Numerical Semigroups and Basically Full Closures

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Abstract

In investigating the basically full closure of monomial ideals of rings we use numerical semigroups as a proxy. A numerical semigroup is a subset of the natural numbers S with a binary operation such that it is, associative, abelian, has an identity and has a finite complement with the Natural Numbers which we call its gaps G(S). Numerical semigroups can be denoted by their minimally generating elements $a1, ..., an$ such that

$$< a_1, a_2, ..., a_n >= \{x \in N | x = k_1 * a_1 + ... + k_n * a_n, k_1, ..., k_n \in \mathbb{N}\}$$

In particular we will discuss basically full ideals in numerical semigroups, their closures and the structure of basically full closures associated with different types of ideals.

1 Intro

Our first questions about basically full closures come from Heinzer, Ratliff, Rush 2002 where they discuss the basically full closures of commutative rings.

Definition 1.1 *A commutative ring R is a set with two binary operations $+$ and $\times$ (called addition and multiplication) with an abelian group under addition, associativity and commutativity under multiplication, and the regular distributive laws.* (Dummit, pg224)

When we decided to investigate basically full closures it was clear that tackling the broadest case of any commutative ring R was too cumbersome. Instead we looked to the more particular case of the formal power series and the numerical semigroups they generate.

Definition 1.2 *A formal power series $R[[x]]$ with coefficients a_i and b_i in the ring R are sums* $\sum\limits_{i=1}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$
with addition defined as
$\sum\limits_{n=0}^{\infty} a_n x^n + \sum\limits_{n=0}^{\infty} b_n x^n = \sum\limits_{n=0}^{\infty} (a_n + b_n) x^n$
and multiplication as
$\sum\limits_{n=0}^{\infty} a_n x^n \times \sum\limits_{n=0}^{\infty} b_n x^n = \sum\limits_{n=0}^{\infty} \sum\limits_{k=0}^{n} (a_k b_{n-k}) x^n$

Definition 1.3 *Numerical Semigroup Rings. Any numerical semigroup generates a numerical semigroup ring, a formal power series ring, $k[[t^S]]$ who's elements are* $\sum\limits_{s \in S} c_s t^s$

Basically full closures are an operation done on ideals of a commutative ring so we investigate ideals and more specifically monomial ideals in the semigroup ring $k[[t^S]]$. We start by discussing numerical semigroups in their own right, then discuss ideals in rings and their operations in rings and connect those ideals and operations to the equivalent in numerical semigroups.

2 Numerical Semigroups

Definition 2.1 *A Numerical semigroup is a subset of the natural numbers S with the binary operation addition that is commutative, associative and has an identity.*

Definition 2.2 *An important feature of a numerical semigroup is its finite complement with the natural numbers which we call its gaps, G(S).*

Definition 2.3 *Multiplicity of a numerical semigroup S is the size of the smallest generating element i.e. for $S =< a_1, \dots, a_n >$ the multiplicity is a_1.*

Numerical semigroups can be denoted minimally by generating elements $a_1, \dots, a_n$ with $\gcd(a_1, a_2, \dots, a_n) = 1$ such that

$< a_1, a_2, \dots, a_n >= \{x \in \mathbb{N} | x = j_1 a_1 + ... + j_n a_n, j_1, \dots, j_n \in \mathbb{N}\}$.

Example 2.4

$S1 =< 2, 3 >= \{0, 2, 3, 4, \ldots\}$ $\quad$ $G(S1) = \{1\}$
$S2 =< 4, 7 >= \{0, 4, 7, 8, 11, 12, 14, 15, 16, 18, \ldots\}$ $\quad$ $G(S2) = \{1, 2, 3, 5, 6, 9, 10, 13, 17\}$
$S3 =< 3, 5, 7 >= \{0, 3, 5, 6, 7, 8, \ldots\}$ $\quad$ $G(S3) = \{1, 2, 4\}$

