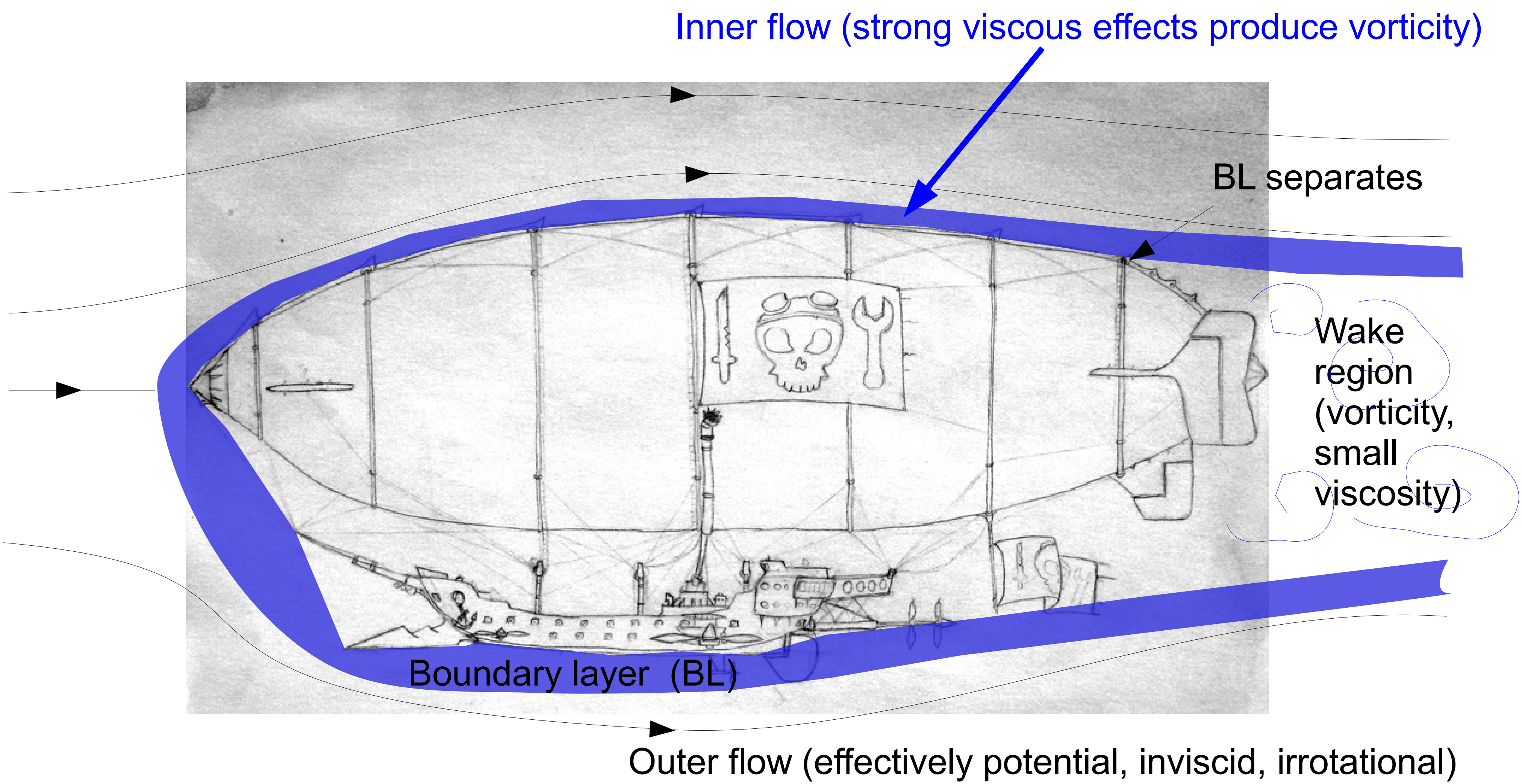
9. Boundary layers

Flow around an arbitrarily-shaped bluff body



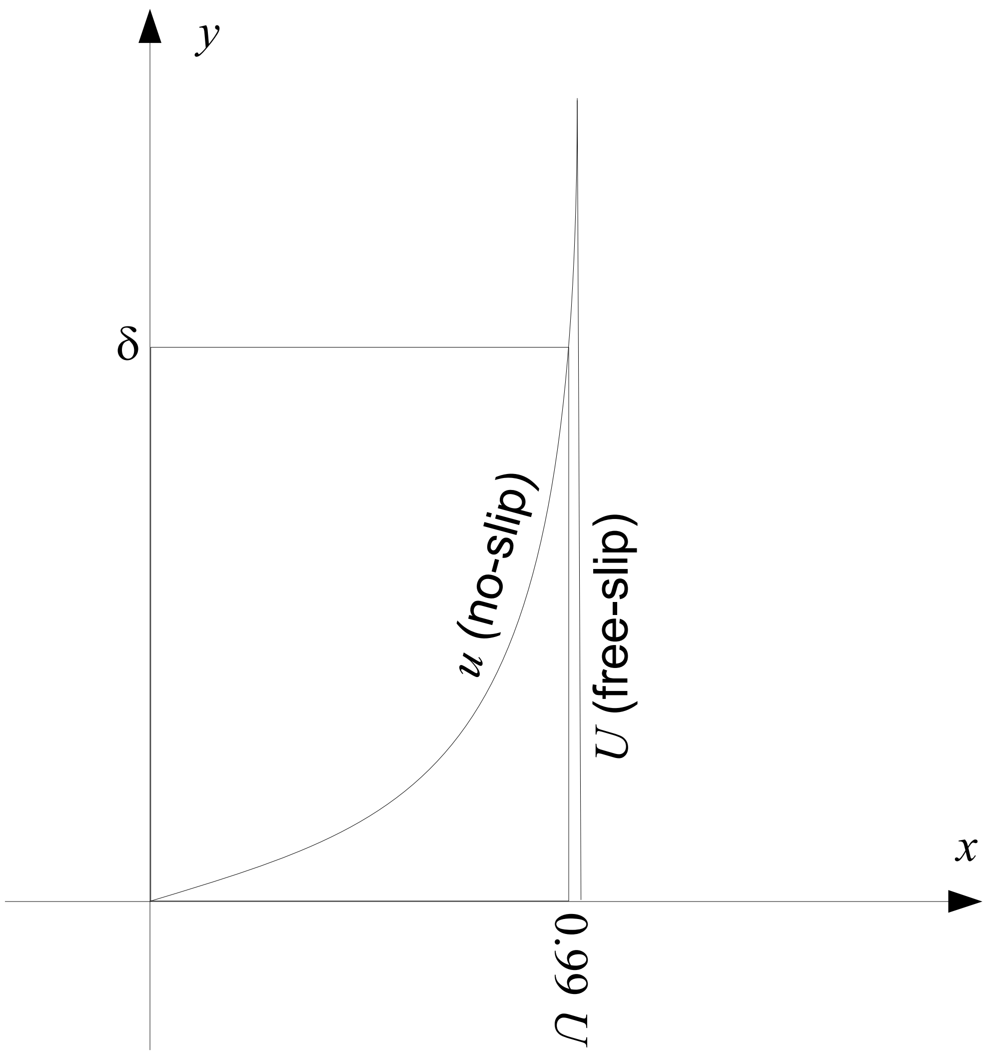
9.1. Boundary layer thickness

- Outer flow solution (ideal): U
- Inner flow: u
- Arbitrary threshold to mark the viscous layer boundary:

