Analyzing a Non-linear Black-Scholes Option Pricing Model: The Cost of Constant Volatility[1]

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Abstract

The financial crisis of 2008-2009 exposed the inaccuracies of derivative pricing models. This revelation was seen in the diffusion of financial risk on a global scale. We assume the models were imprecise because of unrealistic underlying volatility inputs. Therefore, we consider a modification to the ideal condition of constant and known volatility in the Black-Scholes (B-S) option pricing model. The modified B-S model defines volatility as a function of the highest spatial derivative, where volatility is a stochastic variable contained in an empirically determined interval. We test the modification by comparing observed option prices with B-S and modified B-S estimates.

Financial risk has the potential to be systematically diffusive. Most recently, we observed its far reaching effects during the financial crisis of 2008-2009. It has been well-documented that a primary cause of the crisis resulted from corporate mismanagement of risk due to an inability to accurately price derivative securities. Rajun, Seru, and Vig (2010) have shown a prodigious forecasting error resulting from the application of historical data-driven models that do not account properly for human decision-making. Moreover, banks and corporations employing these inaccurate models found themselves incurring decimating losses as their risk management strategies were based on the assumption that their derivative holdings were accurately priced. For example, see Morgenson (2008), Murphy (2008), Nocera (2009), and Taleb (2008).

With modelers unable to capture the price variation in derivatives induced by human decision-making, the results were disastrous consequences for the financial markets. Therefore, we assume the root cause in the models occurred from a mishandling of volatility. We aim to mitigate the effects of inaccurate derivative pricing by focusing on the key component of volatility. We analyze the well-known Black-Scholes (B-S) option pricing partial differential equation (PDE) model put forth by Black, and Scholes (1973). We concern our analysis with the improvement of the unrealistic assumption of known and constant volatility for the life of the option. Thus, we make use of a proposed modification and analysis from Avellaneda, Levy, and Parás (1995), and Lorenz, and Qiu (2009), in which the volatility is a function of the highest spatial derivative. Using this strategy, it is assumed that volatility is an unknown stochastic variable lying in a known range.

We use a numerical method strategy to estimate implied volatility values to input into the B-S and modified B-S model. Armeanu, and Vasile (2009) present a Newton-Raphson algorithm which we incorporate for this purpose. Then we compare our systematic volatility patterns to their work as a means of verification. Finally, we compare the results of the classic B-S model with that of the modified B-S model with observed option prices for Apple Incorporated (AAPL), and test the systematic pricing patterns we observe with that of Macbeth, and Merville (1979). We conclude with closing remarks about our empirical findings and its tentative application.

I. Deriving the classic B-S model

We derive the classic B-S model using the approach of Black, and Scholes (1973), and Lorenz, and Qiu (2009). An option is a contract between a buyer and a seller that gives the buyer the right to purchase or sell an underlying asset at a specified price (the strike price) and future time (the expiration date). Thus, we can model the value v of an option as a function of certain parameters, expressed as:

$$v(s,t;\mu,\sigma;E,T;r) \equiv v(s,t) \qquad (1)$$

We have that s is the market price of the underlying asset (underlying); t is the date in which the option contract is purchased; μ is the drift or return of s; σ is the volatility or variation of s; E is the strike price specified by the contract; T is the expiration date; r is the risk-free interest rate.

The classic B-S model is derived in a stock and option market of "ideal" conditions (assumptions). These include:

i. The short-term interest rate is known in advance and is constant through the life of the option.
ii. The stock price of the underlying follows a mathematical random walk in continuous time, and the possible stock price distribution is log-normal and the variance rate of returns is constant.
iii. The underlying does not make dividend payments.
iv. The option contract can only be exercised at maturity, i.e., a "European" option.
v. Transaction costs are not associated with the purchase and selling of the stock or option.
vi. The underlying is perfectly divisible such that fractional shares of the underlying can be purchased and held at the short-term interest rate.
vii. Short selling is not subject to penalty and there no arbitrage opportunities.

These requirements lead to a valuation model for options that depends only on the price of the underlying and time. This allows for the construction of a hedged portfolio that contains a long position on the underlying and a short position on the option. It is this combination that is conducive to dynamic hedging and thus the elimination of risk.

Using identity (1), it is possible to create a hedged portfolio Π consisting of one long call option position $v(s,t)$and a short position of quantity Δ in the underlying s. This is denoted by:

$$\Pi = v(s,t) - \Delta s. \tag{2}$$

We now employ the assumption that the underlying follows a lognormal random walk represented by:

$$ds = \mu s dt + \sigma s dX, \tag{3}$$

where X is Brownian motion. The change on the value of the portfolio from t to dt can be expressed as:

$$d\Pi = dv - \Delta ds. \tag{4}$$

Assuming that the portfolio employs a dynamic hedging strategy, in which the short position is continuously adjusted, allows us to use stochastic calculus or Ito's calculus which yields the following result:

$$dv = \frac{\partial v}{\partial t} dt + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} dt + \frac{\partial v}{\partial s} ds. \tag{5}$$

Therefore, equations (4) and (5) give the change in the portfolio as:

$$d\Pi = \frac{\partial v}{\partial t} dt + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} dt + \frac{\partial v}{\partial s} ds - \Delta ds. \tag{6}$$

From equation (6) we see that $\frac{\partial v}{\partial s} ds - \Delta ds$ is the random component in the model. This uncertainty is considered financial risk and it is what we want to address. Therefore, if we choose $\Delta = \frac{\partial v}{\partial s}$ then the uncertainty in the model is eliminated. This delta hedging strategy is an example of dynamic hedging. Furthermore, it assumes that no arbitrage opportunities exist in the market for the underlying and call option. We can now modify the equation to account for the elimination of risk by:

$$d\Pi = \left(\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2}\right) dt. \tag{7}$$

It then follows from the fact that the change in the portfolio is without risk that an equivalent amount of growth would be returned from an equal valued position in a risk-free interest bearing account. This can be represented by:

$$d\Pi = r\Pi dt. \tag{8}$$

We then combine equations (6) and (8) to achieve the following result:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{9}$$

We recognize equation (9) as the B-S PDE model with the following end-condition at T:

$$v(s,T) = \begin{cases} \max(s - E, 0) & \text{for a call option;} \\ \max(E - s, 0) & \text{for a put option.} \end{cases} \tag{10}$$

It is possible to find an explicit solution to (9) using (10). We append this process and its corresponding formula as the analytical solution is not necessary for our purpose.