Generally elements of $G(S)$ can be ordered $b_1, b_2, \ldots, b_n$ where we have b_n called the frobenius of S, $F(S)$. When S is generated by two elements a and b, $S =< a, b >$, it is know that $F(S)$ can be determined by the formula

$F(S) = (a - 1)(b - 1) - 1.$

Definition 2.5 *The conductor, $C(S)$, is the first element of S for which all integers greater than $C(S)$ are in S.*

3 Ideals and Ideal Operations in rings

3.1 Ideals

Definition 3.1 *Let R be a commutative ring and I a subset of R, I is an ideal if and only if for all $a \in I$ and $r \in R$ we have $ar \in I$. (Dummit, pg 243)*

Definition 3.2 *Let S be a semigroup ring and I a monomial ideal, then $I := (t^s)$ for some $s \subset S$.*

Monomial ideals of semigroup rings can be investigated by looking at ideals of numerical semigroups which meet all the usual ideal criteria but in terms of numerical semigroups as opposed to rings in general. We must also define the maximal ideal

Definition 3.3 *For S a semigroup ring M a monomial ideal, then M is the maximal monomial ideal of the semigroup ring if $M := (t^S)$.*

Before we can define basically full closure of an ideal we must define two operations, Multiplication and the Colon Operation.

Definition 3.4 *For a commutative ring we define multiplication for ideals as I and J as*
$IJ := \sum_{k=1}^{n} i_k j_k$

Definition 3.5 *For a commutative ring we define the colon operation for ideals I and J as* $I : J := \{x \in R | xJ \subseteq I\}$.

Definition 3.6 *An ideal I in a semigroup ring A is basically full if* $I = MI : M$.

3.2 Ideals and Operations in numerical semigroups

Now that we have defined, monomial Ideals, and Ideal operations in terms of semigroup rings we can create equivalent definitions for numerical semigroups and ideals of numerical semiogroups rings.

Definition 3.7 *Let S be a numerical semigroup and I a subset of S, I is an ideal if and only if for all $a \in I$ and $s \in S$ we have $a + s \in I$.*

Definition 3.8 *For a numerical semigroup the equivalent to multiplication for monomial ideals in rings is addition of ideals I and J in numerical semigroups as*
$I + J := \{x \in S | x = i + j \text{ with i} \in \text{I and j} \in \text{J}\}$

Here are some examples of addition of ideals under a given numerical semigroup.

Example 3.9 $T =< 3, 7 >= \{0, 3, 6, 7, 9, 10, 12, 13, 14, 15, \ldots\}$ with
$I = (6) = \{6, 9, 12, 13, 15, 16, 18, 19, 20, \ldots\}$ and
$J = (7, 9) = \{7, 9, 10, 12, 13, 14, \ldots\}$ we have that
$I + J = \{13, 15, 16, 18, 19, 20, \ldots\} = (13, 15)$

Example 3.10 $U =< 3, 20 >= \{0, 3, 6, 7, 9, 10, 12, 13, 14, 15, \ldots\}$ with
$A = (23) = \{23, 26, 29, 32, 35, 38, 41, 43, 44, 46, 47, 49, 50, 52, 53, 55, 56, 58, 59, 61, 62, 63, 64, \ldots\}$
and
$B = (3) = \{3, 6, 9, 12, 15, 18, 21, 23, 24, 26, 27, 29, 30, 32, 33, 35, 36, 38, 39, 41, 42, 43 \ldots\}$ we have that
$A + B = \{26, 29, 32, 35, 38, 41, 44, 46, 49, 50, 52, 53, 55, 57, 59, 61, 62, 65, 66, 67, \ldots\} = (26)$

Definition 3.11 *Similarly for numerical semigroups we define the equivalent operation, called difference, for ideals I and J as* $(I - J) := \{x \in R | x + J \subseteq I\}$

Here are some examples of difference of ideals under a given numerical semigroup.