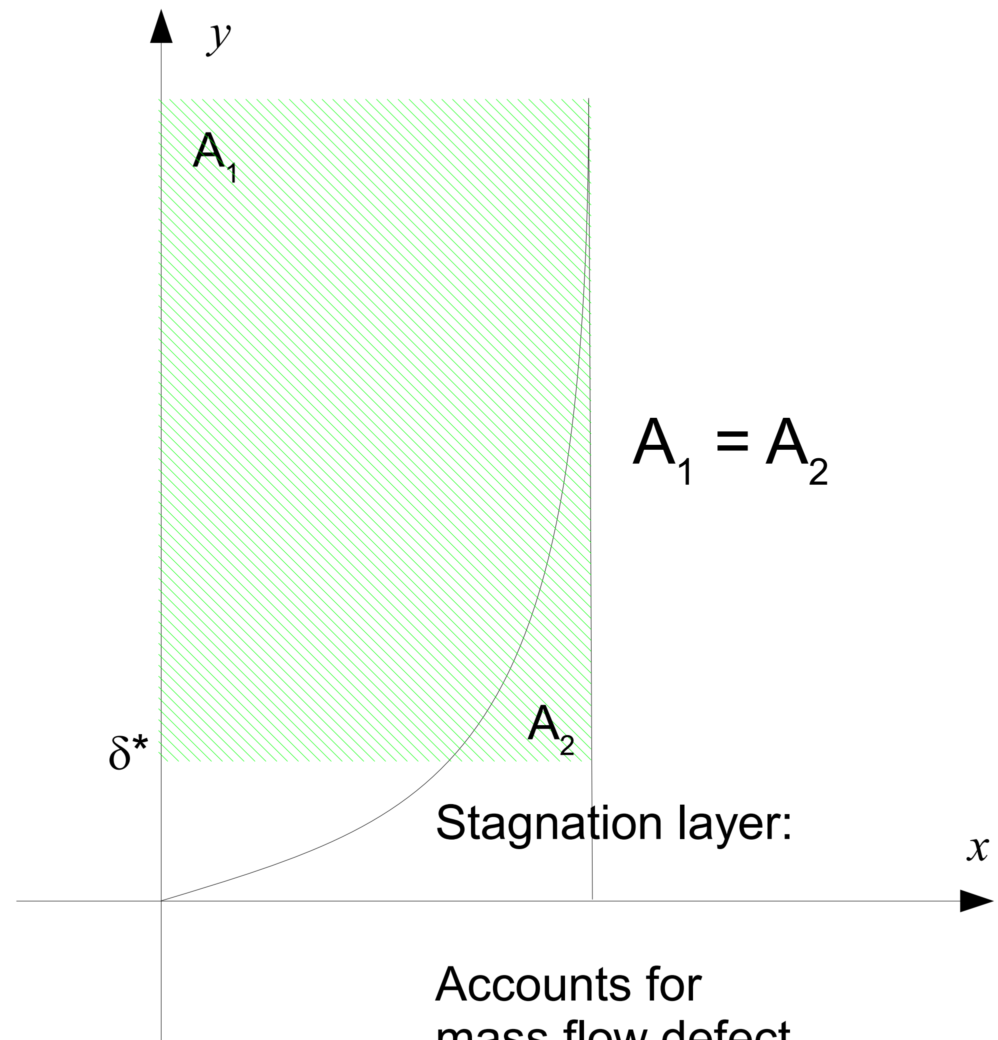
$$y = \delta \text{ for } u\,(x, \delta) = 0.99\ U$$

- δ: BL thickess (or velocity BL thickness)

Velocity boundary layer thickness



Displacement thickness



Stagnation layer:

Accounts for mass flow defect due to BL

Displacement (stagnation) layer thickness

$$A_1 = A_2$$

$$\int_0^\infty (U - u)\, dy = U \delta^*$$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

Momentum thickness θ

Similar to displacement thickness, but accounts for momentum transfer defect in BL:

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

9.2. Boundary layer equations

- Start with full Navier-Stokes (2D steady) near a flat surface
- Main assumption: thin boundary layer ($\delta/x << 1$)
- Order of magnitude analysis for terms of Navier-Stokes equation
  - $u \sim U$
  - $\partial/\partial x \sim 1/x$
  - $\partial u/\partial x \sim U/x$
  - $\partial/\partial y \sim 1/\delta$

- From contunuity equation...

$$\frac{U}{x} \sim \frac{\partial u}{\partial x} \sim \frac{\partial v}{\partial y} \sim \frac{v}{\delta}$$

$$v \sim U \frac{\delta}{x}$$

Examine the orders of terms in momentum equations

$$\underset{U^2/x}{\underbrace{\overset{U}{u}\,\overset{\frac{U}{x}}{\frac{\partial u}{\partial x}}}}+\underset{U^2/x}{\underbrace{\overset{\frac{U\delta}{x}}{v}\,\overset{\frac{U}{\delta}}{\frac{\partial u}{\partial y}}}}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\overset{\frac{U}{x^2}}{\cancel{\frac{\partial^2 u}{\partial x^2}}}\;\ll\;\overset{\frac{U}{\delta^2}}{\frac{\partial^2 u}{\partial y^2}}\right)$$

$$\underset{\frac{U^2\delta}{x^2}}{\underbrace{\overset{U}{u}\,\overset{\frac{U\delta}{x^2}}{\cancel{\frac{\partial v}{\partial x}}}}}+\underset{\frac{U^2\delta}{x^2}}{\underbrace{\overset{\frac{U\delta}{x}}{v}\,\overset{\frac{U\delta}{x\delta}}{\cancel{\frac{\partial v}{\partial y}}}}}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\overset{\frac{U\delta}{x^3}}{\cancel{\frac{\partial^2 v}{\partial x^2}}}+\underset{\frac{U}{x\delta}}{\overset{\frac{U\delta}{x\delta^2}}{\cancel{\frac{\partial^2 v}{\partial y^2}}}}\right)$$

$p = p(x)$!

Much smaller than U^2/x terms we keep in first equation

What remains of the continuity and momentum equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{U^2}{x} \qquad\qquad \nu\frac{U}{\delta^2}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{d\,p}{d\,x}+\nu\frac{\partial^2 u}{\partial y^2}$$

Boundary layer equations

Notable scalings

$$\frac{U^2}{x}\sim\nu\frac{U}{\delta^2}, \quad \text{thus} \quad \delta\sim\sqrt{\frac{\nu}{U}x}$$

$$Re_x=\frac{U\,x}{\nu}\sim\frac{x^2}{\delta^2}\gg 1$$

Pressure – same as in outer (ideal) flow

Bernoulli equation for outer flow

$$\frac{p}{\rho}+\frac{U^2}{2}=const$$

Thus

$$-\frac{1}{\rho}\frac{d\,p}{dx}=U\,\frac{d\,U}{d\,x}$$

Plug this into momentum equation to get rid of pressure

Boundary conditions

$u(x,0) = 0$

$v(x,0) = 0$

$u(x,y) \to U$ as $y \to \infty$

9.3. Blasius solution

- Flat plate, $U = const$, $p = const$
- Boundary layer equations become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

Reformulate for streamfunction ψ

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Continuity satisfied automatically, momentum equation is

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}$$

No length scale!

Dimensional variables: $x, y, \nu, U \rightarrow n = 4$

Dimensionally independent units: L, t $\rightarrow k = 2$

For Buckingham's π-theorem, $n - k = 2$

Look for

$$\psi/\nu = f(\pi_1, \pi_2)$$

Look for

$$\pi_1 = y^{a_{11}} U^{a_{21}} x^{a_{31}} \nu^{a_{41}}, \quad \pi_2 = y^{a_{12}} U^{a_{22}} x^{a_{32}} \nu^{a_{42}}$$

Construct a dimensional matrix

$$M = \begin{array}{cc} \begin{array}{cccc} y & U & x & \nu \end{array} & \\ \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & -1 \end{bmatrix} & \begin{array}{c} \mathsf{L} \\ \mathsf{t} \end{array} \end{array}$$

Find its kernel vectors $a_i = (a_{1i}, a_{2i}, a_{3i}, a_{4i}),\ i = 1,2$:

$$M a_i^{\mathsf{T}} = (0,0)$$

$$a_1 + a_2 + a_3 + 2\,a_4 = 0$$

$$a_2 + a_4 = 0$$

This simplifies to

$$a_1 + a_3 + a_4 = 0$$

$$a_2 = -a_4$$

Let a_{21} = 1/2, then a_{41} = -1/2 and

$$a_{11} + a_{31} = 1/2$$

a_{11} = 1, a_{31} = -1/2 would fit, so

$$\pi_1 = y U^{1/2} x^{-1/2} \nu^{-1/2} = \frac{y}{\sqrt{\frac{\nu}{U} x}} \sim \frac{y}{\delta}$$

Let a_{12} = 0, a_{32} = 1, then a_{42} = -1, a_{22} = 1:

$$\pi_2 = \frac{Ux}{\nu} = Re_x \sim \frac{x^2}{\delta^2}$$

Look for

$$\frac{\psi}{\nu} = Re_x^n \, f\left(\frac{y}{\delta}\right)$$

Not how Blasius did it though...

Blasius approach -

Similarity variable – clearly $\eta = y/\delta$

How to nondimensionalize ψ?

Let $\psi \sim f(\eta)$, then

$$u = \frac{\partial \psi}{\partial y} \sim \frac{d f}{d \eta} \frac{\partial \eta}{\partial y} = f\,' \frac{1}{\delta} = f\,' \sqrt{\frac{U}{\nu x}}$$

For $\eta = const$, $u = const$ (otherwise velocity profiles would not be self-similar), thus $\psi \sim x^{1/2} f(\eta)$

Rewrite this as $\psi \sim Re_x^{\ 1/2} f(\eta)$

dimensional dimensionless

Easy to fix that...

$$\frac{\psi}{\nu}=\sqrt{Re_x}\,f(\eta)$$

So, look for

$$\psi=\nu\sqrt{\frac{U\,x}{\nu}}\,f\left(\frac{y}{\sqrt{\nu\,x/U}}\right)=\sqrt{U\,x\,\nu}\,f\left(\frac{y}{\sqrt{\nu\,x/U}}\right)$$

Plug this into momentum equation and BC to get...

$$f'''+\frac{1}{2}f\,f''=0$$ Blasius equation

$$f(0)=f'(0)=0$$
$$f'(\eta)\rightarrow 1,\quad \eta\rightarrow\infty$$

Solve numerically to get some notable results

For a plate of length x, drag coefficient

Drag force

$$C_D = \frac{F}{\frac{1}{2}\rho U^2 x} \approx \frac{1.328}{\sqrt{Re_x}}$$

$$\frac{\delta}{x} \approx \frac{5}{\sqrt{Re_x}}$$

9.4. Falkner-Skan solutions

Look for solutions in the form (generalized from Blasius solution)

$$u(x, y) = U(x) f'(\eta), \quad \eta = \frac{y}{\xi(x)}$$

Outer flow solution

The corresponding streamfunction form is

$$\psi(x, y) = U(x)\xi(x) f(\eta)$$

Continuity satisfied, plug ψ into x-momentum equation...