II. Relaxing the assumption of constant volatility and deriving the modified B-S model

We modify the B-S model, relaxing the assumption of constant volatility, using the approach of Avellaneda, Levy, and Parás (1995), and Lorenz, and Qiu (2009). We consider the volatility parameter σ as an uncertain stochastic variable in which we assume σ lies in the following range:

$$0 < \sigma^- \leq \sigma \leq \sigma^+, \tag{11}$$

where σ^- and σ^+ are estimated minimal and maximal values of σ, respectively. Thus, we have:

$$\min_{\sigma^- \leq \sigma \leq \sigma^+} \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} = \begin{cases} \frac{1}{2}(\sigma^+)s^2 \frac{\partial^2 v}{\partial s^2} < 0 \\ \frac{1}{2}(\sigma^-)s^2 \frac{\partial^2 v}{\partial s^2} \geq 0 \end{cases} \tag{12}$$

Then, we can define the discontinuous function as:

$$\sigma_d\left(\frac{\partial^2 v}{\partial s^2}\right) = \begin{cases} \sigma^+ < 0 \\ \sigma^- > 0 \end{cases}. \tag{13}$$

Using the same strategy of delta hedging, $\Delta = \frac{\partial v}{\partial s}$ in the derivation of the classic B-S model we get:

$$d\Pi = \left(\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2}\right) dt. \quad (14)$$

Assuming that r is the minimum return of the portfolio with σ varying over the range (11) and that it equals $r\Pi dt$. We then have:

$$\left(\frac{\partial v}{\partial t} + \frac{1}{2}\sigma_d^2\left(\frac{\partial^2 v}{\partial s^2}\right) s^2 \frac{\partial^2 v}{\partial s^2}\right) dt = \ r\Pi dt = r\left(v - \frac{\partial v}{\partial s}\right) dt\ . \quad (15)$$

Hence, we have the following non-linear PDE, which will serve as our comparison model to the classic B-S version:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma_d^2\left(\frac{\partial^2 v}{\partial s^2}\right) s^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \quad (16)$$

III. Describing the data and empirical results

The sample consists of 544 daily closing prices for call options for AAPL from January 28th, 2014 to February 12th, 2014. The option prices and prices of the underlying stocks were retrieved from Yahoo! Finance. We take the risk-free interest rate from bid and ask yields reported on the Wall Street Journal website for the 30-day Treasury Bill. We then aggregate this number to correspond to the number of days until maturity for each call option.

Next, we employ the Newton-Raphson method to calculate implied volatilities σ for each day t and option price. This is a sufficient numerical method since we have a closed form volatility derivative $v(\sigma)$ that is never negative. We chose an initial starting value of σ_0 and iterate using:

$$\sigma_{n+1} = \sigma_n - \frac{BS(\sigma_n) - E}{v(\sigma_n)} \quad (17)$$

until we reached a solution of sufficient accuracy, where $BS(\sigma_n)$ describes the B-S formula evaluated at σ_n. We define sufficient accuracy as the number of correct digits necessary to get the call option price correct to two digits using the B-S formula. The time to expiration and the risk-free rate of return is measured on a daily basis. Furthermore, we assumed that no dividends were paid and the option contracts are European.

Using the Newton-Raphson method, we find implied values of σ in the interval 0.16 to 0.26. We observe that σ lies between 0.16 and 0.26 for roughly 95% of the 544 call option prices, where the implied value of σ is greater than 0.26 in just 8 cases and is below 0.16 in only 17 cases. This is depicted in Figure I.

i. Systematic implied volatility pattern

We provide the implied volatility estimates for the 10-days of data for AAPL in Table 1. After careful inspection of the σ estimates we notice that, in general, the implied volatility decreases as the strike prices increase for in-the-money options and reaches its minimum for an at-the-money option. Then the volatility estimate begins increasing with increasing strike prices for out-the-money options. We observe this systematic pattern holds for most of our option data and is mostly consistent with a less defined volatility smile described by Vasile and Armeanu (2009). Therefore, we find the implied variance is different, and depends on whether the option is in, at, or out of the money, as depicted in Figure II, where moneyness m is defined as:

$$m = \frac{\ln\left(\frac{S}{E}\right)}{\sigma\sqrt{t}} \tag{18}$$

We also include Figures III and IV in the Appendix III, which plot the implied volatility against moneyness and strike price, respectively.

