Minicourse 3:
An Introduction to Fluid Dynamics
Part 1

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Contents

0 Introduction

My goal for this week is to introduce you to several topics in fluid dynamics, and give you an appreciation of the beauty of the field, its diversity, and most importantly its interplay with various areas of mathematics. Starting with elementary vector fields, by the end I hope you have a sense for a variety of fluid flows, ranging from inviscid, potential, viscous, to highly viscous creeping flows. To study these flows we will use mathematics in several areas, such as Calculus, Vector Analysis, Complex Variables, Ordinary differential equations, Partial differential equations, Linear algebra, Numerical methods. I will review what I feel is necessary from these areas, but please interject and interrupt me whenever you would like to see more explanations.

To understand and visualize many of these flows we will use MATLAB in the afternoon laboratories. I will give you detailed instructions on how to prepare your codes. Also, there are several of you that have some experience with MATLAB and by working in groups we can help each other. A useful summary of MATLAB commands is posted on the web, at `http://www.math.unm.edu/ nitsche/courses/375/handouts/mattutorial.pdf` [3]

To start with, let me show you and discuss some sample fluid flows

- **Flow past cylinder**, note dependence of $Re = LU/\nu$
  (from Milton Van Dyke, An album of fluid motion, Parabolic Press, 1982)
- **Flow in a narrowing channel**, see particle trajectories
  (http://www.youtube.com/watch?v=DOUfyDHxkYQ&feature=related, go to minute 3:10)
- **Flow past a flapping plate**, see instantaneous streamlines, trajectories, streaklines
  (from my research)

The topics I will introduce are:
Day 1: Basic concepts of velocity fields. Trajectories, streamlines, streaklines. Streamfunction. Point vortex motion.
Day 2: Vorticity. Governing Equations, inviscid and viscous flow. Boundary layers.
Day 3, morning: Potential flow (potential function, conformal mapping)

Day 3 afternoon, and Days 4 and 5, Professor Vorobieff will lecture on fluid flow, applied to aerodynamics.

Strouhal number $St = fA/U$

What are the blue curves??

1 Basic concepts

We begin today introducing some basic concepts and looking at some apparently randomly made-up velocity fields. On Day 2 we will derive the equations that real-life fluid flows satisfy (approximately). The holy grail in fluid dynamics is to find solutions to these equations, that is, velocity fields that satisfy them given certain initial and boundary conditions. Such solutions, if we can find them, can help better understand fluid flow and help to obtain better designs in engineering applications.

We will then show that the velocity field induced by a set of *point vortices* is a solution of the *inviscid planar Euler Equations*, and use them to construct solutions that satisfy certain boundary conditions.

To start with however, we will simply inspect velocity fields without making any direct connection to real fluid flows.

1.1 Velocity fields

Let D be a region in two or three-dimensional space filled with a fluid. Let $\mathbf{x} \in D$. It is described by its Cartesian coordinates $\mathbf{x} = (x, y, z) = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$. Alternatively, it could be described by cylindrical coordinates $\mathbf{x} = (r, \theta, z) = r\mathbf{e}_r + \theta\mathbf{e}_\theta + z\mathbf{e}_z$, spherical coordinates $\mathbf{x} = (\rho, \theta, \psi) = \rho\mathbf{e}_\rho + \theta\mathbf{e}_\theta + \psi\mathbf{e}_\psi$, or in terms of any other basis of $\Re^3$. So, if you write $\mathbf{x} = (1, 2, 3)$ we don't know what point this refers to unless the basis is specified. But, unless otherwise mentioned, we will imply Cartesian coordinates.

Imagine a fluid particle at position $\mathbf{x}$ at time t. It moves with a well-defined velocity $\mathbf{u}(\mathbf{x}, t) = \langle u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)\rangle$. The velocity $\mathbf{u}$ is a **vector field**. At a given time t, it is a function that assigns to each position $\mathbf{x}$ a vector $\mathbf{u}$. Here are some example vector fields and their graphs

Example 1: $\mathbf{u}(x, y, z) = (U, 0, 0)$, some positive constant U

Example 2: $\mathbf{u}(x, y, z) = (x, y, z)$

Example 3: $\mathbf{u}(x, y, z) = (-y, x, 0)$

Example 4: $\mathbf{u}(x, y, z) = \dfrac{1}{2\pi}\dfrac{(-y, x, 0)}{x^2 + y^2}$ (this flow is induced by what is called a **point vortex**)

Exercise 1: Sketch graph of (a) $\mathbf{u}(x, y, z) = \dfrac{(-y, x, 0)}{x^2 + y^2}$, (b) $\mathbf{u}(x, y, z) = -\dfrac{(x, y, 0)}{\sqrt{x^2 + y^2}}$.

The above examples are very simple velocity fields which we can easily draw by hand. Why? Besides there simple form:

- All but the second examples are **2-dimensional** vector fields. The velocity (i) does not depend on z, and (ii) has zero component in the third dimension. Thus the velocity is fully described by its graph in the 2d plane (even though velocity exists at every point in space).
- For convenience all examples chosen are independent of time. Time independent velocities are called **steady** velocity fields. In such fields it is easy to view particle trajectories (see below). Flows that are not steady are called **unsteady**.

1.2 Streamlines, particle trajectories, streaklines

Suppose you freeze a velocity $\mathbf{u}(\mathbf{x}, t)$ at a fixed time and look at the instantaneous vector field at that instant. Then you can visualize curves $\mathbf{x}(s)$ that are tangent to the velocity at every point in space. That is, they satisfy that their tangent vector equals $\mathbf{u}(\mathbf{x}, t)$ at the given time. If in addition we specify one point on such a curve, the curves $\mathbf{x}(s)$ are given by the unique solution to the initial value problem (IVP)

$$\frac{d\mathbf{x}}{ds} = \mathbf{u}(\mathbf{x}, t = constant) \ , \quad \mathbf{x}(s_0) = \mathbf{x}_0 \ . \tag{1.1}$$

These curves are called **integral curves** of the vector field, or **streamlines** of the flow.

Exercise: Draw some streamlines of the above vector fields.

Exercise: Find the streamlines of the above vector fields. by solving the ordinary differential equation (1.1).