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = U\frac{dU}{dx} + \nu \frac{\partial^3 \psi}{\partial y^3}$$

The momentum equation becomes...

$$f''' + \left[\frac{\xi}{\nu}\frac{d}{dx}\left(U\xi\right)\right] f f'' + \left[\frac{\xi^2}{\nu}\frac{dU}{dx}\right]\left(1 - \left(f'\right)^2\right) = 0$$

x-dependent parts

For a similarity solution to exist, must have:

$$\alpha = \frac{\xi}{\nu}\frac{d}{dx}\left(U\xi\right) = const$$

$$\beta = \frac{\xi^2}{\nu}\frac{dU}{dx} = const$$

The Falkner-Skan approach (counterintuitive but neat)

- Choose α, β
- Solve $\alpha = \ldots$, $\beta = \ldots$ for U, ξ: does $U(x)$ correspond to any useful outer flow?
- If yes, solve this system for f with chosen α, β

$$f''' + \alpha f f'' + \beta\left[1 - \left(f'\right)^2\right] = 0$$
$$f(0) = f'(0) = 0$$
$$f'(\eta) \to 1, \eta \to \infty$$

- Combine U, y, f to construct streamfunction:

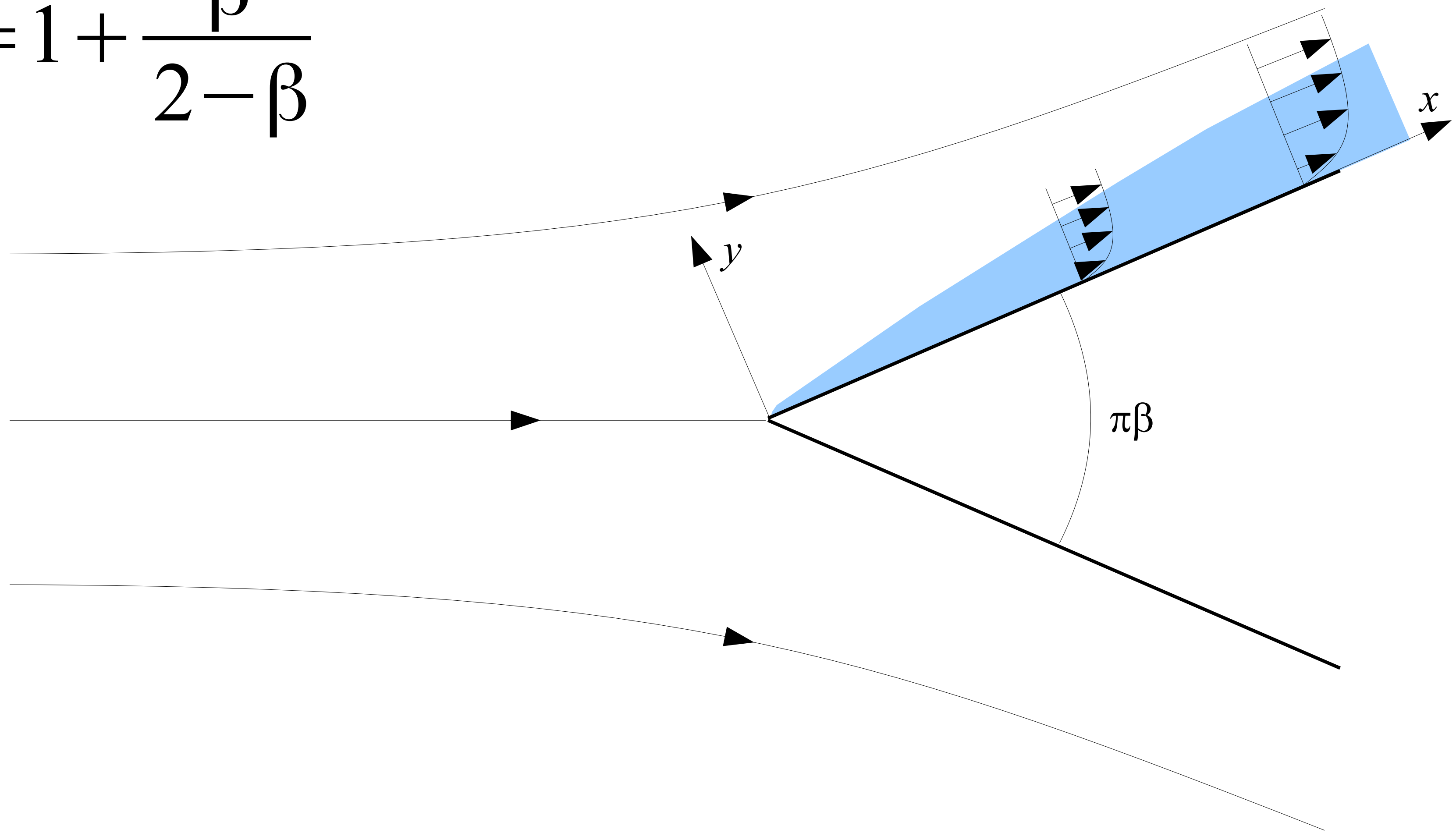
$$\psi(x, y) = U(x)\xi(x) f\left(\frac{y}{\xi(x)}\right)$$

Example: $\alpha = 1/2$, $\beta = 0$: Blasius solution

9.5. Flow over a wedge

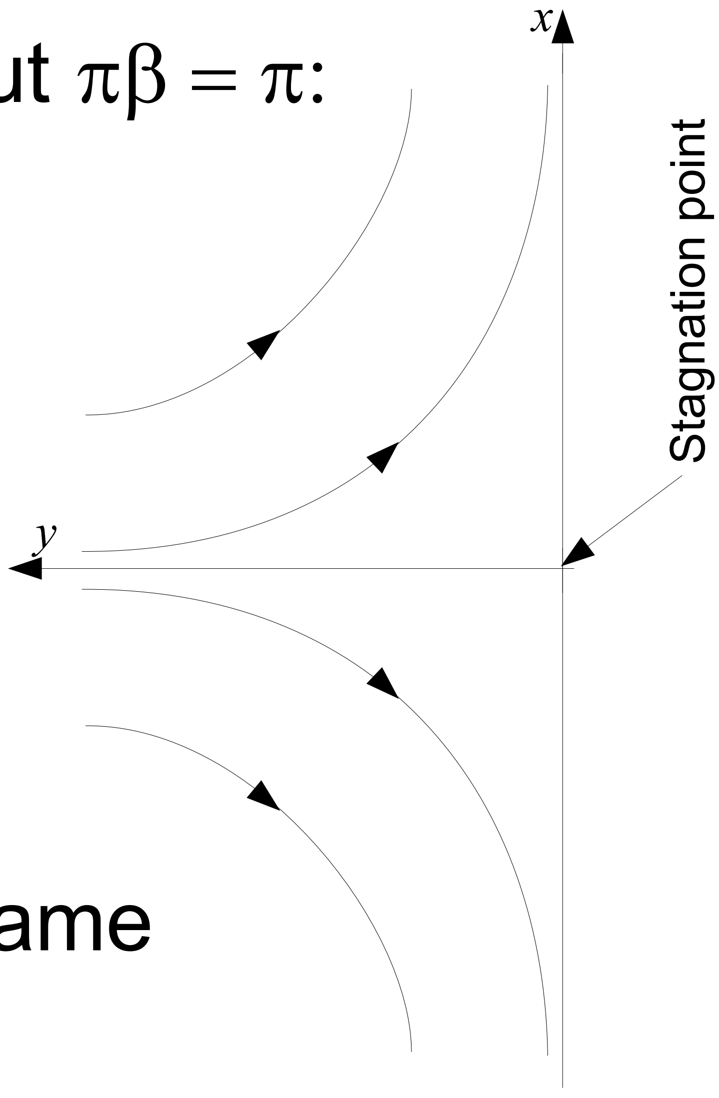
- $\alpha = 1,\ 0 < \beta < 1$:
- $U(x) = nUx^{n\text{-}1},\ V = 0$ – wedge flow!

$$n = 1 + \frac{\beta}{2 - \beta}$$



9.6. Stagnation-point flow

- $\alpha = 1, \;\; \beta = 1$:
- Same as previous problem, but $\pi\beta = \pi$:



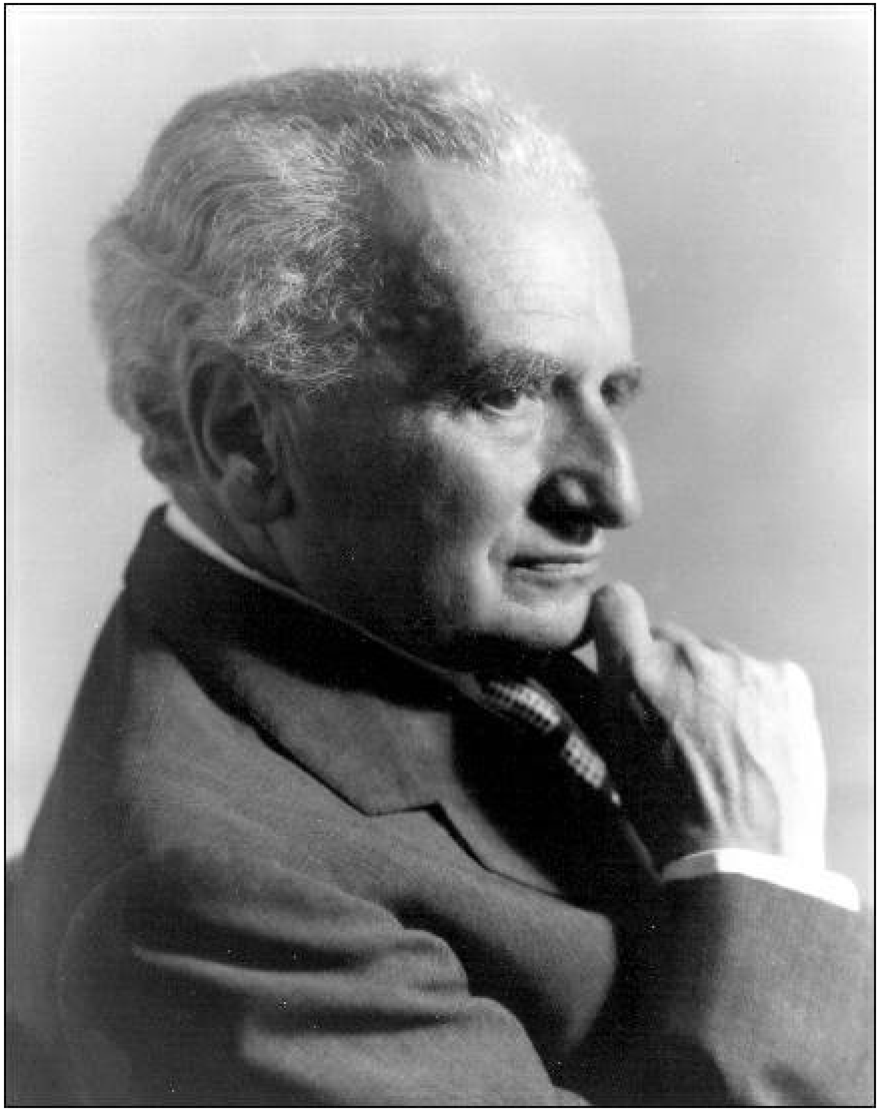
Boundary layer solution is the same as exact Hiemenz solution!

9.7. Flow in a convergent channel

- $\alpha = 0, \;\; \beta = 1$:
- $U(x) = -c/x$, $V = 0$ – limit case ($Re \to \infty$) for convergent wedge flow!
- No BL solution for divergent flow exists (which is physically correct!)

9.8. Approximate solution for a flat surface

- A demonstration of the widely applicable integral method developed by von Kármán (later refined by Ernst Pohlhausen)



Theodore von Kármán, 1881-1963

Flow over a flat plate, $U = p = const$

BL equations for this case...

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

Rewrite first term in momentum equation...

$$u\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}\left(u^2\right) = \frac{\partial}{\partial x}\left(u^2\right) - u\frac{\partial u}{\partial x} =$$

$$= \frac{\partial}{\partial x}\left(u^2\right) + u\boxed{\frac{\partial v}{\partial y}}$$

From continuity equation

Momentum equation becomes

$$\frac{\partial}{\partial x}\left(u^2\right)+\boxed{u\frac{\partial v}{\partial y}+v\frac{\partial u}{\partial y}}=\nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial}{\partial y}\left(u\,v\right)$$

Integrate this in y from surface to BL edge

$$\int_{y=0}^{\delta}\left[\frac{\partial}{\partial x}\left(u^2\right)+\frac{\partial}{\partial y}\left(u\,v\right)\right]dy=\nu\int_{y=0}^{\delta}\frac{\partial^2 u}{\partial y^2}\,dy$$

$$\int_{y=0}^{\delta}\frac{\partial}{\partial x}\left(u^2\right)dy+uv\Big|_{y=0}^{y=\delta}=\nu\frac{\partial u}{\partial y}\Bigg|_{y=0}^{y=\delta}$$

Note that

$$u\big|_{y=0} = v\big|_{y=0} = 0$$ No slip on surface

$$u\big|_{y=\delta} = U$$ Transition to outer flow at BL edge is continuous

$$\left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0$$ Transition to outer flow at BL edge is smooth

Define surface shear stress τ_0 as

$$\tau_0 = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

The integral becomes

$$\int_{y=0}^{\delta} \frac{\partial}{\partial x}\left(u^2\right) dy + U\, v(x, \delta) = -\frac{\tau_0}{\rho}$$

Integrate continuity equation to evaluate the Uv term in momentum equation

$$\int_{y=0}^{y=\delta}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dy=0$$

$$\int_{y=0}^{y=\delta}\frac{\partial u}{\partial x}dy+v\Big|_{y=0}^{y=\delta}=0$$

$$v(x,\delta)=-\int_{y=0}^{y=\delta}\frac{\partial u}{\partial x}dy$$

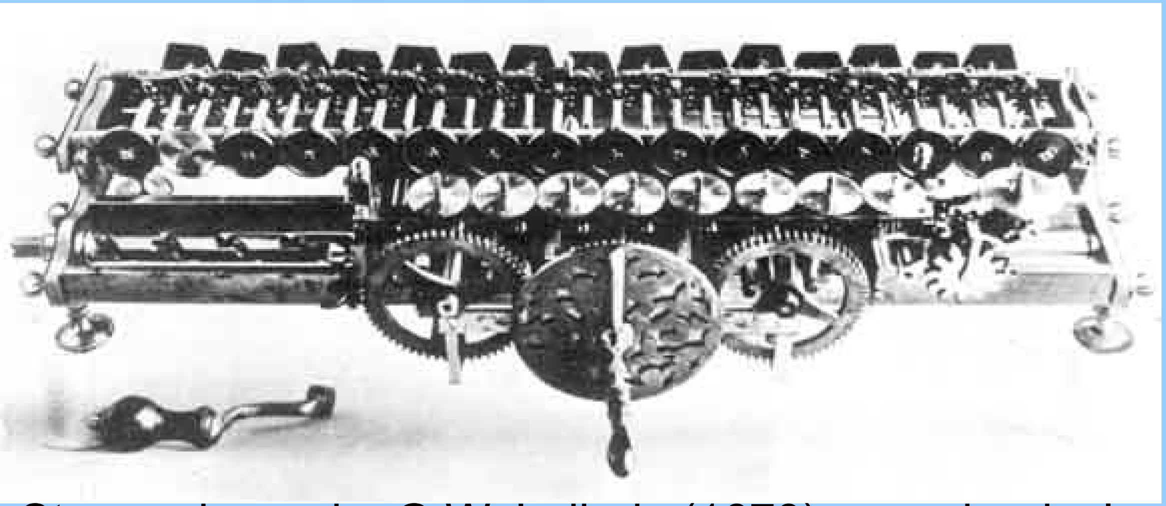
Substitute this into the momentum equation -

$$\int_{y=0}^{\delta(x)}\frac{\partial}{\partial x}\left(u^2\right)dy-U\int_{y=0}^{\delta(x)}\frac{\partial u}{\partial x}dy=-\frac{\tau_0}{\rho}$$

Leibniz integral rule

For an integral of $f(x,y)$ with variable limits,

$$\int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial x}\, dy = \frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} f\, dy - f(x,\beta)\frac{d\beta}{dx} + f(x,\alpha)\frac{d\alpha}{dx}$$



Step reckoner by G.W. Leibniz (1673) – mechanical computer for addition and multiplication

Gottfried Wilhelm Leibniz (1646-1716)

Apply Leibniz integral rule to momentum equation integral...

$$\int_{y=0}^{\delta(x)} \frac{\partial}{\partial x}\left(u^2\right) dy - U \int_{y=0}^{\delta(x)} \frac{\partial}{\partial x} u \, dy = -\frac{\tau_0}{\rho}$$

$$\frac{d}{dx}\int_{y=0}^{\delta(x)} u^2 \, dy - u^2\Big|_{y=\delta} \frac{d\delta}{dx} -$$

$$-U\left[\frac{d}{dx}\int_{y=0}^{\delta(x)} u \, dy - u\Big|_{y=\delta} \frac{d\delta}{dx}\right] = -\frac{\tau_0}{\rho}$$

$$\frac{d}{dx}\int_{y=0}^{\delta(x)} u^2 \, dy - \cancel{U^2 \frac{d\delta}{dx}} - U\frac{d}{dx}\int_{y=0}^{\delta(x)} u \, dy + \cancel{U^2 \frac{d\delta}{dx}} = -\frac{\tau_0}{\rho}$$