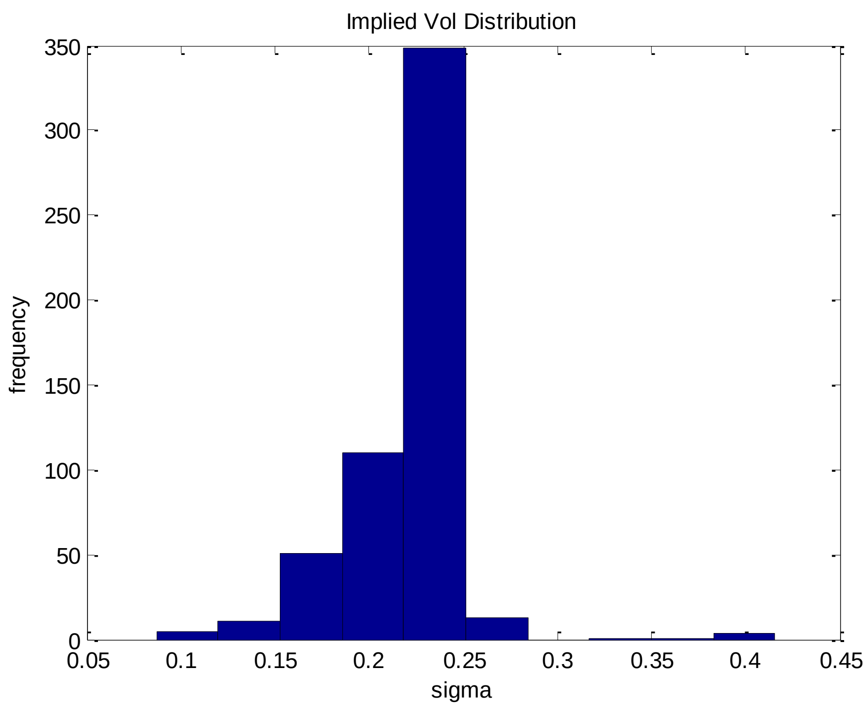

Figure I: Implied Volatility Distribution for all AAPL option data

Table 1: 10-day sample of implied volatility estimates for AAPL

Date	Exercise Price	March	July	October	January 15'	January 16'	Stock Price
1/28/2014	$ 460.00	0.33670	0.25658	0.19933	0.24176	0.22580	$506.50
1/28/2014	$ 480.00	0.21973	0.22550	0.19366	0.23455	0.22927	$506.50
1/28/2014	$ 500.00	0.21568	0.23344	0.20348	0.23969	0.23053	$506.50
1/28/2014	$ 520.00	0.21868	0.23472	0.20558	0.23670	0.23138	$506.50
1/28/2014	$ 540.00	0.22816	0.23876	0.20793	0.23932	0.23918	$506.50
1/29/2014	$ 460.00	0.41580	0.24110	0.20345	0.22967	0.22974	$500.75
1/29/2014	$ 480.00	0.20206	0.22027	0.19070	0.24608	0.23775	$500.75
1/29/2014	$ 500.00	0.21203	0.22063	0.19925	0.23277	0.23540	$500.75
1/29/2014	$ 520.00	0.21940	0.22322	0.19639	0.23387	0.23549	$500.75
1/29/2014	$ 540.00	0.22240	0.23239	0.20419	0.23456	0.23768	$500.75
1/30/2014	$ 460.00	0.25225	0.23876	0.19271	0.23037	0.22408	$499.78
1/30/2014	$ 480.00	0.20980	0.22417	0.18049	0.22537	0.22614	$499.78
1/30/2014	$ 500.00	0.21817	0.22434	0.19197	0.22812	0.23070	$499.78
1/30/2014	$ 520.00	0.22265	0.22401	0.19767	0.23070	0.23295	$499.78
1/30/2014	$ 540.00	0.22549	0.22814	0.20250	0.23193	0.23609	$499.78
2/3/2014	$ 460.00	0.24474	0.23019	0.21348	0.23630	0.23299	$501.53
2/3/2014	$ 480.00	0.21410	0.22515	0.21757	0.22166	0.23136	$501.53
2/3/2014	$ 500.00	0.18026	0.22171	0.21933	0.22300	0.22880	$501.53
2/3/2014	$ 520.00	0.18106	0.22590	0.22684	0.23168	0.23250	$501.53
2/3/2014	$ 540.00	0.22662	0.22687	0.22690	0.23088	0.23686	$501.53
2/4/2014	$ 460.00	0.14221	0.21430	0.17393	0.21781	0.22113	$508.79
2/4/2014	$ 480.00	0.12285	0.21568	0.21620	0.22112	0.21992	$508.79
2/4/2014	$ 500.00	0.14633	0.21832	0.21852	0.22079	0.22684	$508.79
2/4/2014	$ 520.00	0.15930	0.21937	0.21878	0.22212	0.23126	$508.79
2/4/2014	$ 540.00	0.21447	0.21821	0.22131	0.22639	0.23195	$508.79
2/5/2014	$ 460.00	0.24992	0.20936	0.21321	0.21869	0.21081	$512.59
2/5/2014	$ 480.00	0.20948	0.22279	0.21775	0.21632	0.22222	$512.59
2/5/2014	$ 500.00	0.16875	0.20999	0.22157	0.23033	0.22614	$512.59
2/5/2014	$ 520.00	0.18645	0.21970	0.22441	0.23029	0.22714	$512.59
2/5/2014	$ 540.00	0.17268	0.22121	0.22195	0.23083	0.23595	$512.59
2/6/2014	$ 460.00	0.25502	0.22866	0.21034	0.22687	0.23115	$512.51
2/6/2014	$ 480.00	0.20294	0.22769	0.19393	0.22815	0.23065	$512.51
2/6/2014	$ 500.00	0.10825	0.22493	0.22229	0.23193	0.23370	$512.51
2/6/2014	$ 520.00	0.17468	0.22469	0.22421	0.23310	0.23789	$512.51
2/6/2014	$ 540.00	0.16789	0.22668	0.22902	0.23282	0.24125	$512.51
2/7/2014	$ 460.00	0.26377	0.25326	0.24257	0.23937	0.23305	$519.68
2/7/2014	$ 480.00	0.25260	0.25074	0.23655	0.24231	0.23508	$519.68
2/7/2014	$ 500.00	0.18382	0.23410	0.23284	0.23355	0.23666	$519.68
2/7/2014	$ 520.00	0.17576	0.23296	0.23869	0.23586	0.23895	$519.68
2/7/2014	$ 540.00	0.17250	0.23618	0.23773	0.23745	0.24043	$519.68
2/10/2014	$ 460.00	0.26431	0.22730	0.21908	0.24075	0.20616	$528.99
2/10/2014	$ 480.00	0.26568	0.23867	0.22429	0.22956	0.23551	$528.99
2/10/2014	$ 500.00	0.17110	0.23137	0.22700	0.22942	0.22800	$528.99
2/10/2014	$ 520.00	0.18761	0.22462	0.22481	0.22860	0.22352	$528.99
2/10/2014	$ 540.00	0.16181	0.22471	0.22987	0.23060	0.23650	$528.99
2/11/2014	$ 460.00	0.26562	0.23632	0.18205	0.20460	0.22881	$535.96
2/11/2014	$ 480.00	0.21750	0.23251	0.22848	0.23345	0.23080	$535.96
2/11/2014	$ 500.00	0.17884	0.22627	0.22605	0.23105	0.21979	$535.96
2/11/2014	$ 520.00	0.16571	0.21734	0.22764	0.23124	0.23020	$535.96
2/11/2014	$ 540.00	0.16692	0.22486	0.23008	0.23438	0.23771	$535.96
2/12/2014	$ 460.00	0.37922	0.23373	0.22040	0.22956	0.22910	$535.92
2/12/2014	$ 480.00	0.22225	0.23355	0.22701	0.22608	0.23249	$535.92
2/12/2014	$ 500.00	0.15896	0.22193	0.22320	0.23047	0.23454	$535.92
2/12/2014	$ 520.00	0.14803	0.22193	0.22795	0.23116	0.23369	$535.92
2/12/2014	$ 540.00	0.15609	0.22572	0.23521	0.23688	0.23492	$535.92

