# Nonlinear theory of the excitation of waves by a wind due to the Kelvin–Helmholtz instability

E. A. Kuznetsov and P. M. Lushnikov

*L. D. Landau Theoretical Physics Institute, Russian Academy of Sciences, 117334 Moscow, Russia*

(Submitted 6 March 1995)
Zh. Éksp. Teor. Fiz. **108**, 614–630 (August 1995)

We give a nonlinear theory of the Kelvin–Helmholtz instability based on a Hamiltonian description and the small-angle approximation of the boundary surface of the two fluids. The basic nonlinear process is connected with a wave-wind interaction which differs significantly from the nonlinear interaction when there is no wind. We show that the nonlinearity does not saturate the linear instability but, on the contrary, leads to an explosive growth of the amplitude. Near the instability threshold we obtain an equation for the envelopes which is the same as the (2+1)-dimensional nonlinear Klein–Gordon equation. We find for this equation the sufficient conditions for the collapse of the integral form. 

## 1. INTRODUCTION

For a long time the tangential discontinuity instability (see, e.g., Ref. 1) which was discovered in the last century by Kelvin and Helmholtz has been considered the main mechanism for exciting sea waves by wind. The two stabilizing factors, the surface tension and the gravitational field, imply that there is a threshold for this instability. The threshold value $V_{cr}$ of the velocity above which the surface oscillations begin to be excited is determined by the minimum phase velocity $V_{\min}=\min(\omega_k/k)$ of the surface gravitational-capillary waves when there is no wind:

$$V_{\rm cr}=\frac{1}{\sqrt{\varepsilon}}V_{\min}. \tag{1}$$

Here

$$\omega_k^2+\frac{k}{1+\varepsilon}[g(1-\varepsilon)+\alpha k^2] \tag{2}$$

is the dispersion law for the surface waves, $g$ is the acceleration of the field of gravity, $\alpha$ is the surface tension coefficient, and $\varepsilon=\rho_2/\rho_1$ is the ratio of the densities $\rho_2$ and $\rho_1$ of the upper and lower fluids, which we shall assume to be small. For instance, for air and water we have $\varepsilon=1.24\times10^{-3}\ll1$ and the critical velocity $V_{\rm cr}=6.4$ m/s is consequently large compared to the minimum phase velocity $V_{\min}=23$ cm/s. It is important that this instability is aperiodic; this fundamentally changes the small-oscillation spectrum, especially in the region where the growth rate is the greatest. The maximum growth rate occurs at the transition between the gravitational and the capillary sections of the spectrum, where $k\sim k_0=\sqrt{(g/\alpha)}$. Far from the instability region (in the gravitational and the capillary regions) the dispersion law for small oscillations goes over to (2). If the velocity is just above the threshold value, perturbations with wavevectors in a small neighborhood of $\mathbf{k}=\mathbf{k}_0$, where the direction of the vector $\mathbf{k}_0$ is the same as the direction of the wind velocity $\mathbf{V}$ and the magnitude is equal to $k_0=\sqrt{(g/\alpha)}$, are unstable.

It is important that the Kelvin–Helmholtz instability occurs for ideal liquids, i.e., when one neglects their viscosity. This, indeed, explains the fact that the value (1) of the critical velocity for this instability is overestimated by a factor $1/\sqrt{\varepsilon}$ in comparison with the minimum phase velocity $V_{\min}$ of the gravitational-capillary waves. The spectrum (2) of the surface waves is similar to the spectrum of the Landau excitations of liquid helium.[2] Therefore, exactly as in the case of the destruction of the superfluidity of liquid helium, the excitation of waves by a wind must start for $V>V_{\min}$. The corresponding linear theory was developed in papers by Miles.[3] According to this theory the wave generation process is possible for $V>V_{\min}$ if viscosity is taken into account. The viscosity leads to the formation of a boundary layer in the air near the wave surface. Waves are generated because of the existence of a shear flow $\mathbf{V}=\mathbf{V}(z)$ in the boundary layer, where the coordinate $z$ is measured in the vertical direction from the unperturbed boundary between the two fluids. The growth rate of this instability is small compared to the frequency $\omega_k$ close to $V_{\min}$, and increases with wind speed and has no singularity for velocities $V\sim V_{\rm cr}$, which significantly distinguishes this instability from the Kelvin–Helmholtz one.