Momentum equation integral is...

$$\frac{d}{dx}\int_{y=0}^{\delta(x)} u^2\,dy - U\frac{d}{dx}\int_{y=0}^{\delta(x)} u\,dy = -\frac{\tau_0}{\rho}$$

...or...

$$-\frac{d}{dx}\int_{y=0}^{\delta(x)}\left(u^2 - U\,u\right)dy = \frac{d}{dx}\int_{y=0}^{\delta(x)} u\left(U-u\right)dy = \frac{\tau_0}{\rho}$$

Momentum integral for Blasius BL

Physical meaning: momentum change in BL is due to surface shear

General procedure for the von Kármán-Pohlhausen method

- Represent the unknown velocity profile with a polynomial (a general profile should have a polynomial series expansion?)
- Fit the polynomial constants to match known boundary conditions

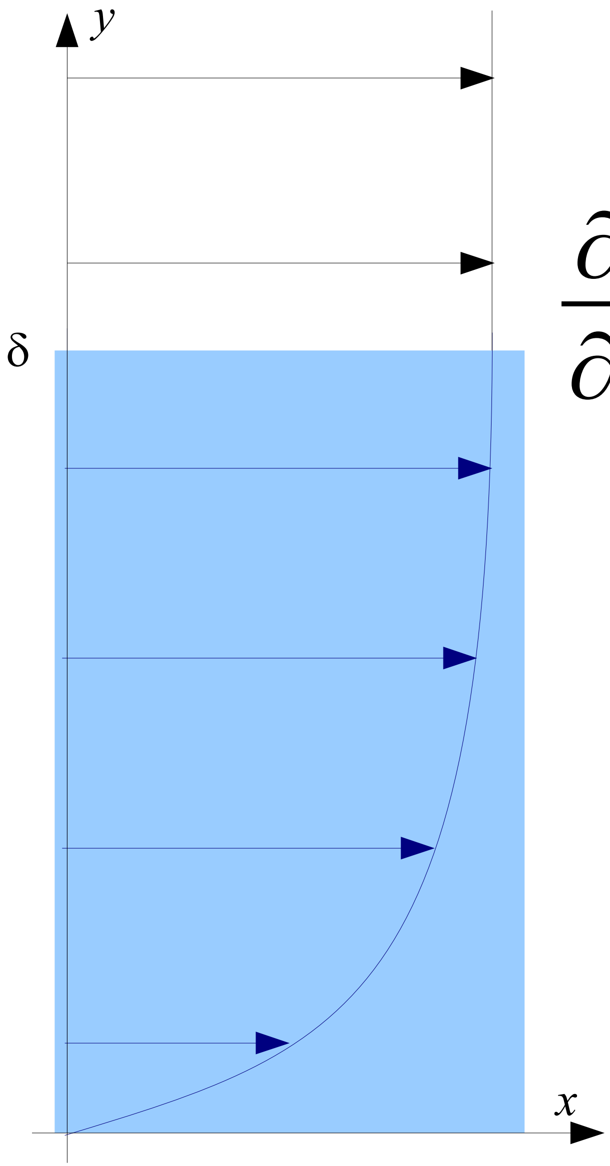
$$u(x,0) = 0$$

$$u(x,\delta) = U$$

$$\partial u/\partial y(x,\delta) = 0$$

- Can impose further boundary conditions (as needed to determine polynomial coefficients)

Additional boundary conditions



$$\left.\frac{\partial u}{\partial y}\right|_{y=\delta} = \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=\delta} = \ldots = 0$$

Apply momentum equation at $y = 0$

$$\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right]_{y=0} = \nu \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} = 0$$

(For more BC, apply derivatives of momentum equation, etc.)

- Apply the BC to determine polynomial coefficients (as functions of δ)
- Plug the velocity profile polynomial into momentum integral, integrate, solve resulting ODE for $\delta = \delta(x)$
- Find drag coefficient, etc.

  Similar procedure can be applied to free-surface and other flows (replace unknown functions with polynomials, satisfy BC, satisfy conservation eqs.)

For a flat-plate BL, look for

$$u = a_0 + a_1 y + a_2 y^2$$

This only works for zero pressure gradient!

$$u(0) = 0$$

$$u(\delta) = U$$

$$\left. \frac{\partial u}{\partial y} \right|_{y=\delta} = 0$$

From BC at $y = 0$, $a_0 = 0$

$$\frac{\partial u}{\partial y} = a_1 + 2 a_2 y, \quad \left. \frac{\partial u}{\partial y} \right|_{y=\delta} = a_1 + 2 a_2 \delta = 0$$

Thus from second BC at $y = \delta$,

$$a_1 = -2 a_2 \delta$$

Now use first BC at $y=\delta$

$$u(\delta)=a_0+a_1\delta+a_2\delta^2=U$$

$$\underset{0}{} \qquad \underset{-2a_2\delta}{}$$

$$a_2\left(-2\delta^2+\delta^2\right)=U$$

$$a_2=-\frac{U}{\delta^2}, \quad a_1=2\frac{U}{\delta}$$

Polynomial expression for u to plug into momentum integral

$$\frac{u}{U}=2\frac{y}{\delta}-\left(\frac{y}{\delta}\right)^2=2\eta-\eta^2$$

$$\eta=\eta(x,y)=\frac{y}{\delta(x)}$$

We've seen this one before...

Rewrite the momentum integral a bit...

$$\frac{d}{dx}\int_{y=0}^{\delta(x)} u(U-u)\,dy = U^2\frac{d}{dx}\int_{y=0}^{\delta(x)} \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \frac{\tau_0}{\rho}$$

Plug in expression for u/U

$$\frac{d}{dx}\int_{y=0}^{\delta(x)} \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \frac{d}{dx}\int_{y=0}^{\delta(x)} \left(2\eta-\eta^2\right)\left(1-2\eta+\eta^2\right)dy$$

Evaluate the integral

$$\int_{y=0}^{\delta(x)} \left(2\eta-\eta^2\right)\left(1-2\eta+\eta^2\right)dy =$$

$$= \int_{y=0}^{\delta(x)} \left(2\eta-\eta^2-4\eta^2+2\eta^3+2\eta^3-\eta^4\right)dy$$

Collect terms

$$= \int_{y=0}^{\delta(x)} \left(-\eta^4 + 4\eta^3 - 5\eta^2 + 2\eta\right) dy$$

Variable substitution $y \to \eta$, $y = \delta\eta$, dy = $\delta\, d\eta$ and
$y = \delta \to \eta = 1$

$$= \delta \int_{\eta=0}^{1} \left(-\eta^4 + 4\eta^3 - 5\eta^2 + 2\eta\right) d\eta =$$

$$= \delta \left(-\frac{1}{5}\eta^5 + \eta^4 - \frac{5}{3}\eta^3 + \eta^2\right)_0^1 =$$

$$= \delta \left(-\frac{1}{5} + 1 - \frac{5}{3} + 1\right) = \delta \frac{2}{15}$$

Plug the evaluated integral back...

$$U^2 \frac{d}{dx}\left(\frac{2}{15}\delta\right) = \frac{\tau_0}{\rho}$$

Use the definition of surface shear stress

$$\tau_0 = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

Recall that

$$u = U\left(2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right)$$

$$\frac{\partial u}{\partial y} = U\left(\frac{2}{\delta} - 2\frac{y}{\delta^2}\right), \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = 2\frac{U}{\delta}$$

Plug that into the ODE for δ

$$U^2 \frac{d}{dx}\left(\frac{2}{15}\delta\right) = \frac{\mu}{\rho}\frac{2U}{\delta}$$

A more compact form

$$\delta'\delta = 15\frac{\nu}{U}$$

$$\frac{1}{2}\left(\delta^2\right)' = 15\frac{\nu}{U}$$

Integrate...

$$\delta^2 = 30\frac{\nu x}{U} + C$$

Since at $x = 0$ $\delta = 0$, $C = 0$ and

$$\delta = \sqrt{30 \frac{\nu x}{U}}$$

Note that

$$\frac{\delta}{x} = \sqrt{30 \frac{\nu}{U x}} = \frac{\sqrt{30}}{\sqrt{Re_x}} \approx \frac{5.48}{\sqrt{Re_x}}$$

Compare with exact result:

$$\frac{\delta}{x} \approx \frac{5}{\sqrt{Re_x}}$$

Error < 10%, despite a very crude approximation

9.9. General momentum integral

Similar reasoning, but for an arbitrary BL with non-zero pressure gradient in the x-direction

Momentum equation (with pressure eliminated using Bernoulli equation for outer flow)...