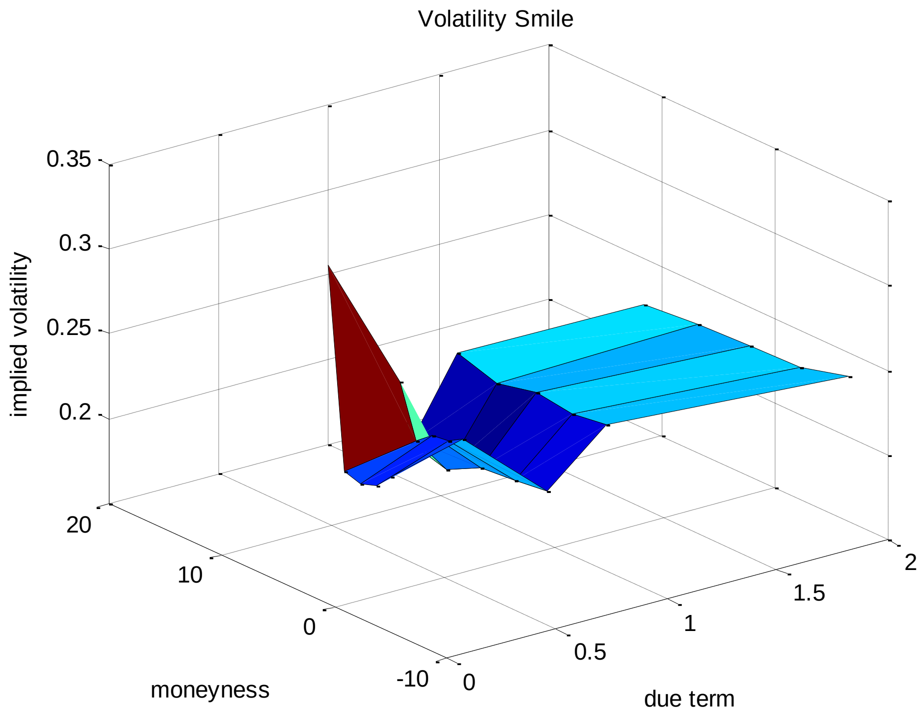

Figure II: Volatility Smile for AAPL

There also appears to be a relationship between the time to maturity of the option and implied volatility. For instance, if we look down the columns of the March and January 16' columns for in-the-money options we recognize a tendency for the shorter term options to have the higher σ values than longer to expiration options with equivalent strike prices. Moreover, we observe that out-the-money options with less time to maturity tend to have smaller σ estimates than the longer term calls with identical strike prices.

We list the minimum and maximum volatility values in Table 2, as well as the estimated σ that was used for the B-S formula estimates. We take the average minimum and maximum values for all days and all call options for each associated strike price. These values were input into the model and depending on the sign of $\frac{\partial^2 v}{\partial s^2}$ went into the calculation of the corresponding modified B-S estimate. The σ for each strike price was taken as the average of all volatility values corresponding to the strike price.

ii. Systematic B-S model, modified B-S model, and observed market price pattern

We estimate the B-S prices using both the formula as well as the implicit Euler finite difference method. This is carried out in order to test the validity of our MATLAB program. We then use the same numerical method, but modify volatility as described in (13).

Table 3 provides the data for observed market prices in Panel A, their corresponding implied volatility values in Panel B, B-S formula price estimates and B-S finite difference method price estimates in Panel C and D, respectively, and the Modifies B-S finite-difference method estimates in Panel E on January 28th, 2014. Table 4 lists the error estimates, where the error is defined as the difference between the observed price and the model estimates. Therefore, a negative error represents an overestimation by the model relative to the observed market price.

Table 2: Volatility Interval for AAPL

	Sigma	Sigma Min	Sigma Max
$460.0	0.21417	0.11270	0.31564
$480.00	0.21241	0.11200	0.31281
$500.00	0.21448	0.14800	0.28097
$520.00	0.22782	0.19260	0.26305
$540.00	0.23553	0.20190	0.26917

Table 3: AAPL Option Data for January 28, 2014

	Exercise Price	March	July	October	January 15'	January 16'
	Market Prices					
Panel A	$460.00	$54.70	$62.10	$65.50	$72.75	$92.20
	$480.00	$33.00	$45.15	$55.75	$59.85	$78.55
	$500.00	$20.10	$35.70	$43.52	$50.86	$71.27
	$520.00	$11.15	$25.95	$34.38	$41.20	$59.68
	$540.00	$6.10	$19.41	$27.90	$34.00	$54.25
	Implied Values of σ					
Panel B	$460.00	0.336700	0.25658	0.19933	0.24176	0.22580
	$480.00	0.2973	0.22550	0.19366	0.23455	0.22927
	$500.00	0.21568	0.23344	0.20348	0.23969	0.23053
	$520.00	0.2188	0.23472	0.20558	0.23670	0.23138
	$540.00	0.22816	0.2376	0.20793	0.23932	0.23918
	Black-Scholes Formula Prices					
Panel C	$460.00	$48.91	$58.36	$65.08	$71.52	$83.95
	$480.00	$32.65	$44.68	$52.10	$58.94	$72.68
	$500.00	$19.93	$33.61	$41.49	$48.59	$63.54
	$520.00	$11.88	$26.29	$34.57	$42.00	$69.38
	$540.00	$6.58	$20.06	$28.25	$35.67	$63.12
	Black-Scholes PDE Model Prices					
Panel D	$460.00	$48.99	$58.37	$65.08	$71.50	$83.95
	$480.00	$32.67	$44.69	$52.15	$58.94	$72.66
	$500.00	$19.97	$33.63	$41.51	$48.59	$63.54
	$520.00	$11.89	$26.34	$34.60	$42.05	$69.35
	$540.00	$6.59	$20.07	$28.32	$35.68	$63.12
	Modified Black-Scholes Model Prices					
Panel E	$460.00	$46.74	$49.59	$52.86	$56.70	$75.67
	$480.00	$27.66	$32.96	$37.10	$41.35	$61.06
	$500.00	$14.92	$24.55	$30.34	$35.71	$57.61
	$520.00	$14.60	$31.53	$40.91	$49.24	$59.68
	$540.00	$8.73	$24.77	$34.12	$42.34	$72.43

Table 4

Estimation Error for Observed vs. B-S and Modified B-S Estimates

	March			**July**			**October**		
	BS	BS PDE	M BS PDE	BS	BS PDE	M BS PDE	BS	BS PDE	M BS PDE
$460.00	$5.79	$5.71	$7.96	$3.74	$3.73	$12.51	$0.42	$0.42	$12.64
$480.00	$0.35	$0.33	$5.34	$0.47	$0.46	$12.19	$3.65	$3.60	$18.65
$500.00	$0.17	$0.13	$5.18	$2.09	$2.07	$11.15	$2.03	$2.01	$13.18
$520.00	-$0.73	-$0.74	-$3.45	-$0.34	-$0.39	-$5.58	-$0.19	-$0.22	-$6.53
$540.00	-$0.48	-$0.49	-$2.63	-$0.65	-$0.66	-$5.36	-$0.35	-$0.42	-$6.22

	January 15'			**January 16'**		
	BS	BS PDE	M BS PDE	BS	BS PDE	M BS PDE
$460.00	$1.23	$1.25	$16.05	$8.25	$8.25	$16.53
$480.00	$0.91	$0.91	$18.50	$5.87	$5.89	$17.49
$500.00	$2.27	$2.27	$15.15	$7.73	$7.73	$13.66
$520.00	-$0.80	-$0.85	-$8.04	-$9.70	-$9.67	$0.00
$540.00	-$1.67	-$1.68	-$8.34	-$8.87	-$8.87	-$18.18

We observe that in-the-money options prices are greater than the estimates of both the classic B-S and modified B-S model. For example, the observed market price for the \$460 March option is \$54.70, while the B-S formula estimates a price of \$48.91, and the modified B-S model estimates a price of \$46.74; an underestimation of \$5.79 and \$7.96, respectively. We have that this systematic price pattern holds for all in-the-money options and the degree of underestimation increases dramatically as the time to maturity increases. For instance, the \$460 January 16' options has an observed price of \$92.20, while the B-S formula and modified B-S model estimate respective prices of \$83.95 and \$75.67, an underestimation of \$8.25 and \$16.53, respectively.

