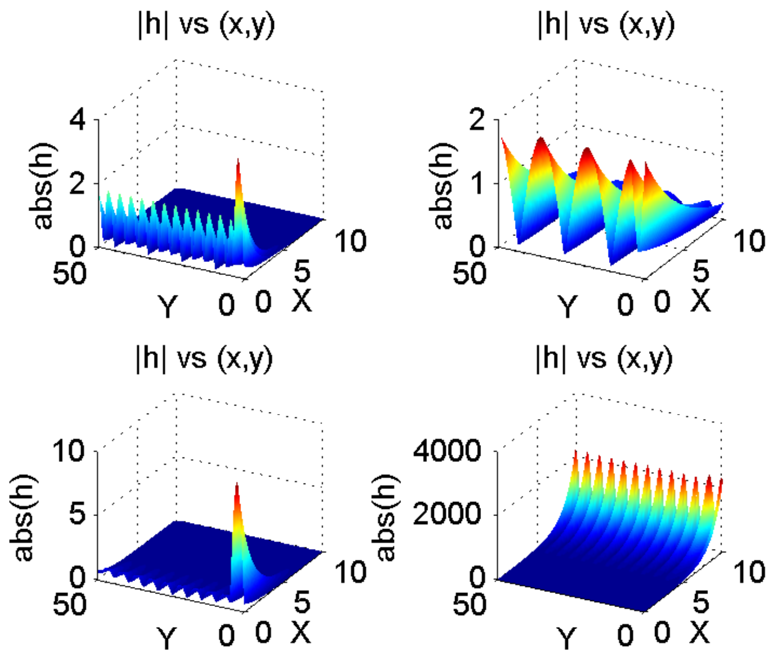


Figure 5.6: Magnetic Fields for the seventh 4 k_x's

The electric fields for different k_x:

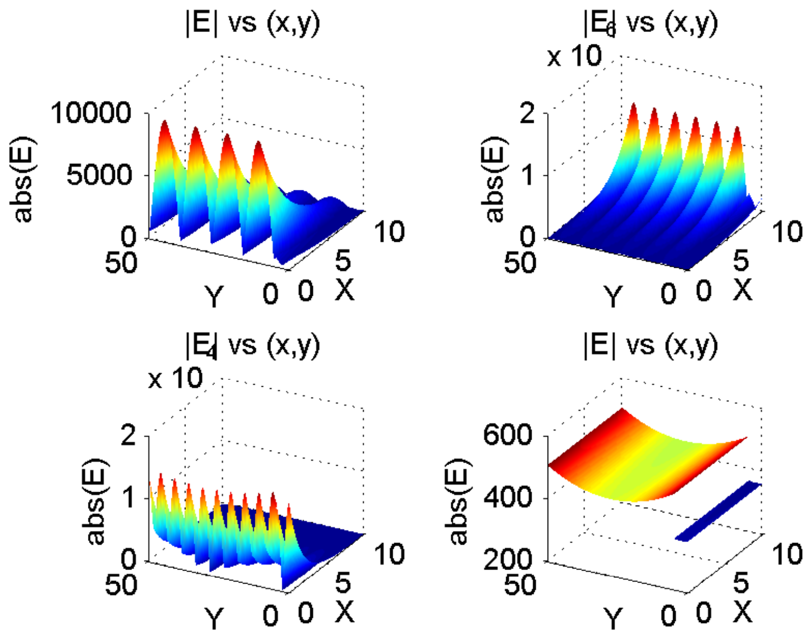


Figure 5.7: Electric Fields for the second 4 k_x's

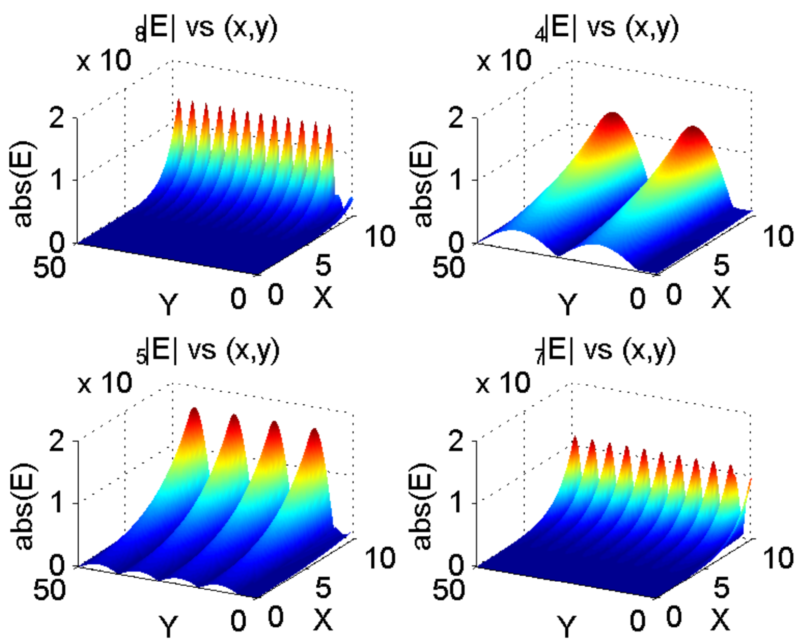


Figure 5.8: Electric Fields for the third 4 k_x's

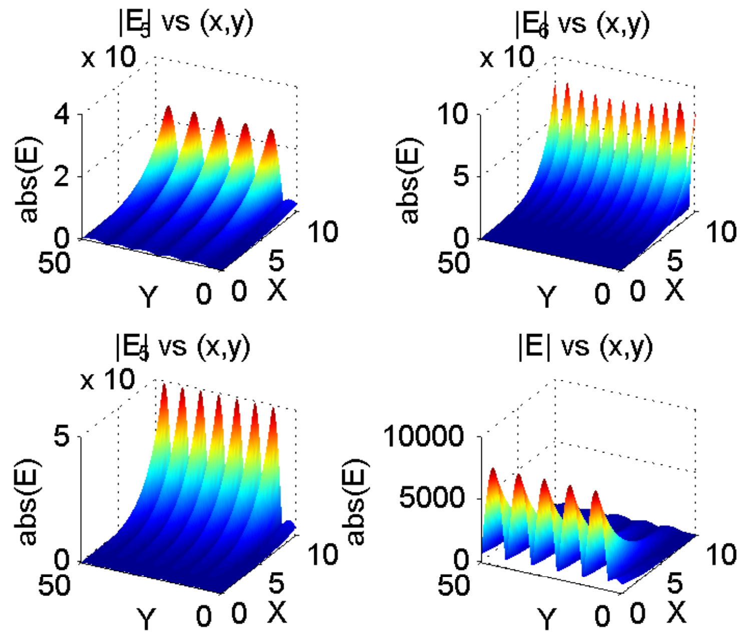


Figure 5.9: Electric Fields for the fourth 4 k_x's

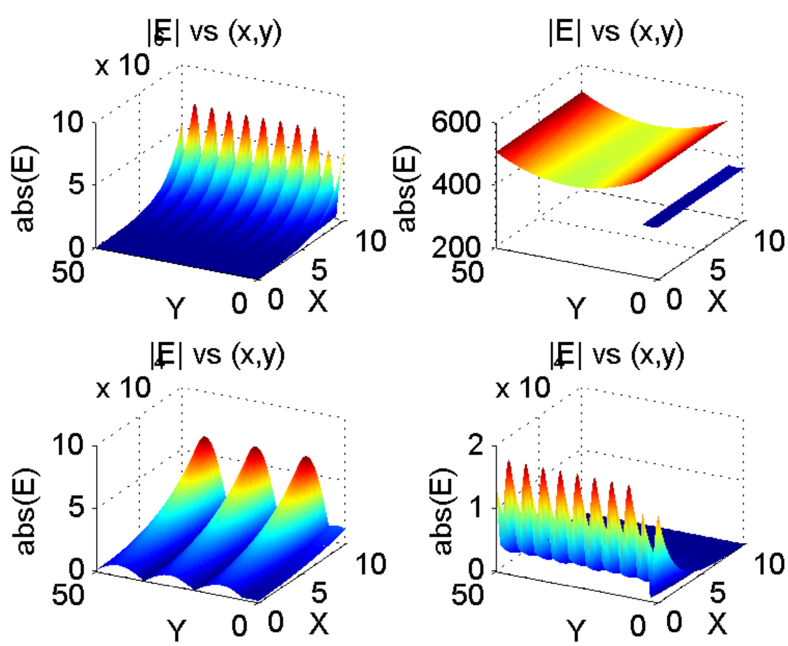