If a flow is steady, the streamlines do not change in time. If a flow is unsteady, the streamlines change with time. The streamlines at a fixed time t are called the **instantaneous streamlines** at time t.

Now, suppose a fluid particle at time t has position $\mathbf{x}(t)$. Then it moves with velocity $\mathbf{u}(\mathbf{x}, t)$. That is, it satisfies the ordinary differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t) \ , \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1.2}$$

This implies that at every instant in time its motion is tangent to the velocity field. The solution to this IVP is the **particle trajectory** of the particle initially at x_0.

If the velocity field is steady (time-independent) as in the above examples, then the particle trajectories are simply the streamlines of the velocity and are easy to visualize. However, if the velocity is not steady, that is, the velocity at a fixed point $\mathbf{x}$ changes in time, then the particle trajectories are not equal to the streamlines (the solutions to 1.1 and 1.2 are not the same) and are harder to visualize. We can visualize both streamlines and trajectories for any given field $u(\mathbf{x}, t)$ by solving the odes (1.1,1.2) numerically. Will do this in afternoon.

Finally, you may have heard of the word **streakline**. Suppose you have a possibly unsteady flow, such as the flow behind an oscillating rudder. If you insert dye into the flow at a fixed point x_0 in space at all times, the dye will show you the position at time t of all the particles which at time $t < t_0$ where at x_0. This is referred to as a streakline.

A streakline is also used to denote the position at time $t > 0$ of all particles which at time $t = 0$ lay on a curve. This is what you would see if you inject dye along a curve at time 0 and observed where this curve moves to at time $t > 0$. For us, what is important is: streaklines are not trajectories (nor streamlines) but are sample of a whole set of trajectories at a given instant, and reflect what you would see experimentally by injecting dye into the fluid. Of course, if the flow is steady, streaklines=trajectories=streamlines.

1.3 Divergence of a velocity field

The **divergence** of a vector field $\mathbf{u} = (u, v, w)$ is defined as

$$\text{div}\,\mathbf{u} = \nabla \cdot \mathbf{u} = u_x + v_y + w_z \tag{1.3}$$

Let's compute the divergence of the sample vectorfields in §1.1. Does anyone know the meaning of this scalar function associated to the vector field $\mathbf{u}$? The meaning of $\nabla \cdot \mathbf{u}$ can be deduced from the **Divergence theorem**,

$$\int_V \nabla \cdot \mathbf{u}\, dV = \int_D \mathbf{u} \cdot \mathbf{n}\, dS \tag{1.4}$$

where V is a closed volume bounded by D, and $\mathbf{n}$ is the outward unit normal vector. In words: "the volume integral of the divergence of $\mathbf{u}$ equals the flux (or flow rate) of $\mathbf{u}$ out of D." (Why flux? Draw figure showing that RHS = volume of fluid leaving domain per time interval.) Now imagine V to be a small volume within which $\nabla \cdot \mathbf{u}$ is approximately constant. Taking the limit as the volume size $\to 0$ you see that the divergence is the net outflux per unit volume.

In particular, if $\nabla \cdot \mathbf{u} > 0$ at a point, then there is a positive net outflux out of small neighbourhood, that is, the flow is **expanding**. If $\nabla \cdot \mathbf{u} < 0$, the flow is **compressing**. A flow for which $\nabla \cdot \mathbf{u} = 0$ everywhere is called **incompressible**. Check the divergence of sample vector fields.

1.4 Vorticity, circulation, point vortex

The **vorticity** of a vector field is defined as the curl of $\mathbf{u}$,

$$\omega = \nabla \times u \tag{1.5}$$

The meaning of the vorticity can be deduced from **Stokes Theorem**,

$$\int_C \mathbf{u} \cdot \mathbf{T}\, ds = \int_D (\nabla \times \mathbf{u}) \cdot \mathbf{n}\, dS \tag{1.6}$$

where D is a surface bounded by C, $\mathbf{T}$ is a unit vector tangent to C, $\mathbf{n}$ is a unit vector normal to S, and $\mathbf{T}$ points counterclockwise with respect to $\mathbf{n}$ ($\mathbf{T}, \mathbf{n}$ satisfies right hand rule). If we now consider a small circular surface D of radius R *normal to the vorticity* on which the vorticity is approximately constant, then the magnitude of the RHS is $|\omega|\pi R^2 = u_{av} 2\pi R$, where u_{av} is the average counterclockwise velocity component. Therefore the flow has an average component that rotates around the vector ω with angular velocity

$$\frac{u_{av}}{R} \approx \frac{|\omega|}{2}$$

The approximation becomes = in the limit as $R \to 0$. Thus, vorticity measures the amount of **rotation** of the flow.

A flow in which $\nabla \times \mathbf{u} = 0$ everywhere is called **irrotational**.

For any oriented curve C, the line integral $\Gamma_C = \int_C \mathbf{u} \cdot \mathbf{T}\, ds$ is called the **circulation** of the flow around C. Note that the orientation of the curve matters, since it determines the sign of the tangent vector, and $\Gamma_{-C} = -\Gamma_C$. By Stokes Theorem the circulation equals the integral normal vorticity in a surface bounded by the curve.

Finally, some comments about planar 2-dimensional flows of the form $\mathbf{u}(x, y, z) = (u(x, y), v(x, y), 0)$. Since the velocity does not depend on z we commonly drop the z-dependence and write $\mathbf{u}(x, y) = (u(x, y), v(x, y))$. However, remember that these flows are defined at all point in space. Planar 2d simply means invariance and zero flow in z-direction. The vorticity of a planar flow is

$$\omega = \nabla \times \mathbf{u} = (0, 0, v_x - u_y) \tag{1.7}$$

Thus it is a vector that points in the z-direction. This makes sense since the flow rotates in a plane normal to this vector and it can only rotate in the x-y plane. In planar flow the convention is to let $\mathbf{n} = (0, 0, 1)$ and $\mathbf{T}$ point in the counterclockwise direction (so the pair $\mathbf{n}, \mathbf{T}$ satisfies the right hand rule). As a result the circulation in the plane is defined with counterclockwise orientation and, by Stokes Theorem,

$$\oint_C \mathbf{u} \cdot \mathbf{T}\, ds = \oint_C u\, dy - v\, dx = \int_D v_x - u_y dA \tag{1.8}$$

The small circle around the integral sign denotes counterclockwise orientation of a closed curve. You may recognize this planar version of Stokes Theorem as Greens Theorem that we covered in vector calculus (Math 264).