We also observe that this in-the-money option underestimation generally decreases as the strike price increases and the option approaches at-the-money status. On January 28th, 2014, the underlying for AAPL was trading at \$506.50. Then if we look at a \$500 March option we have an observed price of \$20.10. We compare this with the B-S formula price of \$19.93, and the modified B-S price of \$14.92, which is an underestimation of \$0.17 and \$5.18, respectively. In addition, the \$500 January 16' estimation error is \$7.73 and \$13.66, respectively. Thus, both of the models estimate errors decrease as an in-the-money option approaches a nearly at-the-money strike price.

Finally, we notice that out-the-money options have a similar but inverse relationship to that of the in-the-money options described above. We conclude that observed prices for out-the-money options are less than both of the B-S and modified B-S model estimates. For example, a \$520 March option has an observed price of \$11.15, and the B-S formula estimates a price of \$11.88, while the modified B-S gives a result of \$14.60. This is an overestimation difference of \$0.73 and \$3.45, respectively. Also, we note that this overestimation error appears to decrease as the option becomes more out-the-money. This is evidenced in the case where a \$540 March observed option is \$6.10, while the B-S formula price is \$6.58, and the modified B-S price is \$8.73; a respective difference of \$0.48 and

$2.63. If we look at the $540 January 16' option we find an observed price of $54.25, while the B-S indicates a price of $63.12, and the modified B-S results a price of $72.43; estimation error of $8.87 and $18.18, respectively. Thus, it appears that the price pattern holds for out-the-money options as well. This systematic price pattern is displayed in Figure IV and V, where Figure IV compares the error against the option strike price, and Figure V examines the relationship between the estimate error and the amount of time remaining until expiration.

We also include Figure VI to show the relationship between the estimation error and moneyness. Here we see the overestimation by the B-S and modified B-S model for out-the-money options which decreases as the strike price approaches at-the-money status. Similarly, the lower-end of the downward sloping relationship indicates that the B-S and modified B-S underestimate in-the-money options, and that this error decreases as $m \to 0$.

However, we find an interesting break from this systematic behavior for the $520 January 16' option. We have an observed market price of $59.68, a B-S formula price of $69.38, but a modified B-S estimate of $59.68. This means that while the classic B-S formula continues the overestimation pattern, the modified version has an error of $0.00. We also have that, unlike the March out-the-money options, the January 16' out-the-money option estimate error increases as the option becomes more out-the-money. We display the minimum values for this call option in Figure VII and the error relationship in Figure VIII with a blue circle to emphasize that no error is committed by the modified B-S estimate for the $520 January 16' option. Motivated by the success of the modified B-S model for the longer-term (717 days until expiration), moderately out of the money ($13.50) option, we want to test its success on the remaining 9 days of data from January 29th, 2014 to February 12th, 2014.

Table 5 provides the comparison estimates for observed, B-S, and modified B-S estimates over the 10-day sample for AAPL looking at the $520 January 16' option when AAPL is trading below this strike price and then the 3 days after AAPL exceeds $520, we consider the $540 January 16' option.