Example 3.12 $W =< 3, 8, 13 >= \{0, 3, 6, 8, 9, 11, 12, 13, \ldots\}$
$K = (6, 8) = \{6, 8, 9, 11, 12, 14, 15, 16, \ldots\}$
$L = (8, 13) = \{8, 11, 13, 14, 16, 17, 19, 20, 21, \ldots\}$
$L - K = \{8, 11, 13, 14, 15, \ldots\}$

Example 3.13 $V =< 5, 6, 7 >= \{0, 5, 6, 7, 10, 11, 12, 13, 14 \ldots\}$
$C = M = (5, 6, 7)\{5, 6, 7, 10, 11, 12, 13, 14 \ldots\}$
$D = (7, 10) = \{7, 10, 12, 13, 14, 15, 16, 17, \ldots\}$
$C - D = \{5, 6, 7, 10, 11, 12, 13, 14 \ldots\}$

4 Basically Full Closure

Given an ideal I in a numerical semigroup S, we say I is basically full if $I = (I + M) - M$ where M is the maximal ideal of S defined as $S \backslash \{0\}$. We can call the running of the above computation the basically full closure operation as it will reveal the basically full closure of an ideal even if it is not basically full.

Example 4.1 Given a semigroup $S =< 4, 5 >$ and ideals (4) and $(8, 9)$
$(4) + (4, 5) = \{8, 9, 14, 16, 17, 18, 19, \ldots\}$
$[(4) + (4, 5)] - (4, 5) = \{4, 5, 10, 12, 13, 14, 15, \ldots\} = (4, 5)$
$[(8, 9) + (4, 5)] - (4, 5) = (8, 9, 15)$
Hence $(8, 9)$ and (4) are not basically full but their closures are now known as $(8, 9, 15)$ and $(4, 5)$ respectively. For ideal $(8, 9, 10)$ if we run our basically full closure

Example 4.2 $S =< 4, 5 >$ and ideal $I = (8, 9, 10)$
$[(8, 9, 10) + (4, 5)] - (4, 5) = (8, 9, 10)$ hence $(8, 9, 10)$ is basically full.

Example 4.3 Given a semigroup $T =< 3, 7, 8 >= \{0, 3, 6, 7, 8, 9, 10, ...\}$
$N = (6, 7) = \{0, 6, 7, 9, 10, 12, 13, 14, ...\}$
In calculating the basically full closure of $N = (6, 7)$ denoted $(6, 7)^{bf}$ we have that
$(6, 7)^{bf} = (6, 7)$ hence N is basically full.
$P = (6, 11) = \{0, 6, 9, 11, 12, 13, 14, ...\}$
In calculating the basically full closure of $P = (6, 11)$ denoted $(6, 11)^{bf}$ we have that
$(6, 11)^{bf} = \{6, 7, 9, 10, 11, 12, 13, ...\} \neq (6, 7)$ hence P is NOT basically full since 10 was not in P originally

5 "Binary" Addition

In investigating basically full closures we came across an algorithm for calculating closures that, though still lacking a full proof, is promising in it's ability to predict the elements needed in an ideal to generate the basically full closure.

Suppose that
$S =< 3, 7, 11 >= \{0, 3, 6, 7, 9, 10, 11, 12, ...\}$
$I = (3) = \{3, 6, 9, 10, 12, 13, 14, 15, ...\}$

```
S      1 0 0 1 0 0 1 1 0 1 1 1 1
M            1 0 0 1 1 0 1 1 1 1
I            1 0 0 1 0 0 1 1 0 1
________________________________
             1 0 0 1 1 0 1 1 1 1
```

Each place represents an element of the semigroup and/or ideal where a '1' denotes an element that is in the ideal and in the natural numbers and a 0 represents an element that is not in the ideal. The digits in the last line are the result of the "binary" addition where the element in the respective position for M is added to the respective element for I where $0 + 0 = 1, 0 + 1 = 1 + 0 = 1$ and $1 + 1 = 1$. The red 1's in the bottom sequence denote elements which are added to the ideal in order to complete the closure operation. In support of a rigorous proof of these "binary" addition operations we have shown that the gap structure of a basically full closure dose not change when $q, q \in \mathbb{N}$ is added to the ideal (assuming we choose a q such that the ideal plus q is still an ideal).