Note: we refer to these 2d flows as planar to distinguish them from for example axisymmetric or flows on the sphere, in which there are also only two dimensions.

Exercise 4: Compute the vorticity of the sample vector fields above.

Note that in Example 4, even though the flow is rotating, the vorticity is zero everywhere, except at the origin! As we know, vorticity measures local rotation, and in this case an infinitesimal parcel of fluid away from the origin circles about the origin but without rotating about its axis. However, the circulation about any curve enclosing the origin is nonzero.

Exercise 5: (a) Compute the circulation around the unit circle centered at the origin. (b) Use Stokes Theorem to show that the circulation around any other curve inclosing the origin is the same, namely 1.

By Stokes theorem, this implies that the vorticity is zero everywhere except at the origin, but its integral over any region enclosing the origin is 1. This is the definition of the δ-function. The "δ-function" is not really a function. It is defined by what happens when you integrate it:

$$\int_D \delta(x,y)dA = \begin{cases} 0 \text{ if } (0,0) \notin D \\ 1 \text{ if } (0,0) \in D \end{cases} \tag{1.9}$$

It was introduced and used mainly by physicists (Poisson 1815, Fourier 1822, Cauchy 1823, 1827, Kirchoff 1882,1891, Heaviside 1893, 1899, Paul Dirac 1926) but was dismissed by many mathematicians as non-rigorous, until the theory of distributions was developed (Sobolev 1935) which makes it a rigourous mathematical object (not a function, but a distribution) (Schwartz 1945-50). (I got this from an article by Balakrishnan on the web, Resonance, Aug 2003.

1.5 Streamfunction for planar Incompressible Flow

We need to introduce an important quantity that exists in incompressible flows. From Vector Analysis you know that for incompressible 2D flows $\mathbf{u}(x,y,t) = (u(x,y,t), v(x,y,t))$ with $\nabla \cdot \mathbf{u} = u_x + v_y = 0$ there exists a function $\psi(x,y,t)$ such that

$$\frac{\partial \psi}{\partial x} = -v, \frac{\partial \psi}{\partial y} = u. \tag{1.10}$$

or, $\mathbf{u} = (\psi_y, -\psi_x) = \nabla^{\perp}\psi$. (To prove: define $\psi(x,y)$ as a line integral $\int_{(0,0)}^{(x,y)} udy - vdx$, show it is path independent so that this is well-defined, find the partial derivatives.) This function is called the **streamfunction** since for fixed t, its level curves are streamlines of the flow. Why? Let $x(s), y(s)$ be a streamline: $x'(s) = u(x, y, t), y'(s) = v(x, y, t)$, then

$$\frac{d}{ds}\left[\psi(x(s), y(s))\right] = \frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds} = -vu + uv = 0$$

In particular, on the boundary $\psi = 0$.

Exercise: Find streamfunction for the incompressible examples on page 4. and plot its level curves. Write down streamfunction for N point vortices.

A streamfunction also exists in 3D, although we skip the details here.

1.6 LAB PROJECT: Streamlines

- Streamlines
  (a) We visited Professor Vorobieff's laboratory and saw streamlines for flow past cylinder, and flow past a symmetric airfoil at various angles of attack.
  (b) We plotted streamlines for a set of point vortices. These show the instantaneous velocity field induced by these points. A sample of your results: 2 pts, counterclockwise
  (c) 2 pts, $\Gamma_1 = -\Gamma_2$ %item 2 pts, $\Gamma_1 < 0 < \Gamma_2$
  (d) 2 pts, $\Gamma_1 = \Gamma_2$
  (e) 2 pts, $0 < \Gamma_1 < \Gamma_2$
  (f) 3 pts, of your choice
  (g) any other of your choice

  Turn in plots that specify position and stregths of pts

Sample MATLAB code to plot streamlines

```
% Set position and strength of N point vortices.
x=\ldots, y=\ldots, gam=\ldots

% Set grid, and compute streamfunction on grid
xx = -2:01:2;  yy=-2:01:2;
[xg,yg]=meshgrid(xx,yy); %what does this do?
psi=psifun(xg,yg,x,y,gam);

% plot level curves of streamfunction
contour(x,y,z,20), pause    %what does this do?
axis equal
```

2 Inviscid, incompressible flow

Today we will present equations that model inviscid, incompressible flow, the Euler Equations, and interpret them. First, however, lets take another look at a velocity field $\mathbf{u}(\mathbf{x}, t)$, and decompose it using Taylor's theorem, to better understand the local behaviour of a fluid. We then introduce the material derivative of a function evaluated on a particle travelling with the fluid velocity. After this we present the physical principles from which the Euler Equations are derived, and the resulting equations. The derivation is included for completeness (but we did not have time to do this in class).

2.1 Decomposition of velocity field, strain rates

The meaning of the vorticity is also well elucidated by the following. Consider the velocity $\mathbf{u}(\mathbf{y})$, where $\mathbf{y} = \mathbf{x} + \mathbf{h}$ is near a basepoint $\mathbf{x}$. That is, $h = |\mathbf{h}|$ is small. (Here, for simplicity, I omitted the time dependence of $\mathbf{u}$.) Using Taylor series expansion about $\mathbf{x}$ can show that

$$\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) + D(\mathbf{x})\mathbf{h} + \frac{1}{2}\omega(\mathbf{x}) \times \mathbf{h} + O(h^2) \tag{2.1}$$

where D is a symmetric matrix.