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

...can be similarly rewritten as...

$$\frac{\partial}{\partial x}\left(u^2\right)+\frac{\partial}{\partial y}\left(uv\right)=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

Integrate this in y from 0 to δ to obtain

$$\frac{d}{dx}\left(U^2(x)\theta\right)+U\,\delta^*\frac{dU}{dx}=\frac{\tau_0}{\rho}$$

where

$$\delta^*=\int_0^\infty\left(1-\frac{u}{U}\right)dy$$ Displacement thickness

$$\theta=\int_0^\infty\frac{u}{U}\left(1-\frac{u}{U}\right)dy$$ Momentum thickness

9.10. von Kármán – Pohlhausen approximation

- Consider velocity profile in the form of a 4th order polynomial (allows to account for nonuniform freestream velocity and nonzero pressure gradient)
- Apply five boundary conditions to find coefficients (two added conditions – second derivatives at $y = 0$ and $y = \delta$)
- Plug resulting polynomial into expressions for δ^*, θ, τ_0
- Substitute results into general momentum integral, solve ODE for $\delta(x)$

Boundary-layer separation

BL momentum equation

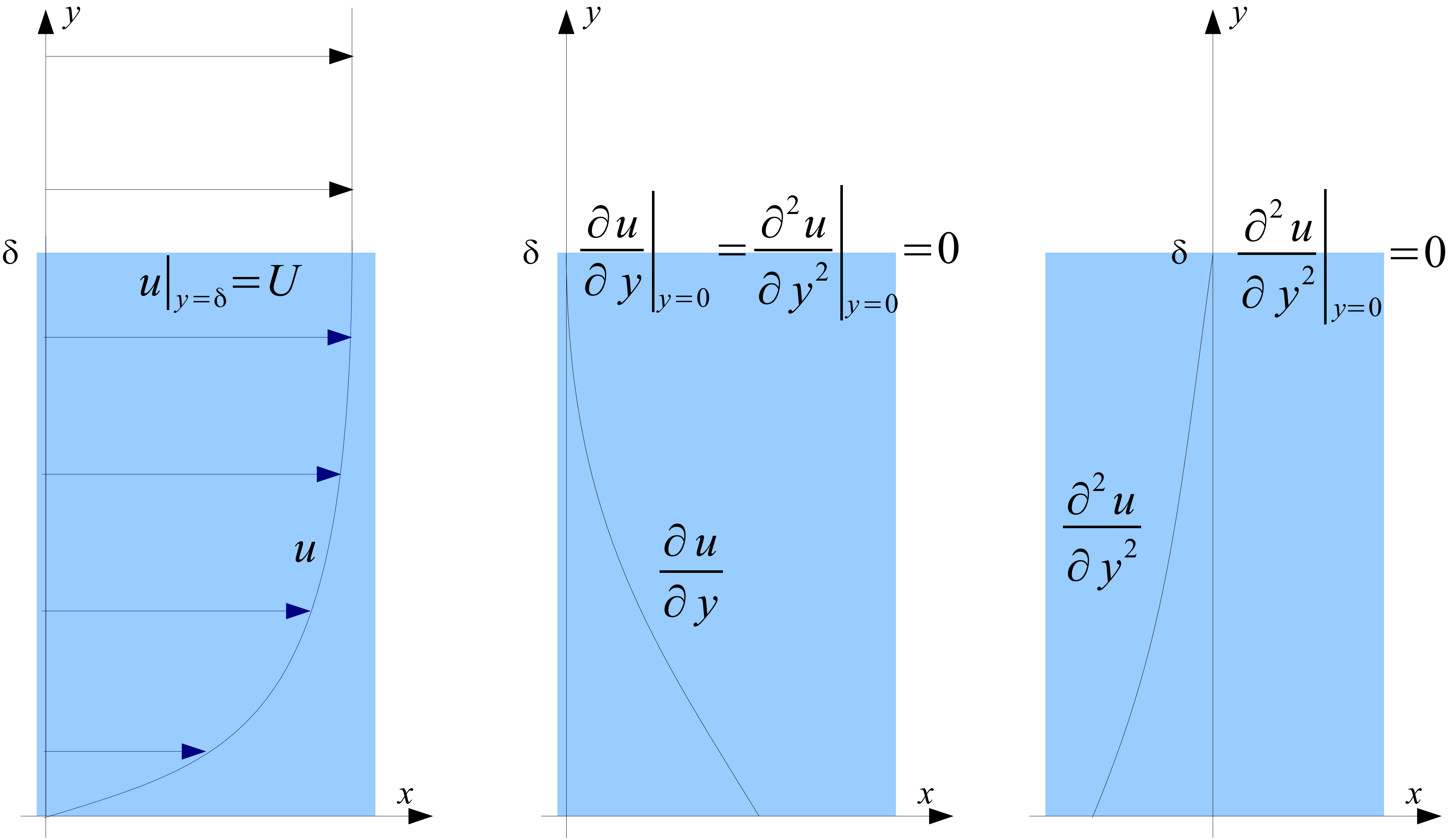
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{d\,p}{d\,x}+\nu\frac{\partial^2 u}{\partial y^2}$$

Apply this equation at $y=0$

$$0=-\frac{1}{\rho}\frac{d\,p}{d\,x}+\nu\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0}$$

Pressure gradient of the outer flow determines velocity profile curvature on the surface!

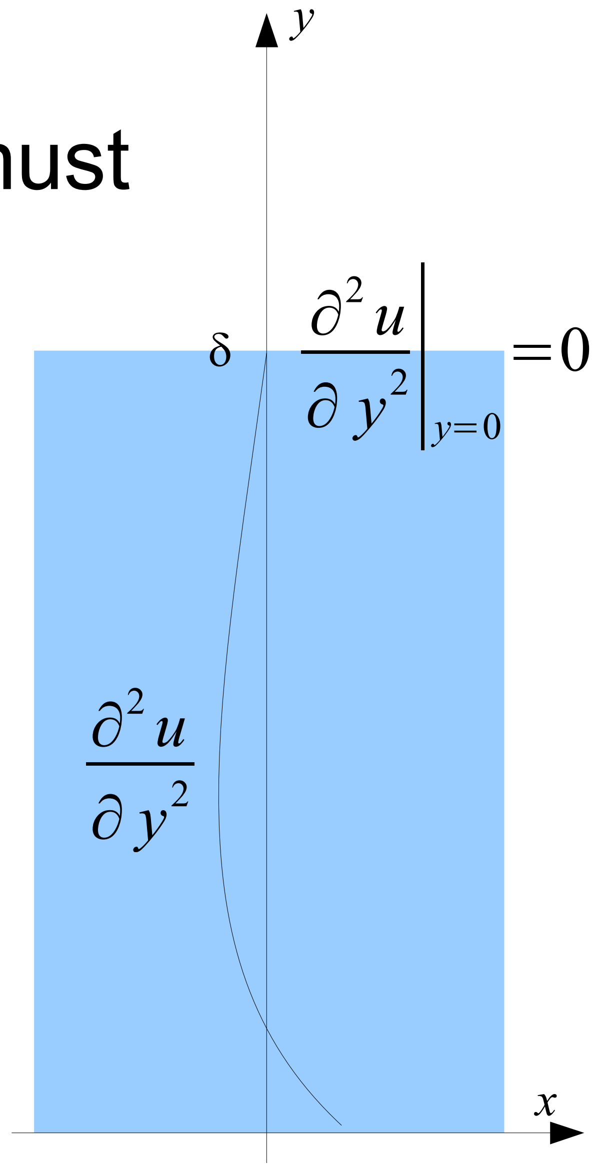
Non-negative dp/dx: non-positive curvature of velocity profile throughout BL