Figure 5.10: Electric Fileds for the fifth 4 k_x's

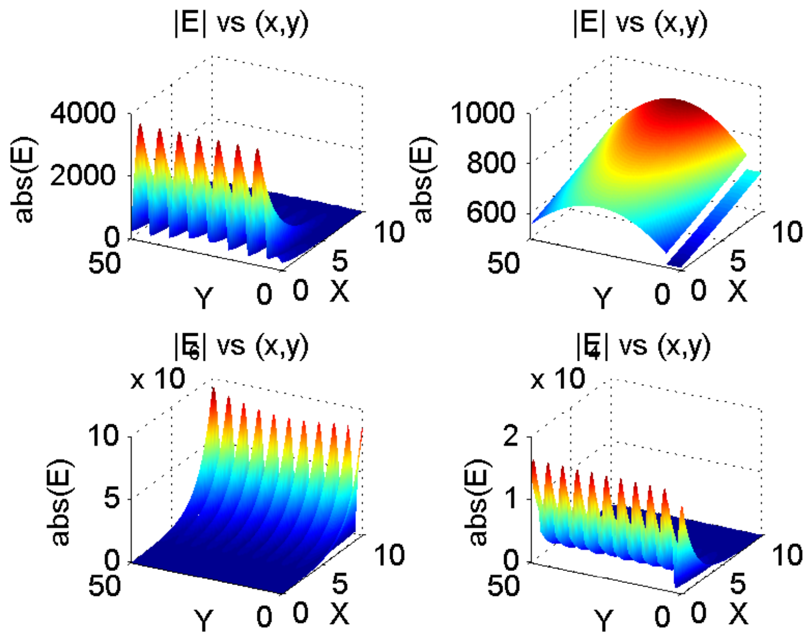


Figure 5.11: Electric Fileds for the sixth 4 k_x's

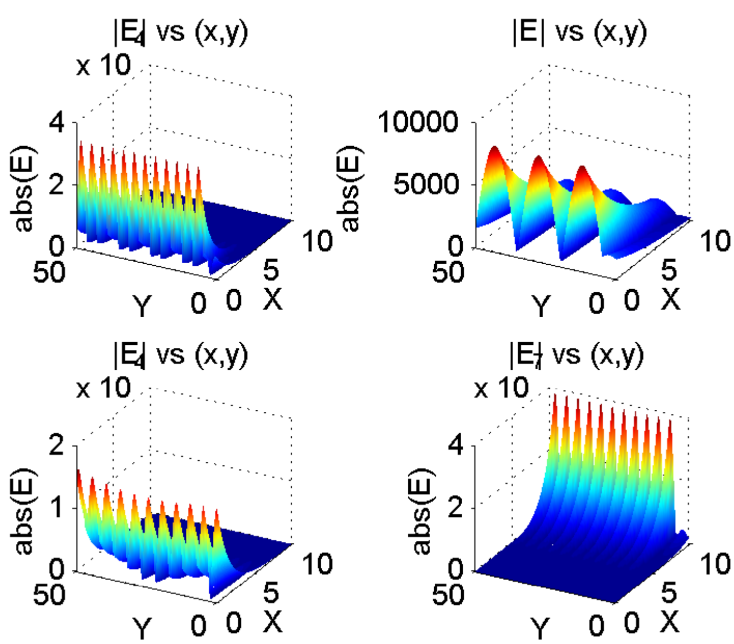


Figure 5.12: Electric Fileds for the seventh 4 k_x's

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