6 Shifting Lemma

Do ideals of any particular class behave the same as each other under basically full closure? Which is to say, do we have the property for any semigroup S and for ideals I and maximal ideal M that. $k + [(I + M) - M] = (k + I + M) - M$? Well call this the shifting lemma.

Lemma 6.1 *Given semigroups S, ideal I of S, and maximal ideal M dose $k + [(I + M) - M] = (k + I + M) - M \forall\ k \in \mathbb{Z}$*

Theorem 6.2 *Proof of Lemma 6.1*

First some definitions. Under a numerical semigroup S we can define the ideal I as $I = (a_1, a_2, ..., a_n)$ where the a_i's , $1 \leq i \leq n$, are the generating elements of the ideal. We have that,

$I = \{x \in S \mid x = y + a_i \ y \in S, \ 1 \leq i \leq n\}$

The maximal ideal is $M = (m_1, ..., m_l)$ where the m_j's , $1 \leq j \leq l$, are generating elements of M and the elements of M are all the elements of S excluding zero.

$M = \{x \in S | x = y + m_j, \ y \in S, \ 1 \leq j \leq l\}$

The basically full closure of I is the same as $[(I + M) - M]$. We can strictly define $[(I + M) - M]$ as $\{x \in S \mid x + m \in (I + M) \ \forall \ m \in M\}$. Notice that,

$z \in k + [(I + M) - M]$

$\iff$

$z \in k + \{x \in S \mid x + m \in (I + M) \ \forall \ m \in M\}$

$\iff$

$z \in \{x \in S \mid x = y + k, y \in \{p \in S \mid p + m \in (I + M), \ \forall \ m \in M\}\}$

$\iff$

$z \in \{x \in S \mid \forall \ m \in M \ , x + m \in k + (I + M)\}$

$\iff$

$z \in \{x \in S \mid \forall \ m \in M \ , x + m \in (k + I) + M\}$

$\iff$

$z \in [(k + I) + M] - M$

Hence we have that $k + [(I + M) - M] = (k + I + M) - M$

7 Going Forward

Next we plan to investigate arithmatic semigroups which are numerical semigroups of the form

$S =< n+k, n+2k, \ldots, n+nk >$ with $n, k \in \mathbb{N}\backslash\{0\}$. We suspect that characterizing elements in the basically full closure will come more readily when we are constricted to arithmatic semigroups. Below are some examples of arithmetic semigroups, some of their ideals, and their basically full closures.

Example 7.1 $S =< 3, 10, 17 >= \{0, 3, 6, 9, 10, 12, 13, 15, 16, 17 \ldots\}$
for an ideal
$I = (6) = \{6, 9, 12, 15, 16, 18, 19, 21, 22, 23, \ldots\}$
the basically full closure of I is
$I^{bF} = \{6, 12, \underline{13}, 15, 16, 18, 19, 20, 21, 22, 23, \ldots\}$

Example 7.2 For $S =< 3, 7, 11 >$ and $k \in S$ we can describe all possible ideals as one of the 11 types directly below. The Maximal ideal will be of the type (k, k + 4, k + 8) and principal ideals will be of the type (k).
$(k) = \{k, k+3, k+6, k+7, k+9, k+10, k+11, k+12, \ldots\}$
$(k, k+1) = \{k, k+1, k+3, k+4, k+6, k+7, k+8, k+9, k+10, k+11, k+12, \ldots\}$
$(k, k+2) = \{k, k+2, k+3, k+5, k+6, k+7, k+8, k+9, k+10, \ldots\}$
$(k, k+4) = \{k, k+3, k+4, k+6, k+7, k+9, k+10, k+11, \ldots\}$
$(k, k+5) = \{k, k+3, k+5, k+6, k+7, k+8, k+9, k+10, k+11, \ldots\}$
$(k, k+8) = \{k, k+3, k+6, k+7, k+8, k+9, k+10, k+11, \ldots\}$ $(k, k+2, k+4) = \{k, k+2, k+3, k+4, k+5, k+6, k+7, \ldots\}$
$(k, k+1, k+4) = \{k, k+2, k+3, k+5, k+6, k+7, \ldots\}$
$(k, k+4, k+5) = \{k, k+3, k+4, k+5, k+6, +7, k+8, \ldots\}$
$(k, k+4, k+8) = \{k, k+3, k+4, k+6, +7, k+8, \ldots\}$
$(k, k+1, k+2) = \{k, k+1, k+2, k+3, k+4, k+5, k+6, k+7, \ldots\}$