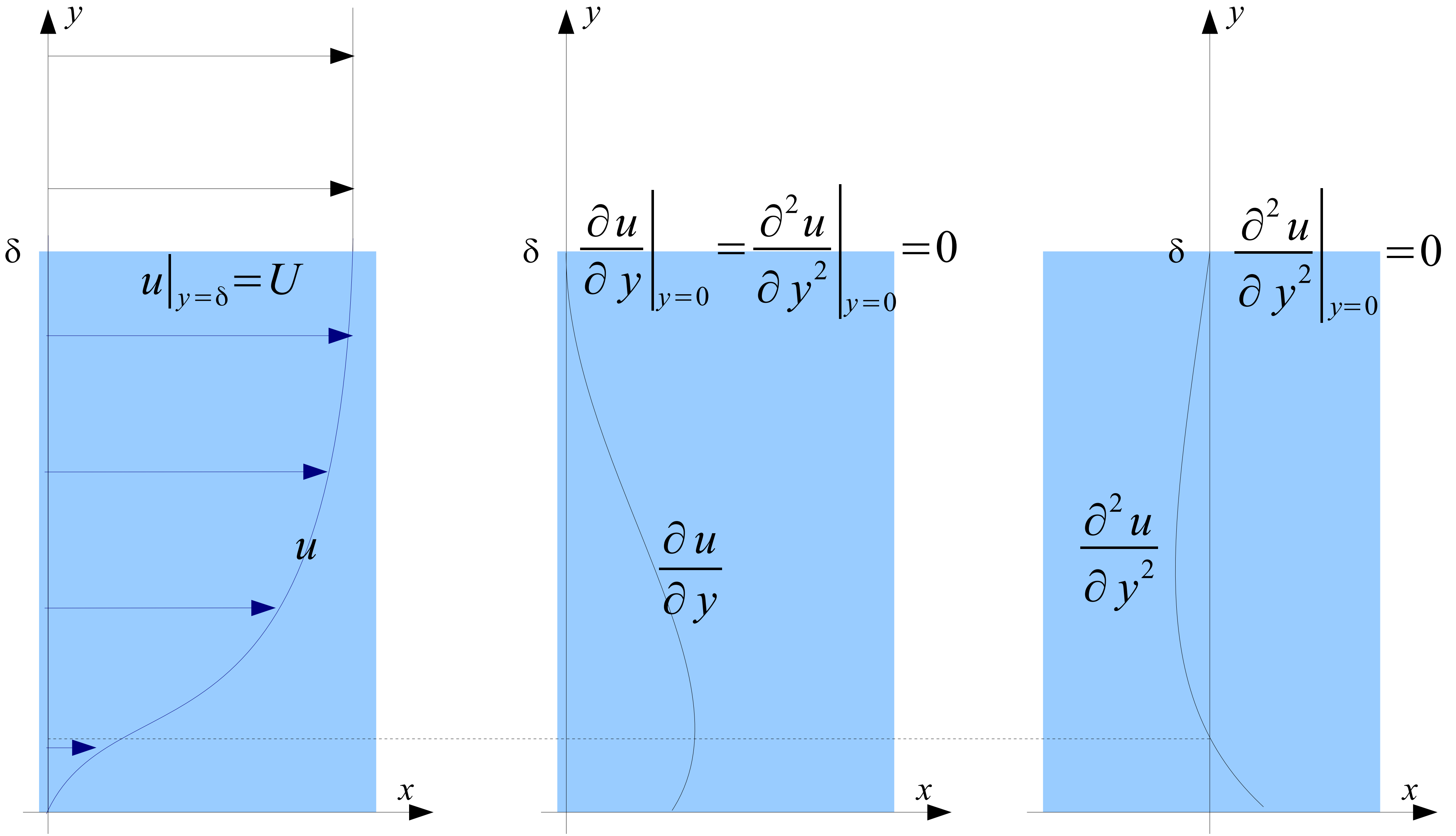
Now suppose we have negative dp/dx:

curvature of velocity profile near the boundary will be positive

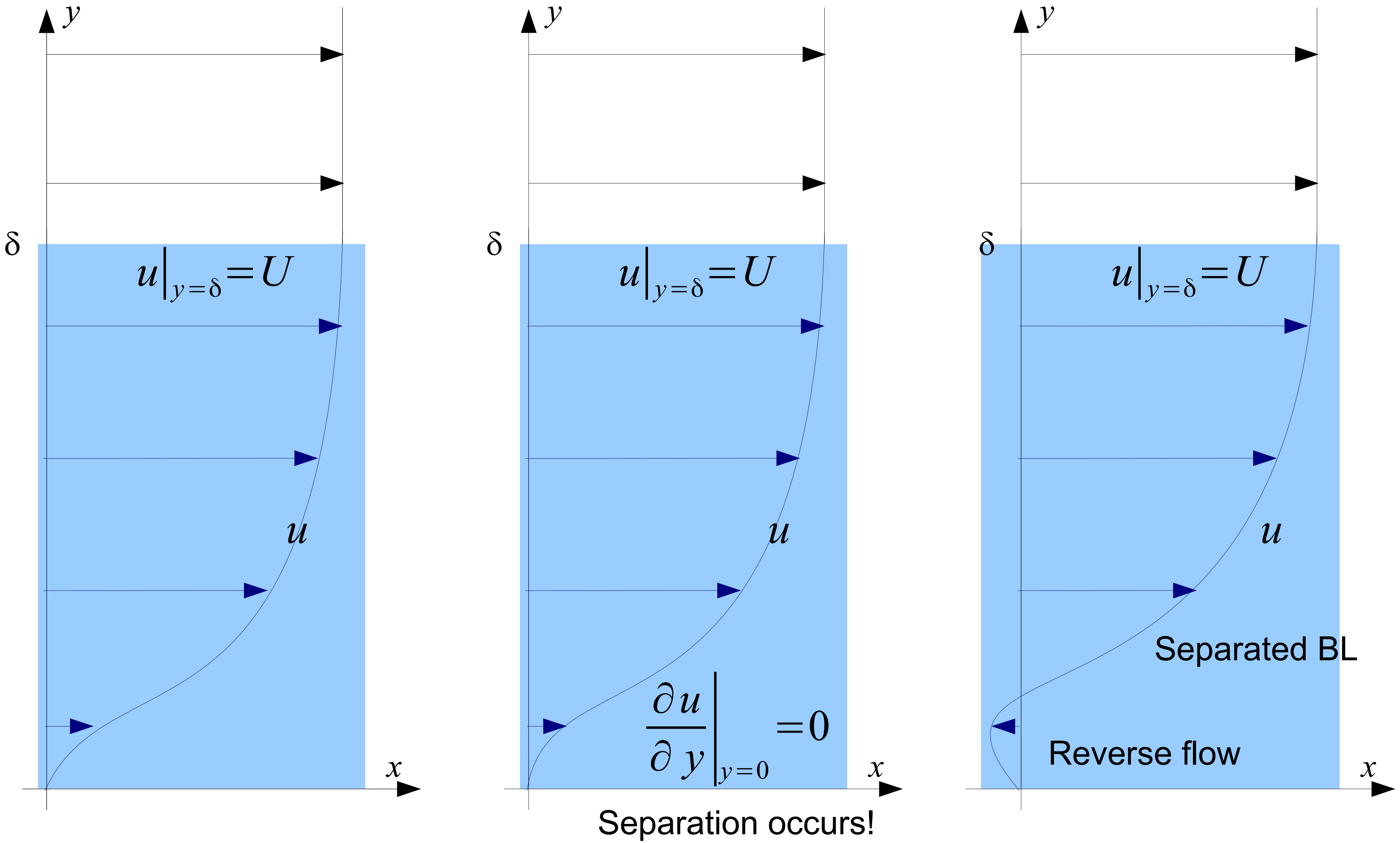
same curvature near the BL edge must approach zero from the negative direction (otherwise – no smooth transition to outer flow!)



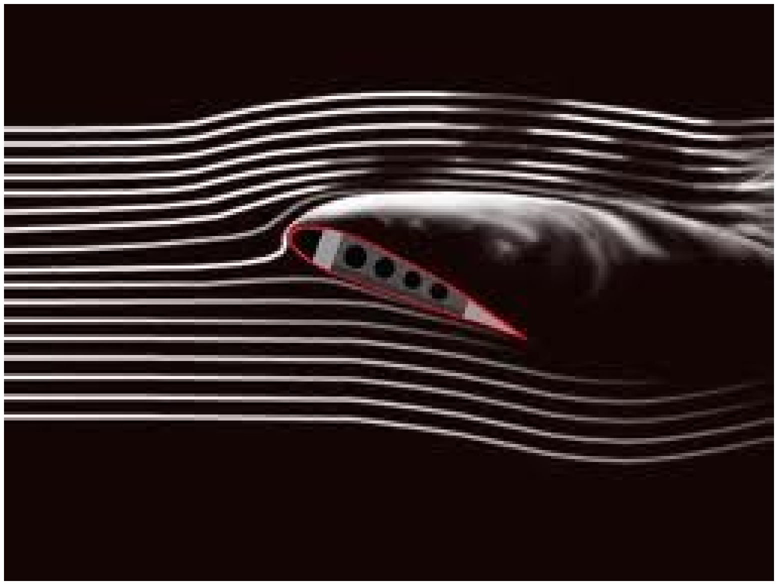
A point of inflection must exist in velocity profile (where curvature changes sign)



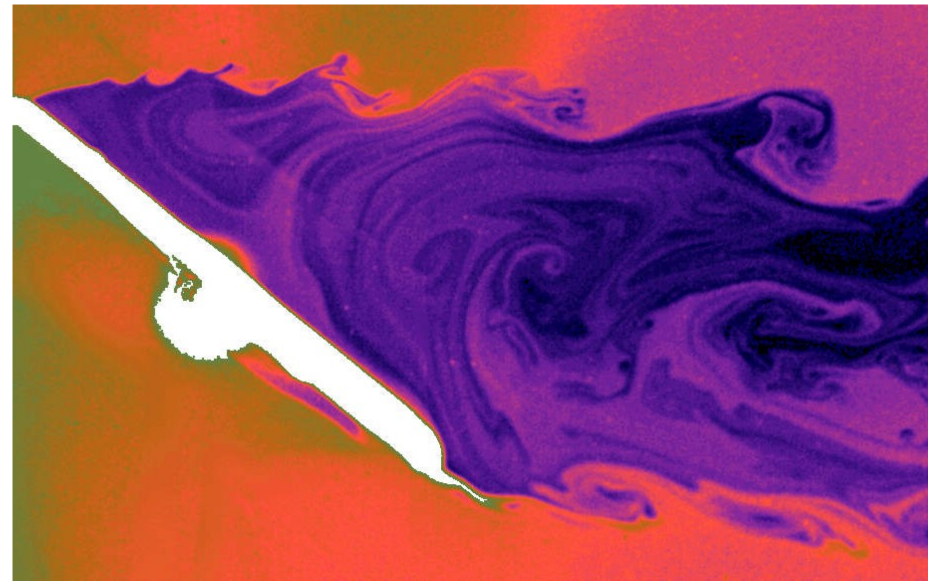
Velocity profiles for increasing adverse pressure gradient