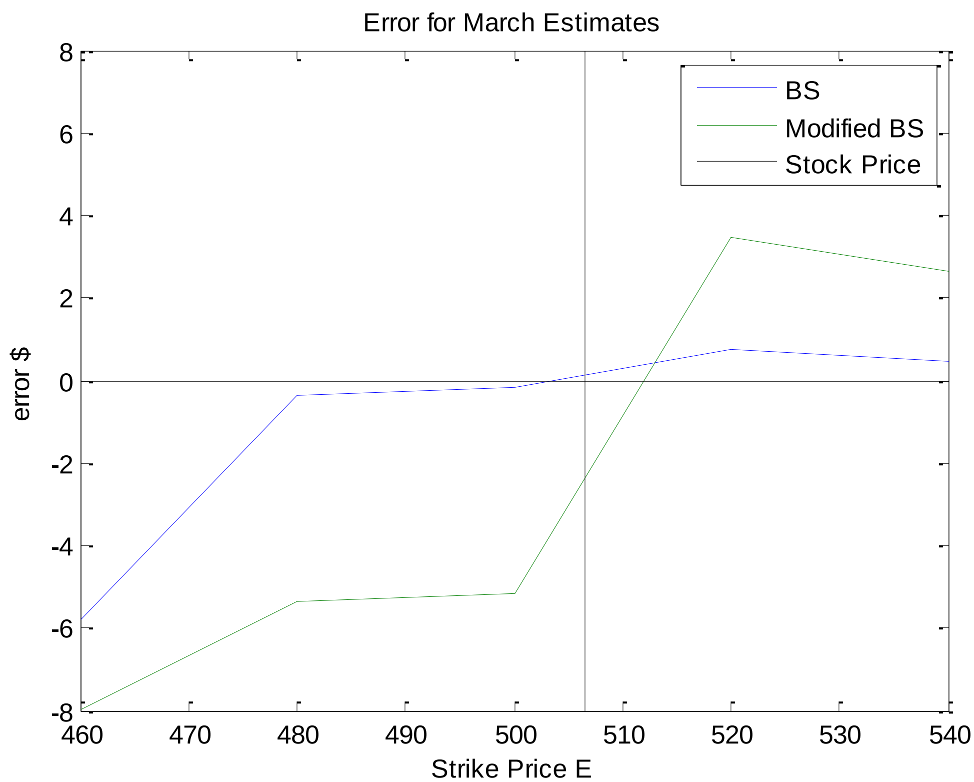

Figure IV: The Estimate Error for AAPL March option

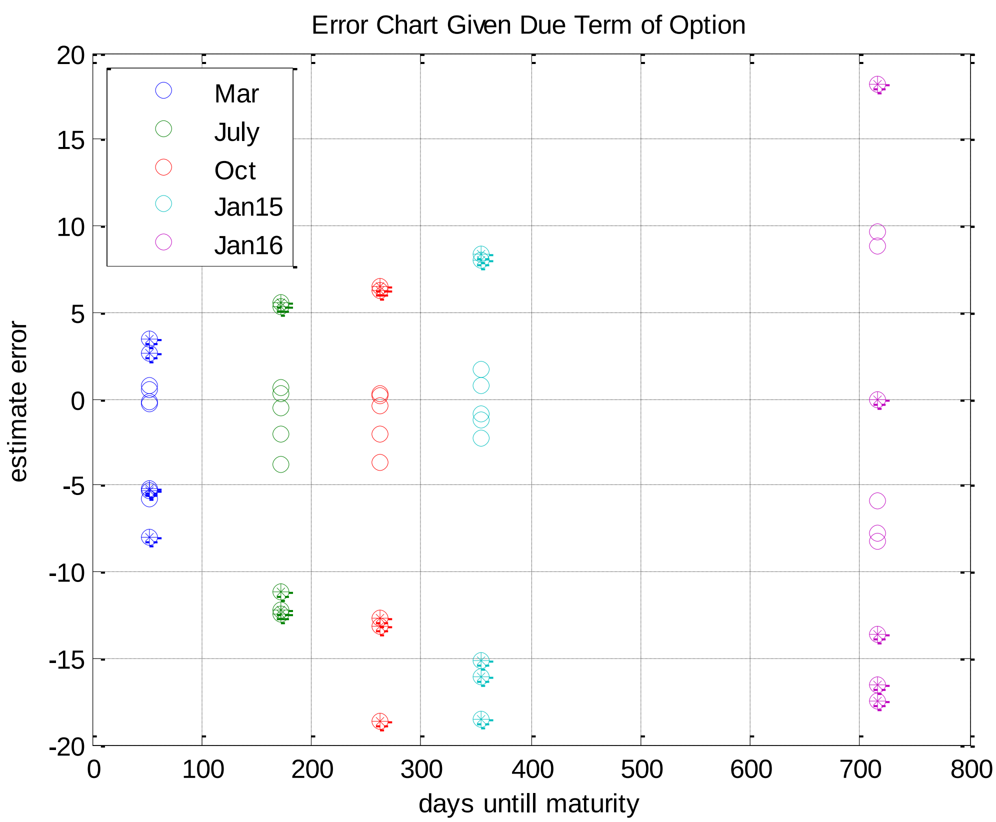

Figure V: Error Chart for AAPL on 1/28/14, where a circle with star represents modified B-S

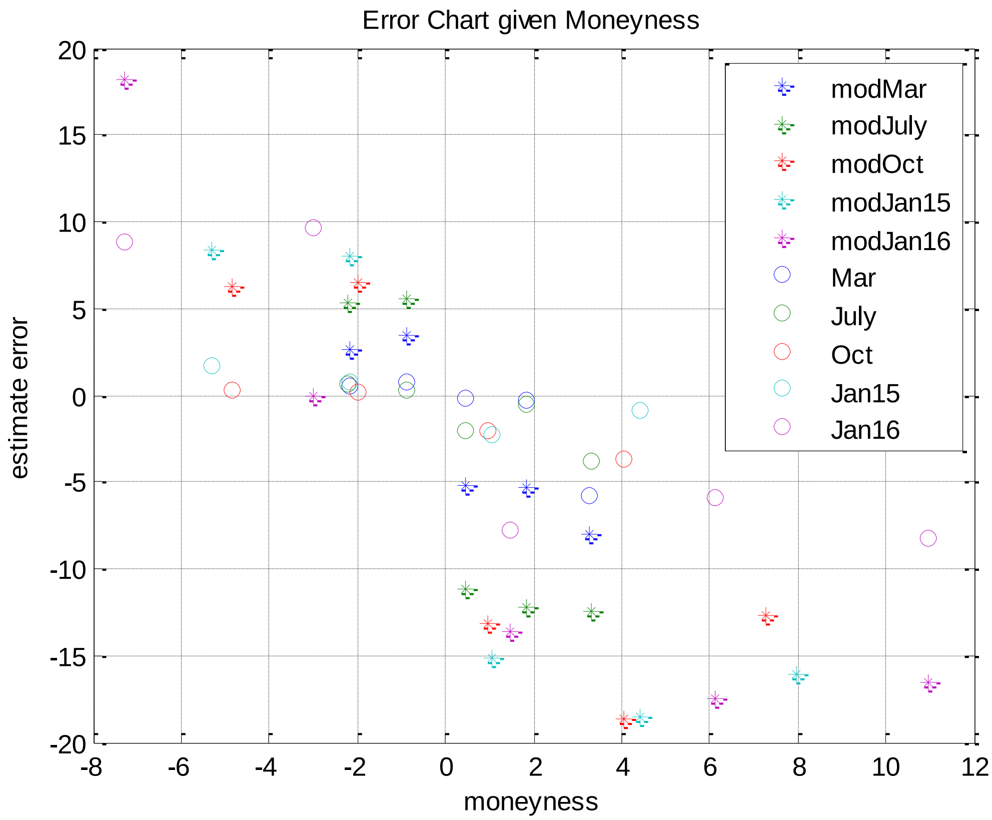

Figure VI: Error Chart for AAPL on 1/28/14

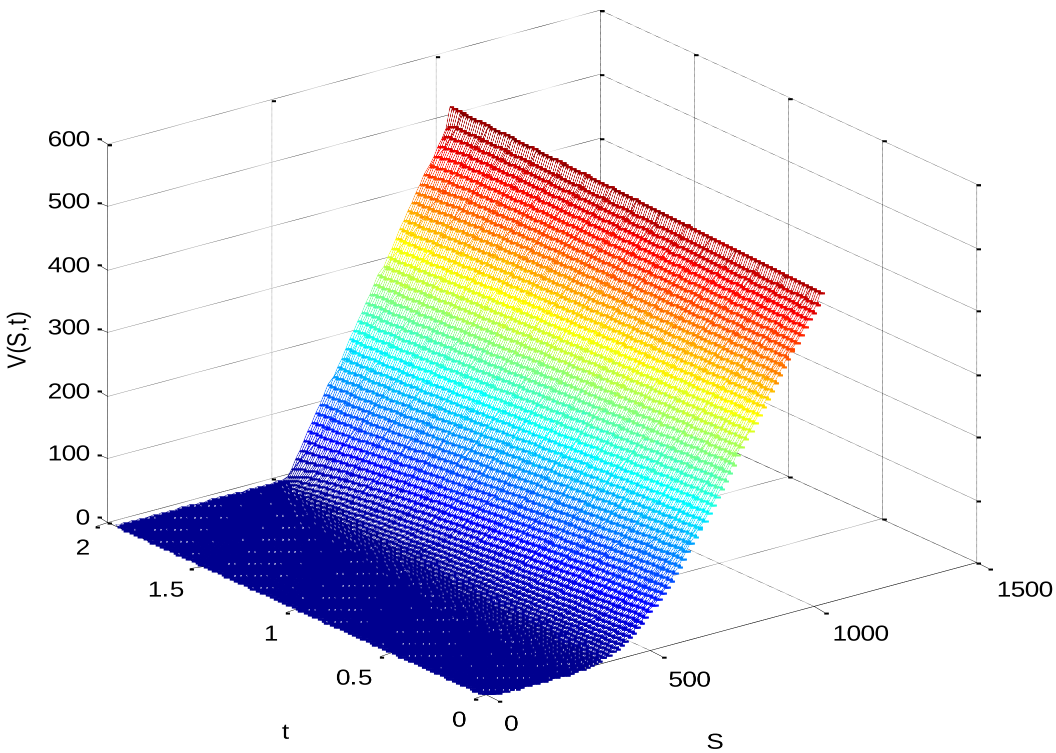

Figure VII: The minimum values for $520 January 16’ call option on AAPL using Modified B-S