Exercise 2: Show (2.1). (a) write down Taylor series in 1 variable; (b) write down Taylor series in 3 variables; (c) write (b) in vector form $\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) + (\nabla\mathbf{u})\,\mathbf{h} + O(h^2)$ (d) split matrix $\nabla\mathbf{u}$ into symmetric and antisymmetric parts

Thus, locally, near $\mathbf{x}$, the velocity is a sum of 3 components. Lets look at each component, by looking at the particle motion due to each component separately. Let $\mathbf{x}$ be a fixed vector, as before. We want to look at the velocity at $\mathbf{y}$, namely $d\mathbf{y}/dt$. Note that since $\mathbf{x}$ is constant

$$\frac{d\mathbf{y}}{dt} = \frac{d\mathbf{h}}{dt}$$

So the question is, what is the solution to the three parts

$$\frac{d\mathbf{h}}{dt} = \mathbf{u}(\mathbf{x}) \tag{2.2a}$$

$$\frac{d\mathbf{h}}{dt} = D(\mathbf{x})\mathbf{h} \tag{2.2b}$$

$$\frac{d\mathbf{h}}{dt} = \frac{1}{2}\omega(\mathbf{x}) \times \mathbf{h} \tag{2.2c}$$

(For (2.2b) we need to diagonalize D. Use linear algebra results for symmetric matrices.) The results of above shows that to first order in $\mathbf{h}$, the velocity is a sum of a translation, a deformation and a rotation. The matrix D is called the **deformation matrix** (or **rate of**

strain tensor), its eigenvalues and eigenvectors are the **principal strainrates** of **principal axes of strain**, respectively.

One can investigate the change of volume of a small parcel of fluid under the strain field D in a small time Δt. Result: to first order in Δt, volume is amplified by factor $1+(\lambda_1+\lambda_2+\lambda_3)\Delta t$. This again indicates significance of divergence of the flow. (Why?)

2.2 The Material derivative

We now derive a differentiation operator we need to be familiar with to understand the following Euler equations.

Suppose $\mathbf{x}(t) = (x(t), y(t), z(t))$ is the trajectory of a fluid particle, and $f(\mathbf{x}(t), t)$ is the value of some quantity assigned to the particle. For example, it could be the density or the temperature at that particle (scalar functions), or the velocity at that particle (a vector valued function). Note that the function $f(\mathbf{x}(t), t) = F(t)$ is a function of time only. We would like to know how this quantity changes in time, that is, we want to find dF/dt. This derivative measures the change of f in time on the particle $\mathbf{x}(t)$. For example, if f is temperature, then dF/dt would be the rate of change of temperature that you would feel if you were sitting on the particle.

Such quantities that are described by their values on a moving particle are called **Lagrangian variables**. Their derivative with respect to time is called the **material derivative** since it denotes changes on a material particle, and is often denoted by Df/DT or df/dt. Using the chain rule and using the fact that $d\mathbf{x}/dt = \mathbf{u}(\mathbf{x}(t), t)$ we find that

$$
\begin{aligned}
\frac{dF}{dt} &= \frac{D}{Dt}\big[f(x(t), y(t), z(t), t)\big] \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t} \\
&= \frac{\partial f}{\partial x}u(x,y,z,t) + \frac{\partial f}{\partial y}v(x,y,z,t) + \frac{\partial f}{\partial z}w(x,y,z,t) + \frac{\partial f}{\partial t} \\
&= \big[(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla)f\big](x(t), y(t), z(t), t)
\end{aligned}
\tag{2.3}
$$

The resulting formula for the differentiation operator D/Dt (or d/dt) can be abreviated as

$$
\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla \tag{2.4}
$$

For example, the acceleration of a particle at $\mathbf{x}(t)$ is the time derivative of its velocity

$$
\frac{D\mathbf{u}}{Dt} = (\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla)\mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} \tag{2.5}
$$

The basic idea is simple (chain rule), just make sure you understand when to use ∂ and when you use d. In (2.5) the brackets around $\mathbf{u}\cdot\nabla$ denote that you take this dot product first. Otherwise you may read it as $\mathbf{u}\cdot(\nabla\mathbf{u})$ and it is a little confusing what the gradient of a vector is.

2.3 The Euler Equations

In this section we derive equations that model the behaviour of real fluid flows. The model does not account for the effects of viscosity and is thus a good approximation for flows in which these effects are negligible. Viscous effects may be negligible for example far from walls bounding the fluid domain, if the viscosity is very small, or in superfluids, where viscosity is practically zero. We will discuss the effects of viscosity, particularly near walls, later in §4.

The equations modelling fluid flows are based on physical principles that real flows satisfy. The equations in this section were derived by Euler in the 1750s. For incompressible flows, which is mainly what we will consider, he obtained the equations in their present form. For compressible equations, one needs a further equation which was obtained only much later.

We will then show that the planar point vortex flows discussed in the previous section are solutions to the Euler Equations (and thus approximate real fluids which are approximately planar and have negligible viscosity).

2.3.1 Conservation of mass

The physical principle is: *mass is neither created nor destroyed.* How do we obtain an governing equation from this? Consider a fixed region W in the fluid. The boundary of W is a surface denoted by ∂W. The physical principle states that *the rate of change of mass in W = the rate at which mass crosses ∂W* or, more precisely

$$\textit{the rate of increase of mass in } W = \textit{the rate at which mass enters } \partial W \tag{PPI}$$

The mass in W is

$$\lim_{||\Delta V||\to 0}\sum_k \rho_k \Delta V_k = \int_W \rho(\mathbf{x},t)dV(\mathbf{x}) = m(t) \tag{2.6}$$

that is, it is the volume integral of density over the region W. Here we are assuming that there is a well-defined density $\rho(x,t)$ at every point x. This is called the **continuum assumption**. We will also assume that the velocity and density are sufficiently smooth to apply the main operations of calculus.

The mass in W can depends on time, since the density at a point depends on time. For example if the mass in W increases the density inside has to increase. The left hand side of the physical principle PPI is the rate of change of the mass

$$\frac{dm}{dt} = \frac{d}{dt}\int\limits_W \rho(\mathbf{x},t)dV(\mathbf{x}) = \int\limits_W \frac{\partial\rho}{\partial t}(\mathbf{x},t)dV(\mathbf{x}) \tag{2.7}$$

(The time derivative enters as a partial derivative since the region of integration W is time independent.)