Recently Newell and Zakharov[4] have proposed a nonlinear theory for the excitation of waves by a wind, taking the Miles instability into account. This theory is based on a weak-turbulence description of the waves using kinetic equations. For $V>V_{\min}$ oscillations are excited in the instability region, which lies in the gravitational part of the spectrum. The growth of the oscillations is limited by cascade processes, as a result of which two Kolmogorov-type turbulent spectra are formed.[5] One of them corresponds to a constant flux of wave action or of the number of waves and develops in the long-wavelength region. The other Kolmogorov spectrum corresponds to a constant energy flux directed to smaller scales. In the gravitational–capillary transition region the spectrum with a constant energy flux must be joined to the Kolmogorov spectrum of the capillary waves.[5] For wind velocities up to velocities comparable to $V_{cr}$ given by (1) such a joining is possible. For higher velocities the energy outflow to the capillary region does not agree with the

 1063-7761/95/080332-09$10.00  

energy flux from the gravitational region. As a result, oscillations build up in the transition region and a condensate is formed, which according to Newell and Zakharov[4] is the cause of the sharp increase in the formation of foam when waves are excited by a wind. This prediction is in agreement with observations in which it was found for wind velocities $V\approx 6$ m/s that the fraction of the sea surface occupied by foam increased (see Refs. 6–10). For the same velocities the Kelvin–Helmholtz instability was observed. Is this a chance coincidence or not? Could it be one of the possible mechanisms corresponding to the formation of foam? In the present paper we attempt to solve these problems by using a small parameter—the angle of inclination of the boundary surface—in perturbation theory to study the nonlinear stage of the Kelvin–Helmholtz instability. Of course, since we use a small parameter we cannot pretend fully to describe the foam formation process. From a mathematical point of view this process corresponds to the development of singularities on the boundary surface, i.e., of angles of inclination of order unity. In the present paper we note the tendency for this instability to develop in the nonlinear stage. We show that the nonlinearity does not stabilize the instability in the first nonvanishing order in the amplitude of the oscillations but leads to an explosive growth of oscillations. For wind velocities below the critical one (1) a stable regime of excitations is possible where small perturbations are stable but when one reaches a critical amplitude and width of the distribution the explosive growth of the oscillations starts.

The plan of the paper is the following: in §2 we give the basic equations of motion which describe the excitation of surface waves due to the Kelvin–Helmholtz instability and we give their Hamiltonian formulation. We restrict ourselves to considering irrotational flows of the liquid. In that case the canonically conjugate quantities are $\Psi=\rho_1\Psi_1-\rho_2\Psi_2$, where the $\Psi_{1,2}$ are the values of the velocity potentials of the first and the second fluid taken at the boundary $z=\eta(x,y,t)$, and $\eta(x,y,t)$:[10,11]

$$\frac{\partial\Psi}{\partial t}=-\frac{\delta H}{\delta\eta},\quad \frac{\partial\eta}{\partial t}=\frac{\delta H}{\delta\Psi}. \tag{3}$$

The Hamiltonian here is the total energy of the system. In Secs. 3 and 4 we construct a consistent perturbation theory in the small parameter $|\nabla\eta|$—the characteristic angle of inclination of the surface—up to fourth-order terms in the Hamiltonian $H$. The analysis shows that the main contribution to the interaction of unstable perturbations comes from the interaction of the surface waves with the wind. It is essential that this interaction does not reduce to the usual interaction of surface waves;[8,10] it vanishes as the wind velocity decreases to zero. Near the instability threshold, if the amount by which the threshold is exceeded,

$$\delta=(V^2-V_{\rm cr}^2)/V_{\rm cr}^2$$

is small, the analysis of the perturbation theory series becomes simpler. In that case a wavepacket which is narrow in $k$-space is excited with a "carrier" wavevector $\mathbf{k}_0$; this makes it possible to change to envelopes in the equations of motion. We show in Sec. 4 that the envelope packet satisfies a nonlinear (2+1)-dimensional relativistically invariant Klein–Gordon equation (the $|\Psi|^4$ model). The square of the mass in that equation is equal to $-\delta$ and the nonlinear interaction describes attraction. This means that in the first nonvanishing order the nonlinearity cannot stabilize the instability; on the contrary, it is the cause for an explosive growth of the oscillations. For this equation we construct a spatially uniform solution and a self-similar asymptotic form which describes the behavior of the singularity after a finite time. In Sec. 5 we use the integral estimate method to obtain a sufficient criterion for collapse. This criterion follows from a second-order differential inequality for the square of the norm of the envelope. After a simple substitution the solution of this inequality reduces to an analysis of the motion of a Newtonian particle in a given potential. In the concluding section we give comparative estimates of the roles of the Kelvin–Helmholtz and the Miles instabilities.

## 2. BASIC EQUATIONS

We consider the motion of the boundary of two ideal fluids in the field of gravity, **g**. We take the direction of the acceleration **g** to be the $-z$ direction.