Directly below are the basically full closures of the ideal types. Elements that were added to the basically full closure that were not in the original ideal are underlined and colored red. $(k)^{bF} = \{k, k+3, \underline{k+4}, k+6, k+7, k+9, k+10, k+11, k+12, \ldots\}$
$(k, k+1)^{bF} = (k, +1)$
$(k, k+2)^{bF} = \{k, k+2, k+3, \underline{k+4}, k+5, k+6, k+7, k+8, k+9, k+10, \ldots\}$
$(k, k+4)^{bF} = \{k, k+3, k+4, k+6, k+7, \underline{k+8}, k+9, k+10, k+11, \ldots\}$
$(k, k+5)^{bF} = \{k, k+3, \underline{k+4}, k+5, k+6, k+7, k+8, k+9, k+10, k+11, \ldots\}$
$(k, k+8)^{bF} = \{k, k+3, \underline{k+4}, k+6, k+7, k+8, k+9, k+10, k+11, \ldots\}$
$(k, k+2, k+4)^{bF} = (k, k+2, k+4)$
$(k, k+1, k+4)^{bF} = (k, k+1, k+4)$
$(k, k+4, k+5)^{bF} = (k, k+4, k+5)$
$(k, k+4, k+8)^{bF} = (k, k+1, k+8)$
$(k, k+1, k+2)^{bF} = (k, k+1, k+2)$

8 Matlab Code for Basically Full Closures

The calculation of basically full closures require a large amount of computation. Naturally we set out to create a program to do these calculations , thus giving a means of verification for hand written computations and an efficient way to generate ideals and their closures revealing patterns and trends. I used Matlab R2013a(8.1.0.604) to write 5 functions . The first was 'semifunc' which generates the elements of a semigroup up to several past the conductor as well as the gaps of that semigroup, 'makeideal' which, given a semigroup and an ideal of that semigroup, will generate the elements of that ideal up to several past the conductor, 'addideal' which given a semigroup and two ideals gives the product of those ideals, 'subideal' which, given a semigroup and two ideals, will complete the ideal difference operation and the 'bfnew' function which generates the basically full closure of an ideal given a semigroup. The code for the various function is listed below.

semifunc

```
function [Semigroup,Gaps] = semifunc(Stemp1)
% Generate a Semigroup %
Stemp2 = unique(Stemp1); %makes sure the semigroup generators are unique

%CHECK semigroup input has 2 or more elements
    if length(Stemp2) < 2

        disp('Semigroup must have AT LEAST 2 generating elements')

    end

iscoprime = zeros(1,length(Stemp2)) ; %MAKING array for varifying coprime

    for k = 1:length(Stemp2) %CHECK that Semigroup elements are coprime

        coprime = gcd(Stemp2(k), setdiff(Stemp2, Stemp2(k)));

        if sum(coprime) > length(Stemp2) - 1

            iscoprime(k) = 1;

        else

            iscoprime(k) = 0;
```

```
    end

        k = k + 1;
end

if sum(iscoprime) ~= 0 % DISPLAY if elements are NOT coprime

    disp('Elements of semigroup MUST be coprime')

else

    minele = min(Stemp2); %min genarating element of the Semigroup
    secminele = min(setdiff(Stemp2, minele)); %2nd smallest generating element of Semig

    maxval = (minele - 1)*(secminele - 1);

    arraysize = floor(maxval/minele);

    terminate = length(Stemp2) + 1;

    k = 1;

    multarray = zeros(length(Stemp2),arraysize);

    while  k < terminate;

        tempmult = [Stemp2(k):Stemp2(k):floor(maxval/Stemp2(k))*Stemp2(k)] ;

        multarray(k,1:length(tempmult)) = tempmult;

       k = k +1 ;

    end

    meshmult = meshgrid(multarray);

    Stemp3 = meshmult + meshmult' ;

    Stemp4 = unique(Stemp3);
```

```
        Stemp5 = Stemp4(Stemp4 < maxval + minele +1) ;

        Semigroup = Stemp5 ;

        Gaps = setdiff(0:max(Semigroup), Semigroup)';

 end

end

%______________________________________________%
```