To find an expression for the right hand side of PPI we look at a small piece of the boundary, ΔS, and assume that ρ, $\mathbf{u}$ and $\mathbf{n}$ are constant on this piece. This assumption holds in the limit as $\Delta S \to 0$. Then (show figure in class) we see that the volume of fluid entering through ΔS in time Δt is $-\mathbf{u}\cdot\mathbf{n}\Delta S\Delta t$. So the mass entering in this time is $-\rho_k\mathbf{u}_k\cdot\mathbf{n}_k\Delta S\Delta t$. Dividing by Δt and taking the limit as $\Delta t \to 0$ (to get the rate at which mass enters) and summing over all ΔS, then taking limit as $\Delta S \to 0$ get

$$-\lim_{||\Delta_S||\to 0}\sum_k \rho_k\mathbf{u}_k\cdot\mathbf{n}_k\Delta S_k = -\int\limits_{\partial W}(\rho\mathbf{u}\cdot\mathbf{n})(\mathbf{x},t)dS(\mathbf{x}) \tag{2.8}$$

According to PPI, we set (2.6) equal to (2.5) to get

$$\int\limits_W \frac{\partial\rho}{\partial t}(\mathbf{x},t)dV(\mathbf{x}) = -\int\limits_{\partial W}(\rho\mathbf{u}\cdot\mathbf{n})(\mathbf{x},t)dS(\mathbf{x}) \tag{2.9}$$

or, after applying the Divergence Theorem

$$\int\limits_W [\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})](\mathbf{x},t)dV(\mathbf{x}) = 0 \tag{2.10}$$

Equation (2.10) states that the integral is zero for any region of integration W. If we assume the integrand is continuous, this implies that the integrand must be identically zero.

Exercise 1: Proof this by contradiction

As a result we get the law of **conservation of mass**.

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{2.11}$$

which can be rewritten as

$$\frac{D\rho}{Dt} = -(\nabla\cdot\mathbf{u})\rho \tag{2.12}$$

Exercise 2: Deduce (2.12) from (2.11). (You are in fact proving a vector identity. There are many identities often used, some are listed in the appendix as examples. Which one of those did you prove in this exercise?).

Again, this makes perfect sense. If $\nabla \cdot \mathbf{u} > 0$ then the fluid near a material particle expands and the density at the material particle decreases. That is, if $\nabla \cdot \mathbf{u} > 0$ then $D\rho/DT < 0$ (using that the density is always positive, $\rho > 0$).

Equation (2.11) is our first equation of motion. In general we do not know the fluid velocity $\mathbf{u} = (u, v, w)$ nor the density function. These are 4 unknowns and the above is only one equation, which is now enough to solve for the unknowns.

2.3.2 Conservation of momentum (we did not cover this in detail in class)

The next equation of motion (whose derivation we will only outline) is based on the physical principle of conservation of momentum *the rate of change of momentum of W = force applied to it.*, or, more precisely

the rate of increase of momentum of W = force applied to it in inward direction. (PPII)

This is Newton's second law, which you may be more familiar with when mass is constant: $F = ma$. (Momentum is mv, so for a nonconstant mass $m(t)$ moving with velocity v this law reads $F = d(mv)/dt$. Here the equation is not that simple because different portions of the fluid move with different velocities.)

We apply this law to a region W_t moving with the fluid. The left hand side of PPII is

$$\frac{d}{dt}\int_{W_t} \rho \mathbf{u} dV = \int_{W_t} \rho \frac{D\mathbf{u}}{Dt} dV \tag{2.13}$$

Proving this equality is not trivial and we will skip it. Having proved the conservation of mass in detail we only wish to indicate how the final equations follow from physical principles.

For the right hand side of PPII we will assume that the force of stress on the boundary of W acts normal to it, and is given by the pressure, so that the

$$\text{force across } \partial W \text{ per unit area} = p(\mathbf{x}(t), t)\mathbf{n} \; .$$

As a result the total inward force on W is

$$-\int_{\partial W_t} p\mathbf{n} dA = -\int_{W_t} \nabla p \, dV \tag{2.14}$$

where again, we used the divergence theorem and skipped some details to obtain the right hand side. By equating (2.13) and (2.14) and assuming the integrands are continuous we obtain the equation of **conservation of momentum**,

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p \tag{2.15}$$

where we ignored other forces such as gravity that could be acting on the body.

2.3.3 Euler equations for incompressible flow

Equations (2.11, 2.15) are a set of 4 equations, but now we have 5 unknowns!, density ρ, velocity $\mathbf{u}$, and pressure p. For incompressible flow, Euler completed the system of equations by $\nabla \cdot \mathbf{u} = 0$. For general flows, the system is completed by enforcing concervation of energy. It turns out that the condition of incompressibility $\nabla \cdot \mathbf{u} = 0$ is equivalent to conservation of kinetic energy. So now we have 5 equations for 5 unknowns that hold in the domain of the fluid D. These need to be complemented by boundary conditions. The boundary conditions $\mathbf{u} \cdot \mathbf{n} = \mathbf{U}_{wall} \cdot \mathbf{n}$ on ∂D, where $\mathbf{U}_{wall}$ is the velocity of the wall, enforce that there is no flow going through the walls. (If wall steady, condition is $\mathbf{u} \cdot \mathbf{n} = 0$.) In summary, Eulers equations for incompressible flow, ignoring other external body forces, are

$$
\begin{aligned}
\frac{D\rho}{Dt} &= 0 && \text{(2.16a)}\\
\rho\frac{D\mathbf{u}}{Dt} &= -\nabla p && \text{(2.16b)}\\
\nabla \cdot \mathbf{u} &= 0 && \text{(2.16c)}\\
\mathbf{u} \cdot \mathbf{n} &= \mathbf{U}_{wall} \cdot \mathbf{n} \text{ on } \partial D && \text{(2.16d)}
\end{aligned}
$$

with initial conditions at $t = 0$, $\mathbf{u}(\mathbf{x}, 0)$, given. Here the first 5 equations hold in the interior D. So, our goal: solve these equations!

2.3.4 Vorticity-streamfunction formulation

The above equations (2.16) are Euler's Equation in the velocity-pressure formulation. They can alternatively be presented in terms of vorticity-streamfunction, which we will do now for the case of constant density flows.