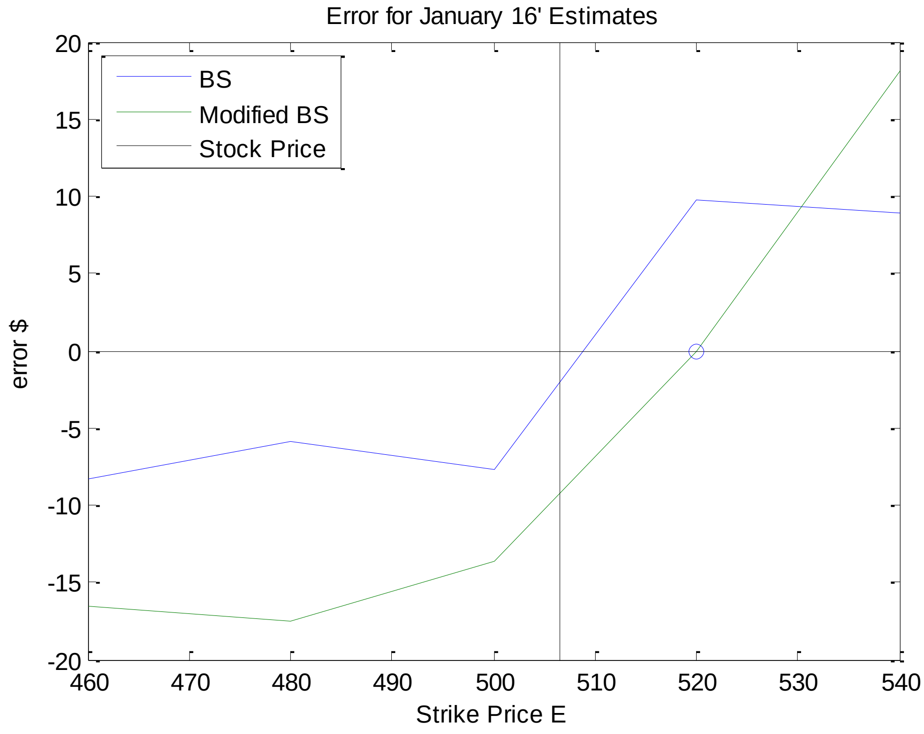

Figure VIII: The Estimate Error for AAPL January 16' option

Table 5: Estimate Comparisons for both Models for AAPL January 16' Option

Date	t	Exercise Price	Price	Observed Price	B-S	Mod. B-S	B-S Error	Mod. B-S Error
1/28/2014	717	520	506.5	59.68	69.3812	59.6795	9.7012	-0.0005
1/29/2014	716	520	500.75	57.76	65.9474	56.2857	8.1874	-1.4743
1/30/2014	715	520	499.78	56.3	65.3147	55.6565	9.0147	-0.6435
2/3/2014	711	520	501.53	57.1	66.0265	56.3951	8.9265	-0.7049
2/4/2014	710	520	508.79	60.55	70.2122	60.5112	9.6622	-0.0388
2/5/2014	709	520	512.59	61.4	72.4158	62.7453	11.0158	1.3453
2/6/2014	708	520	512.51	64.35	72.2911	62.6258	7.9411	-1.7242
2/7/2014	707	520	519.68	68.6	76.6044	66.9409	8.0044	-1.6591
2/10/2014	704	520	528.99	69.4	82.2443	72.6395	12.8443	3.2395
2/11/2014	703	520	535.96	75.4	86.6876	77.1072	11.2876	1.7072
2/12/2014	702	520	535.92	76.33	86.5814	77.0058	10.2514	0.6758
2/10/2014	704	540	528.99	64.46	75.0107	65.4434	10.5507	0.9834
2/11/2014	703	540	535.96	68.59	79.1433	69.5986	10.5533	1.0086
2/12/2014	702	540	535.92	70	79.0376	69.499	9.0376	-0.501

We observe that the modified B-S model does in fact do a much better job of pricing longer-term options (702 days or more) that are moderately out-the-money ($0.32 - $20.22) than the classic B-S model.

iii. Explaining the empirical results

Clearly, the difference in the model estimates has to be a result of how we defined volatility σ. Recall, that we assumed that σ was bounded by a known minimum and maximum value (11), and that it was contained and varied inside this interval. We also allowed σ to be a function of the highest spatial derivative (13) resulting in the discontinuous function σ_d. Thus, the discrepancy in the estimates has to deal with the function input, where we model $\left(\frac{\partial^2 v}{\partial s^2}\right)$ as:

$$\left(\frac{\partial^2 v}{\partial s^2}\right) = \frac{\left(\mathrm{v}_{\mathrm{i+1,j-1}} - 2v_{i,j-1} + v_{i-1,j-1}\right)}{\Delta s^2} \tag{19}$$

We then evaluate the signs of $\left(\frac{\partial^2 v}{\partial s^2}\right)$ for the different strike prices on January 28th, 2014 and provide the sign of the final $\left(\frac{\partial^2 v}{\partial s^2}\right)$ in Table 6.

We observe that the sign of $\frac{\partial^2 v}{\partial s^2}$ is positive for in-the-money option and negative for out-the-money options, except for the $520 January 16' option after the 53rd iteration. That is, the sign of $\frac{\partial^2 v}{\partial s^2}$ for the $520 January 16' is negative up to the 53rd time step, and from then on it is positive. Hence, the modified B-S model had the minimal volatility value as an input from the 54th time step until the end of the simulation. Then, by the way we defined volatility (11) the modified model uses a higher volatility value at the beginning of the pricing process, and then switches to the minimum volatility value until the 80th and final time step. We find that this results in a lower price estimate then the classic B-S model. Also, since we have a systematic pattern in which the B-S model overestimates out-the-money options we have that the modified B-S model will result in a lower estimate and a better approximation of the observed market price when the sign of $\frac{\partial^2 v}{\partial s^2}$ changes from negative to positive for out-the-money options. We also note that if the sign of $\frac{\partial^2 v}{\partial s^2}$ does not change, as was the case for the in-the-money and deep out-the-money options, then volatility will be a constant extreme value as defined by (11) and given in Table 2. It then follows that the classic B-S model will more accurately price the option given the systematic pricing pattern discussed in section II holds.