Separated boundary layers



Roshko, early 1950s



Vorobieff & Ecke, XIIth century

Flometrics.com, 2011

9.12. Boundary layer stability

Consider a narrow strip of a boundary layer and a small perturbation to steady-state velocity and pressure:

$$u(x,y,t)=u(y)+u'(x,y,t)$$
$$v(x,y,t)=v'(x,y,t)$$
$$p(x,y,t)=p(x)+p'(x,y,t)$$

Perturbation – same scale as v (which is small), so consider entire v as perturbation (no loss of generality)

$$\left|\frac{u'}{u}\right|\ll 1,\ \left|\frac{v'}{u}\right|\ll 1,\ \left|\frac{p'}{p}\right|\ll 1$$

Plug these u, v, p into Navier-Stokes (not BL) equations

Linearize

Introduce a perturbation streamfunction

$$u' = \frac{\partial \psi}{\partial y}, \quad v' = -\frac{\partial \psi}{\partial x}$$

Continuity eliminated, rewrite the momentum equations in terms of ψ

Cross-differentiate x-momentum equation in y, y-momentum equation in x

Get rid of the pressure term...

Result: 4^{th} order linear PDE for ψ

Consider the streamfunction in the form

$$\psi = \psi(y) e^{i\alpha(x-ct)}$$

c – speed of perturbation propagation

α – perturbation wavenumber ($\alpha = 2\pi/\lambda$)

If c is real ($\mathrm{Im}\, c = 0$), perturbation is *neutrally stable* (propagates but does not grow)

If $\mathrm{Im}\, c < 0$, perturbation is decaying

If $\mathrm{Im}\, c > 0$, perturbation grows and the **boundary layer is unstable**

Plugging the variable-separated form of y into the momentum equation reduces it to a 4th order ODE

$$(u-c)\left(\psi''-\alpha^2\psi\right)-u''\psi=\frac{\nu}{i\alpha}\left(\psi^{(4)}-2\alpha^2\psi''+\alpha^4\psi\right)$$

The Orr-Somerfeld equation

Boundary conditions

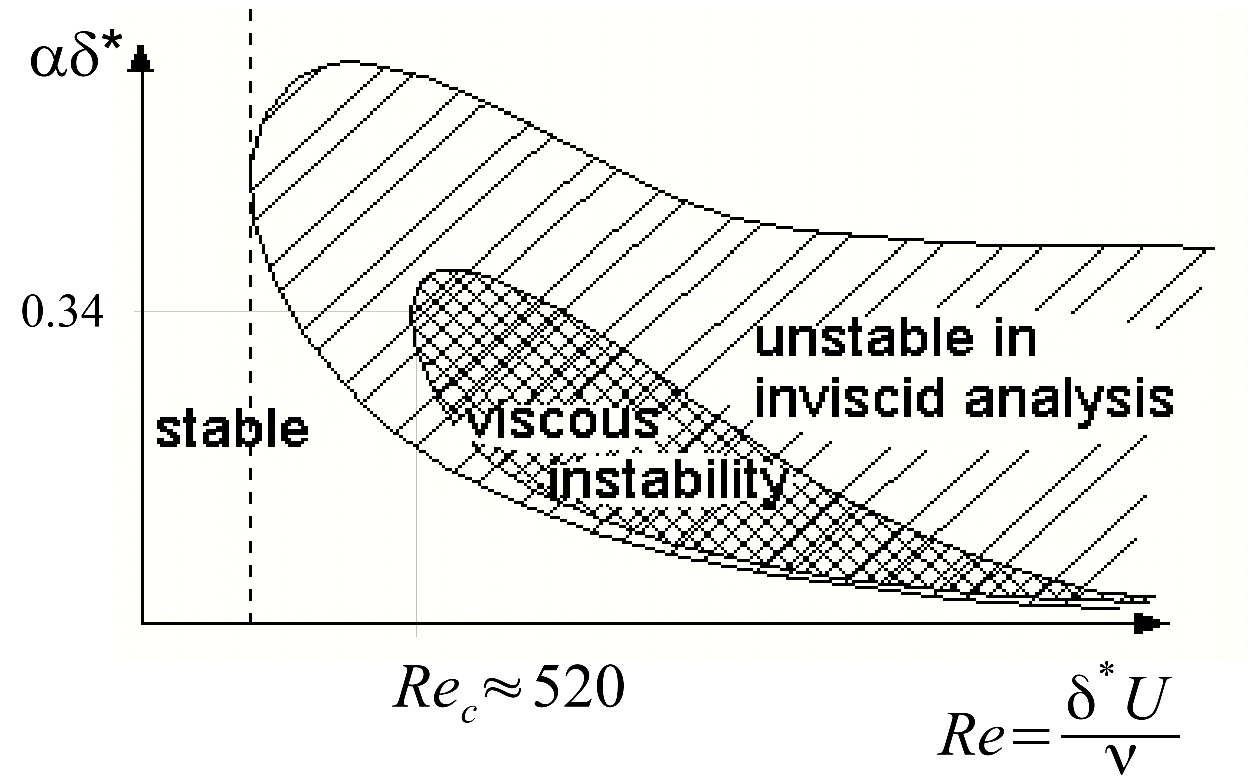
$\psi(0) = \psi'(0) = 0$ (perturbations go to zero on body surface)

$\psi(y) \to 0,\ \psi'(y) \to 0,\ y \to \infty$ (perturbations decay away from the boundary layer)

For every wavelength α, solve for c, determine stability

Results of stability analysis

For $\nu = 0$, Orr-Sommerfeld equation becomes Rayleigh equation

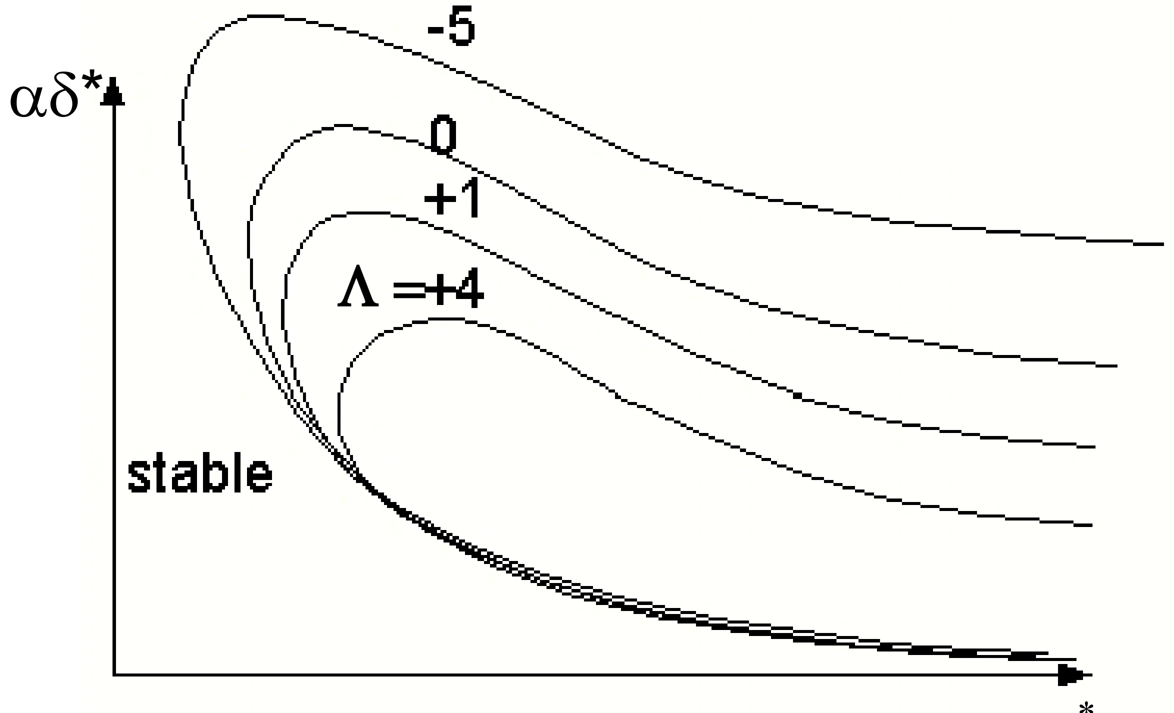


Note. We look for a 2D perturbation...
but what if the flow first becomes unstable in z-direction?
Such instability exists, but luckily, the flow is less stable to xy perturbations

Effects of local pressure gradient on stability

$$\Lambda = \frac{\delta^2}{\nu}\frac{dU}{dx}$$

Pressure parameter
(Pohlhausen, von Kármán)



$$Re = \frac{\delta^* U}{\nu}$$

Favorable pressure gradient expands stability region, adverse pressure gradient shrinks it