Note that (in planar 2d),

$$\Delta\psi = -v_x + u_y = -w \ . \tag{2.17}$$

So if the vorticity is known, the streamfunction is solution to a Poisson equation with Dirichlet boundary conditions $\psi = 0$ on ∂D. This solution can be written down in terms of Green's functions of the Poisson equation, which is known for many domains D. (In particular, if $D = \Re^2$, then $\psi = -\frac{1}{2\pi}\int \omega(\mathbf{x}') \log|\mathbf{x}-\mathbf{x}'|\, d\mathbf{x}' \approx -\frac{1}{2\pi} \sim \Gamma_k \log|\mathbf{x}-\mathbf{x}_k|$, that is, approximately a sum of point vortices.) So, if you know the vorticity, then you can find ψ, therefore can find $\mathbf{u}$. Together with the equation (2.19) derived below, this forms a basis of many numerical methods. A streamfunction also exists in 3D, although we skip the details here.

Now, suppose the flow is **homogeneous**, that is, $\rho = \rho_0$ is constant, and take the curl of the balance of momentum equation,

$$\nabla \times [\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p]$$

Using the identities

$$
\begin{aligned}
\nabla \times (\nabla f) &= 0 && (2.18a)\\
\frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) &= \mathbf{u}\times \text{curl}\,\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} && (2.18b)\\
\nabla \times (\mathbf{u}\times\omega) &= \mathbf{u}\ \text{div}\,\omega - \omega\ \text{div}\,\mathbf{u} + (\omega\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\omega && (2.18c)
\end{aligned}
$$

it follows that

$$
\frac{D\omega}{Dt} = -(\omega\cdot\nabla)\mathbf{u} \qquad (2.19)
$$

This equation holds in 3D. It says that the vorticity of a material particle can be stretched by the term on the right hand side. Thus the right hand side is a called the **stretching term**. Because of this term it is not known whether smooth initial vorticity fields in 3D can blow-up develop and develop singularities in finite time. (This is one of the Million Dollar questions!)

For planar 2D flows, the vorticity evolution equation simplifies significantly since the vorticity vector points in the z-direction and the gradient vector $\nabla = (\partial_x, \partial_y)$ lies in the x, y-plane, so that the operator $\omega\cdot\nabla$ is zero. Thus, for homogeneous inviscid incompressible planar flow the **vorticity streamfunction formulation** of the Euler equations is

$$
\begin{aligned}
\frac{D\omega}{Dt} &= 0, && (2.20a)\\
\Delta\psi &= -\omega, u = \psi_y, v = -\psi_x && (2.20b)\\
\psi = 0 \text{ on walls} & && (2.20c)
\end{aligned}
$$

This means the vorticity of a material particle stays constant. No stretching can occur and as a result it can be shown that smooth solutions remain smooth for all times.

2.4 Bernoulli's Theorem

Bernoulli's Theorem: In stationary, homogeneous, inviscid flow, the quantity $\frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho_0}$ is constant along streamlines.

Outline of proof: use vector identities to show that $\nabla(\frac{1}{2}|u|^2 + \frac{p}{\rho_0}) = \mathbf{u}\times(\nabla\times\mathbf{u})$ Let $x(s)$ be a streamline. Then $\frac{1}{2}|u|^2 + \frac{p}{\rho_0}|_{x(s_1)}^{x(s_2)} = \int_{x(s_1)}^{x(s_2)} \nabla(\frac{1}{2}|u|^2 + \frac{p}{\rho_0})\cdot x'(s)\,ds = \int_{x(s_1)}^{x(s_2)} \mathbf{u}\times(\nabla\times\mathbf{u})\cdot x'(s)\,ds$ Why?

Experiment: Blow over paper and between two papers held closely together. Explain your observations.

Explain: how does an airfoil moving at high speeds induce lift?

2.5 Kelvin's Theorem.

Definition: A vortex line in 3d is an integral curve of the vorticity vector field. (in planar flow, the vortex lines are simply straight lines normal to the plane)

The following theorems follow from the Euler equations derived above.

Kelvin's Theorem: The circulation around vortex lines stays constant in time.

We proved this theorem in class.

2.6 Point Vortex Motion

Any flow that satisfies the equations (2.20) solves the Euler Equation. Note that for the 2D point vortices we have been looking at, (2.20a) implies that their circulation remains constant for all times. Thus the solution of the Euler Equations with point vortex initial data is simply given by the motion of these point vortices in their self-induced velocity field. This velocity field is given by their streamfunction, which we computed yesterday.

The flow in Example 4 induced by a δ-function of vorticity is denoted by a point vortex (vorticity is concentrated in a point). We already know that the flow induced by a point vortex is incompressible, irrotational except at the vortex, decays like 1/R, where R is the distance to the vortex (show this). The circulation of a point vortex is called the **point vortex strength**. By translation, the flow induced by a point vortex at (x_0, y_0) of strength Γ is

$$\mathbf{u}(x,y) = \frac{\Gamma}{2\pi} \frac{(-(y-y_0), x-x_0)}{(x-x_0)^2+(y-y_0)^2} \tag{2.21}$$

By superposition, the flow induced by 2 vortices of strength Γ_1, Γ_2 at positions $(x_1, y_1), (x_2, y_2)$ is

$$\mathbf{u}(x,y) = \frac{\Gamma_1}{2\pi} \frac{(-(y-y_1), x-x_1)}{(x-x_1)^2+(y-y_1)^2} + \frac{\Gamma_2}{2\pi} \frac{(-(y-y_2), x-x_2)}{(x-x_2)^2+(y-y_2)^2} \tag{2.22}$$

and the flow induced by N vortices of strength Γ_k and position (x_k, y_k), $k = 1, \ldots, N$ is

$$\mathbf{u}(x,y) = \sum_{k=1}^{N} \frac{\Gamma_k}{2\pi} \frac{(-(y-y_k), x-x_k)}{(x-x_k)^2+(y-y_k)^2} \tag{2.23}$$

where (x, y) is not one of the point vortices.

What if (x, y) is one of the point vortices? How do the points evolve under their self-induced velocity?? The self-induced velocity of a point vortex is defined to be zero (it doesnt move in its radially symmetric flow field). As a result, it only moves due to the velocity field induced by the other point vortices. Thus, the flow induced by N vortices (as before, of strength Γ_k and position (x_k, y_k), $k = 1, \ldots, N$) at one of the points, say (x_j, y_j) is

$$\mathbf{u}(x_j, y_j) = \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{\Gamma_k}{2\pi} \frac{(-(y_j - y_k), x_j - x_k)}{(x_j - x_k)^2 + (y_j - y_k)^2} \tag{2.24}$$

Analyze: what is self-induced motion of one pt vortex, how do 2 pt vortices move in self-induced field (to start with, look at $\Gamma_1 = \pm\Gamma_2$)?