makeideal

```
function a = makeideal(Semigroup, Ideal)

%This function takes any generating set for an ideal
%in a semigruop and prints out the elements of the Ideal
%to some repetition of the dimension.

Semigrouplist = semifunc(Semigroup);

isSubset = isempty(setdiff(Ideal, Semigrouplist)) ; %MAKE variable telling is Ideal is subs

    if isSubset == 1 %CHECK that ideal generators are subset of Semigroup

      if length(Ideal) < 2

            I = Ideal + Semigrouplist;

      elseif length(Ideal) >= 2

            I = unique(meshgrid(Ideal,Semigrouplist) + meshgrid(Semigrouplist, Ideal)')';

            I = I';
      end

      a = union(I, Ideal);
```

```
    else %DISPLAY that ideal generators NOT subset of Semigroup

        disp('Ideal generators MUST be subset of Semigroup')

    end

end

%_______________________________________________%
```

addideal

```
function a = addideal(Semigroup, FirstIdeal, SecondIdeal)

% This function add's ideals of S

I = makeideal(Semigroup,FirstIdeal);


J = makeideal(Semigroup,SecondIdeal);

%The following makes a meshgrid of each value of I and J respectively
% and adds them to each other. We then take th eunique elements out


IplusJ = unique(meshgrid(I, J) + meshgrid(J, I)');

% This final step limits the number of elements IplusJ contains by capping
% it at the max of S using indexing

GAPIplusJ = setdiff([1:max(IplusJ)], IplusJ); %MAKE gap set of IplusJ

b = IplusJ( IplusJ < max(GAPIplusJ) + 1 + min(Semigroup)); %PRINT min gen past conductor
y = [ ];
bnew = b;

    for k = 1:(length(b) - 1)
        x = b(k);
```

```
        y = x + Semigroup;

        bnew = setdiff(bnew, y);
    end
a = bnew ;
end
%_______________________________________________%
```

subideal

```
% For Semigroup S, and Ideals I & J calculating I - J %

function a = subideal(Semigroup, FirstIdeal, SecondIdeal)

S = semifunc(Semigroup);

I = makeideal(Semigroup, FirstIdeal);
J = makeideal(Semigroup, SecondIdeal);

y = 1;
b = [ ];

TotalMax = max([max(I), max(J), max(S)]) ;
SemiMax = max(S);

BigMax = TotalMax + SemiMax +1 ;

I = union(I, max(I):BigMax);

    for k = 1:length(S);

        x = S(k);

        A = x + J ;

        if isempty(setdiff(A, I)) == 1

            b(y) = x ;
```

```
            y = y + 1;

        end
    end
    if isempty(setdiff(0, b)) ==1

        b = setdiff(b, 0);
    end
bnew = b;

    for k = 1:(length(b) - 1)

        x = b(k);

        y = x + Semigroup;

        bnew = setdiff(bnew, y);

    end

a = bnew;

end

% _______________________________________________ %
```

bfnew

```
% BF Ideal function

function a = bfnew(Semigroup, Ideal)

I = Ideal;
M = Semigroup;

IM = addideal(Semigroup, I, M);

IMcoloM = subideal(Semigroup, IM, M);
```

```
a = IMcoloM;

end
```

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