IV. Conclusion

Employing a sound risk management strategy is essential for the health of individual companies as well as the collective economy. We conjecture that this requires accurate modeling in which human behavior is accounted for by having a stochastic volatility input. We tentatively recommend the modification to the B-S model proposed by Avellaneda, Levy, and Parás (1995), and Lorenz, and Qiu (2009) for companies attempting to price financial options with longer to maturity expiration dates that are moderately out-the-money. This recommendation allows volatility to be a function of the highest spatial derivative in which volatility lies in an empirically determined interval that varies according to the sign of $\frac{\partial^2 v}{\partial s^2}$. However, our findings are treated as tentative observations with the need for further verification because the sample size is relatively small – only 10 days of data for a single company. Still, we are excited by the potential application of a modified B-S model that is more efficient at pricing longer-term, moderately out-the-money options.

Table 6: The final sign of $\frac{\partial^2 v}{\partial s^2}$ using Implicit Euler for AAPL on 1/28/14

Exercise Price	March	July	October	January 15'	January 16'
460	+	+	+	+	+
480	+	+	+	+	+
500	+	+	+	+	+
520	-	-	-	-	+
540	-	-	-	-	-

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VI. Appendix:

I. Finding an exact solution to the classic B-S PDE model

We denote $v(s,T)$ from (10) as $v_0(s)$. Now, using the transformation

$$\left.\begin{aligned} \tau &= T-t, \\ x &= \ln(s) + \left(r - \tfrac{1}{2}\sigma^2\right)(T-t), \\ w(x,\tau) &= e^{r(T-t)}v(s,t) \end{aligned}\right\} \tag{19}$$

the B-S PDE equation (9) transform to the well-known and extensively examined heat equation from physics,

$$w_\tau = \tfrac{1}{2}\sigma^2 w_{xx} \tag{20}$$

and the end-condition (10) becomes the following initial condition:

$$w(x,0) = v(s,T) = v_0(s) = v_0(e^x) = w_0(x). \tag{21}$$

Given (20) and (21) it is possible to find an exact solution of:

$$w(x,\tau) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-\frac{(x-y)^2}{2\sigma^2\tau}} w_0(y)dy \tag{22}$$

and transforming back yields,

$$v(s,t) = e^{-r(T-t)} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-\frac{\left(\ln(s)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)-y\right)^2}{2\sigma^2(T-t)}} v_0(e^y)dy. \tag{23}$$

II. B-S Formula

$$\left.\begin{aligned} C &= SN(d_1) - Ee^{-rt}N(d_2) \\ d_1 &= \frac{\ln\left(\frac{S}{E}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \\ d_2 &= d_1 - \sigma\sqrt{\tau} \end{aligned}\right\} \tag{24}$$

where C is the market value of the call option; $s, E, \sigma, r,$ and τ are defined previously; $N(d_i)$ is the cumulative normal density function evaluated at d_i, $i = 1,2$.

III. Implied Volatility Figures

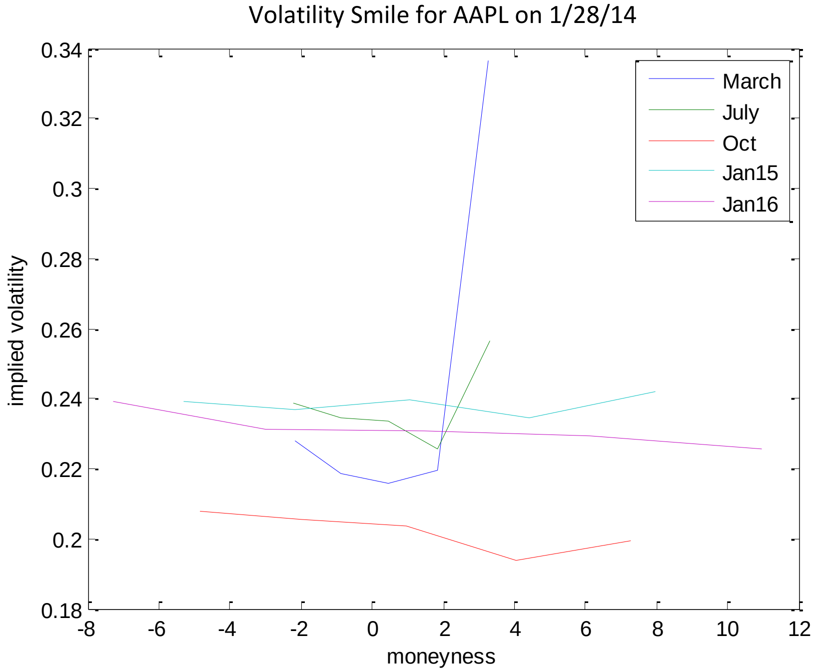

Figure III: Implied volatility against moneyness for AAPL

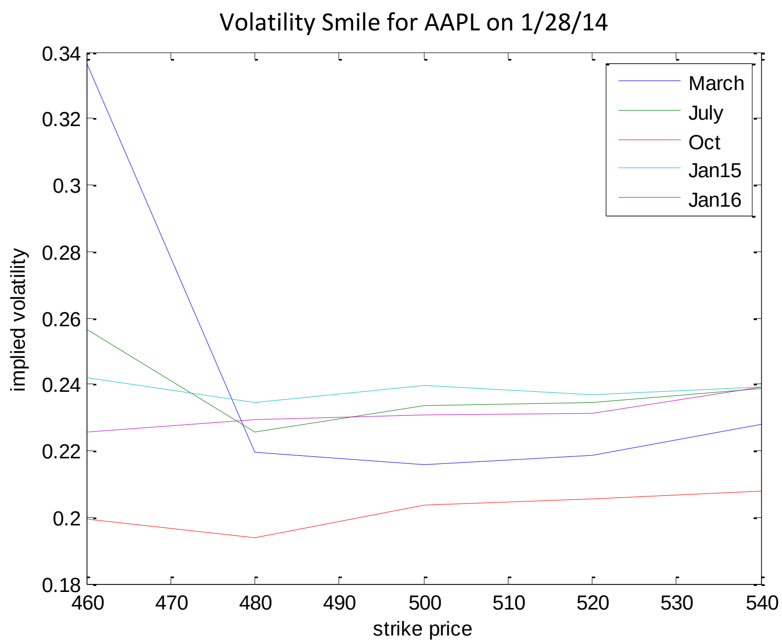

Figure IV: Implied volatility against strike price for AAPL