Fact: The motion of $N \leq 3$ vortices is never chaotic (the flow is said to be integrable). However, the motion of particles in the flow induced by 3 or more vortices can be chaotic.

2.7 LAB PROJECT: Point vortex motion

Euler's method: We reviewed Euler's method to approximate the solution of a scalar equation $x' = f(x, t), x(t_0) = x_0$ for the solution $x(t)$

Point vortex motion: We wrote a Matlab code that implements Euler to solve the motion of N point vortices.
We plotted motion of N=2,3 or more vortices and the evolution of the instantaneous streamlines.

The method of images: How about solutions to the Euler Equations in finite domains?

Example: Suppose D is the upper half-plane and the initial condition again consits of N point vortices in D. How do these points move so as to satisfy the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$? Answer: the boundary condition can be enforced by placing N **image vortices** of opposite sign symmetrically across the boundary to the N point vortices in D. Draw Picture! This gives a solution to the Euler Equations in this domain D: 2.14a-c are satisfied by same reasoning as above. 2.14d is also satisfied.

Placing image vorticity outside the domain to yield an incompressible velocity field that satisfies the boundary conditions is referred to as **the method of images**.

1. Place an arbitrary set of point vortices in a domain D, and place image vorticity outside D so that the no-though-flow boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ on ∂D is satisfied for
   (a) D: upper half-plane
   (b) D: first quadrant

(c) D: circle centered on origin with radius a.

(For b,c: guess the position of the image vortices.)

3 Viscous flow. Potential flow

3.1 The Navier Stokes equations (for viscous incompressible flow)

When we derived the Euler Equations we assumed that the forces across a surface were normal to that surface. This does not allow any transfer of momentum across fluid slabs moving past each other with distinct velocity. Or, for that matter, between a steady wall and fluid moving past it with nonzero velocity. We know in practice that this model is unreasonable. Faster particles in one layer will diffuse and impart momentum to slower particles next to them, slower particle will diffuse, impart momentum and slow down the fluid above. So a better model is to assume that the forces also have nonnormal components, as in

$$\text{force on } S \text{ per unit area} = p(x,t)\mathbf{n} + \sigma(x,t)\mathbf{n}$$

where σ is a matrix. Under some assumptions on σ (symmetric, invariant under rotation, proportional to velocity gradients), and after using divergence theorem as in case of (2.13), **conservation of momemtum** for incompressible flows yields the equation

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mu\Delta\mathbf{u} \tag{3.1}$$

where μd_i are the eigenvalues of σ, where d_i are the eigenvalues of the deformation matrix D. Again, we are skipping a lot of details here. The coefficient μ is called the **viscosity of the fluid**, and measures the transfer of momentum in nonnormal direction due to diffusion.

Comparing equation 3.1 to the inviscid Euler equation 2.14b we see that 3.1 has more derivatives and this requires an additional boundary condition over 2.14d. What is required is that the fluid particles not only move parallel to the wall, but stick to the wall

$$\mathbf{u} = \mathbf{U}_{wall} \text{ on } \partial D \tag{3.2}$$

where $\mathbf{U}_{wall}$ is the wall velocity. For incompressible, viscous, homogeneous ($\rho = \rho_0$) flow, this yields the Navier-Stokes equations

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho_0} + \nu\Delta\mathbf{u} \tag{3.3a}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{3.3b}$$

$$\mathbf{u} = \mathbf{U}_{wall} \text{ on } \partial D \tag{3.3c}$$

Here $\nu = \mu/\rho_0$ is called the **kinematic viscosity**. (Note: if flow homogeneous then conservation of mass autmatically satisfied.)

3.1.1 Vorticity generation at walls and separation

Suppose you have parallel viscous flow next to a flat wall. Since the fluid velocity is zero at the wall, and large non-zero away from the wall, a boundary layer is formed in which velocity gradients are large. That is, vorticity is large! whereas it is almost zero in the region of parallel flow.

That is: viscosity generates vorticity at walls.

It can be shown that the boundary layer thickness grows in time as $\delta \sim \sqrt{\nu t}$. This formula also shows that the boundary layer is thicker for larger values of the viscosity ν.

The boundary layer vorticity can separate from the wall near corners or regions of large curvature, thus introducing vorticity that is convected well into the interior of the fluid.

3.1.2 Nondimensionalization: in class exercise

You may think that the effect of viscosity on the fluid behaviour depends solely on the value of the parameter ν. However, the size of the structures involved and the characteristics speeds of the fluid matter as well. For example, dragging a spoon through water at a small speed may be similar to dragging the same spoon though oil at much faster speeds. To determine the relevant physical parameters that determine the flow we *nondimensionalize* the flow.

First we find the dimensions of all variables. In this case they involve length and time, or equivalently, lengths and velocity. Namely

$$[\mathbf{u}] = U \ , \quad [\mathbf{x}] = L \ , \quad [t] = T = L/U \ , \quad [p/\rho_0] = L^2/T^2 = U^2$$

Here, the brackets denote "the dimensions of" what is inside, and U,L,T simply denote that the dimensions are those of velocity, length, or time, respectively. Note that I treated p/ρ_0 as one variable, to avoid having to introduce mass. The dimensions of p/ρ follow by ensuring that the equation is dimensionally correct. Similary, one can deduce that

$$[\nu] = UL$$

Of course, these dimensions also follow from the definition of p and ν used in the derivation.

Now we introduce dimensionless variables denoted by tilde. Let U and L be characteristic lengths and speeds of the flow. (I am being slightly sloppy by reusing these letters..) For example, U could be the speed with which the spoon is moving, and L could be the size of the spoon. Or, if you are studying flow past a plane, U could be the speed of the plane, and L could be its size, or the width of the wing. The point is this is a choice you make, which needs to be specified. Now let

$$\tilde{\mathbf{u}} = \mathbf{u}/U \ , \quad \tilde{\mathbf{x}} = \mathbf{x}/L \ , \quad \tilde{t} = Ut/L \ , \quad \tilde{p} = \frac{p}{\rho_0}/U^2 \ ,$$

be dimensionless variables. Then the first equation of 3.3(a) states that

$$\frac{\partial \tilde{u}U}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial t} + \tilde{u}U\frac{\partial \tilde{u}U}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} + \tilde{v}U\frac{\partial \tilde{u}U}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial y} + \tilde{w}U\frac{\partial \tilde{u}U}{\partial \tilde{z}}\frac{\partial \tilde{z}}{\partial z} = -\frac{\partial \tilde{p}U^2}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} + \nu\frac{\partial^2 \tilde{u}U}{\partial^2 \tilde{x}}(\frac{\partial \tilde{x}}{\partial x})^2 \tag{3.4}$$

and similarly for the other 2 equations in 3.3(a). This reduces to

$$\frac{\partial \tilde{u}U}{\partial \tilde{t}}\frac{U}{L} + \tilde{u}U\frac{\partial \tilde{u}U}{\partial \tilde{x}}\frac{1}{L} + \tilde{v}U\frac{\partial \tilde{u}U}{\partial \tilde{y}}\frac{1}{L} + \tilde{w}U\frac{\partial \tilde{u}U}{\partial \tilde{z}}\frac{1}{L} = -\frac{\partial \tilde{p}U^2}{\partial \tilde{x}}\frac{1}{L} + \nu\frac{\partial^2 \tilde{u}U}{\partial^2 \tilde{x}}(\frac{1}{L})^2 \tag{3.5}$$

After dividing by U^2 and multiplying by L get

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} + \tilde{w}\frac{\partial \tilde{u}}{\partial \tilde{z}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{1}{Re}\frac{\partial^2 \tilde{u}}{\partial^2 \tilde{x}} \tag{3.6}$$

where $Re = \frac{LU}{\nu}$ is called the **Reynolds number**. Similarly we can check that equation 3.3(b) reduces to

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0 \tag{3.7}$$

Now we remove the tildes for convenience and state the Navier-Stokes equation in nondimensional variables

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho_0} + \frac{1}{Re}\Delta\mathbf{u} \tag{3.8a}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{3.8b}$$

$$\mathbf{u} = \mathbf{U}_{wall} \text{ on } \partial D \tag{3.8c}$$

As a result we deduce that the solution to the Navier-Stokes equation remains the same for different flows, up to scaling, as long as the Reynolds number is the same. For example, slow flow past a thin cylinder in water is a small sized version of fast flow past a thick cylinder in water. The nondimensionalization process reveals the relevant (nondimensional) combination of parameters determining the solution of the equation.

3.2 Potential flow

3.2.1 Potential function. Laplace equation.

An irrotational, incompressible flow is called *potential* flow, that is, in potential flow $\nabla \cdot \mathbf{u} = 0$ and $\nabla \times \mathbf{u} = 0$.

If $\nabla \cdot \mathbf{u} = 0$ in a domain D then, as we already know, there exists a streamfunction ψ such that $\mathbf{u} = \nabla \times \psi$. (In 3D, ψ is a vector. In 2d, ψ is a scalar with $\mathbf{u} = \nabla^{\perp}\psi$.) For this, D has to be simply connected, which means that it contains no holes.

If $\nabla \times \mathbf{u} = 0$ in a simply connected domain D then there exists a **potential function** ϕ such that $\mathbf{u} = \nabla\phi$. Can you prove this? It follows that the circulation around any closed curve C in D is zero. (Why?)

The level curves of ϕ are the potential curves. How are the potential curves and streamlines related??

Exercise: Find the streamfunction and the potential functions for
(a) uniform flow $\mathbf{u} = (U, 0, 0)$
(b) The point vortex flow (which is irrotational and incompressible and thus potential away from the origin)

It follows that

$$\Delta\psi = 0 \text{ in D, with } \psi = 0 \text{ on } \Delta D$$

and

$$\Delta\phi = 0 \text{ in D, with } \frac{\partial\phi}{\partial n} = U_{wall} \cdot \mathbf{n} \text{ on } \Delta D$$

That is, both ϕ, ψ satisfy the **Laplace equation** (with different boundary conditions, Dirichlet for ψ and Neumann for ϕ). Such functions are called *harmonic*.

The Laplace equation is a fundamental equation that appears in many applications. It has been much studied and many theoretical results exist about its solutions. There are also many numerical methods to solve the Laplace equation. In 2D (and only in 2D!) there is an analytical method based on complex variables to obtain solutions for certain domains. This is what we'll go over the rest of today.

3.2.2 Review of Complex variables

complex numbers $z = a + ib = r\cos(\theta) + ir\sin\theta = re^{i\theta}$ where $r = \sqrt{a^2 + b^2} = |z|$ and $\theta = \text{atan}(b/a) = arg(z)$. Here we used Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. This follows from Taylor series expansions of these three functions.) So can also write

$$z = |z|e^{iarg(z)} \ .$$

Complex functions $f(z) = u(z) + iv(z), u, v$ real, $z = x + iy$. Examples: z, z^2, polynomials, e^z, $\log z$, $\overline{z}$ (Find real and imaginary components in each case.)

Limits of complex functions. The derivative $f'(z)$. **Cauchy-Riemann Equations**. Expression for the derivative $f'(z) = u_x + iv_x = v_y - iu_y$. If f is differentiable in an (open) neighbourhood of a point, then it is infinitely often differentiable in that region! This is a very strong result. In that case f is said to be **analytic**.

3.2.3 Complex velocity

$z = x + iy$, define **complex velocity** as $F = u - iv$, where $u = u(x, y)$, $v = v(x, y)$

Exercise: Check that if **u** is potential in D, F is analytic in D.

Exercise: Check that (if D simply connected) $W = \phi + i\psi$ satisfies $F = dW/dz$, W analytic. W is called the **complex potential** and ϕ, ψ are called harmonic conjugates.

Exercise: Find complex potential for parallel flow.

Exercise: Find complex potential for pt vortex flow (in domain not including origin)

Exercise: Check that $W(z) = U[z + a^2/z]$ is the complex potential for flow past a circle. Find complex velo and velo at infinity.

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