Let the shape of the boundary be given by the function $z=\eta(x,y,t)$ which is such that the first, heavier fluid (we shall indicate it by the index 1) occupies the region $-\infty<z<\eta$ and the second, light fluid (index 2) occupies the region $\eta<z<\infty$. The normal to the surface $z=\eta\ (x,y,t)$ is given by the vector

$$\mathbf{n}=(-\nabla\eta,1)\frac{1}{\sqrt{1+(\nabla\eta)^2}}.$$

We assume that the flow of the fluids is irrotational and that the upper fluid as a whole moves relative to the lower one with a velocity $\mathbf{V}$ (parallel to the $x$ axis):

$$\mathbf{V}_1=\nabla\phi_1,\quad \mathbf{V}_2=\mathbf{V}+\nabla\phi_2. \tag{4}$$

By virtue of the incompressibility the potentials $\phi_{1,2}$ are determined by the solution of the Laplace equations

$$\Delta\phi_{1,2}=0 \tag{5}$$

with the boundary conditions at infinity

$$\phi_1\to 0\quad \text{for } z\to-\infty,$$

$$\phi_2\to 0\quad \text{for } z\to+\infty. \tag{6}$$

The boundary conditions at the $z=\eta$ surface can be split into kinematic and dynamic conditions.

The kinematic condition at $z=\eta$,

$$\frac{d\eta}{dt}=\frac{\partial\eta}{\partial t}+(\mathbf{V}\nabla)\eta=V_z, \tag{7}$$

which is valid both for the upper and for the lower fluid, gives the condition that the normal components of the velocity are equal:

$$V_{n,1}|_\Gamma=V_{n,2}|_\Gamma,$$

or

$$\left(\frac{\partial\phi_1}{\partial z}-(\nabla\phi_1\nabla\eta)\right)\Bigg|_{z=\eta}=\left(\frac{\partial\phi_2}{\partial z}-(\mathbf{V}\nabla\eta)-(\nabla\phi_2\nabla\eta)\right)\Bigg|_{z=\eta}. \qquad (8)$$

The pressure difference at the boundary of the two media which is determined by the surface tension gives the dynamic boundary condition:

$$p_2-p_1=\alpha\ \mathrm{div}\frac{\nabla\eta}{\sqrt{1+(\nabla\eta)^2}}. \qquad (9)$$

Using the time-dependent Bernouilli equations we can express this difference in terms of derivatives of the potentials on the $z=\eta$ boundary:

$$\left(\frac{\partial\phi_1}{\partial t}+\frac{(\nabla\phi_1)^2}{2}+g\eta\right)\Bigg|_{z=\eta}+\alpha\ \mathrm{div}\frac{\nabla\eta}{\sqrt{1+(\nabla\eta)^2}}=\varepsilon\left(\frac{\partial\phi_2}{\partial t}+\frac{(\nabla\phi_2)^2}{2}+(\mathbf{V}\nabla\phi_2)+g\eta\right)\Bigg|_{z=\eta}. \qquad (10)$$

Here $\varepsilon$ is the ratio of the densities of the fluids, $\varepsilon=\rho_2/\rho_1$. Equations (5)—(10) form a closed set of equations.

We consider the value of the velocity potentials at the boundary of the two fluids, $\Psi_1=\phi_1|_{z=\eta}$, $\Psi_2=\phi_2|_{z=\eta}$, and use them to construct a new quantity

$$\Psi=\rho_1\Psi_1-\rho_2\Psi_2. \qquad (11)$$

Specifying $\Psi_{1,2}$ guarantees the solvability of the Laplace equations (5). Through direct calculations we can then establish that the set (5)—(10) can be written in Hamiltonian form:[11,12]

$$\frac{\partial\Psi}{\partial t}=-\frac{\delta H}{\delta\eta},\quad \frac{\partial\eta}{\partial t}=\frac{\delta H}{\delta\Psi}, \qquad (12)$$

where the Hamiltonian

$$H=\frac{1}{2}\int_{z\geqslant\eta}\varepsilon[(\nabla\phi_2)^2+2(\nabla\phi_2\mathbf{V})]d^3r+\frac{1}{2}\int_{z\leqslant\eta}(\nabla\phi_1)^2d^3r+\int\left[\frac{g\eta^2}{2}(1-\varepsilon)+\alpha(\sqrt{1+(\nabla\eta)^2}-1)\right]d^2r. \qquad (13)$$

is the same as the total energy of the two fluids, apart from a constant. Here and henceforth we shall assume that the density satisfies $\rho_1=1$.

For the later calculations it is convenient to write the Hamiltonian (13) in the form of a surface integral:

$$H=\frac{1}{2}\int[V_n\Psi\sqrt{1+(\nabla\eta)^2}+\varepsilon\Psi_2(\mathbf{V}\nabla)\eta+(1-\varepsilon)g\eta^2+2\alpha(\sqrt{1+(\nabla\eta)^2}-1)]d^2\mathbf{r}. \qquad (14)$$

Here and everywhere in what follows $\mathbf{r}=(x,y)$ is a two-dimensional vector in the horizontal plane and $\nabla$ is the gradient operator with respect to $x$ and $y$.

We must express the normal velocity component $V_n$ and the potential $\Psi_2$ in Eq. (14) in terms of $\Psi$ and $\eta$. To do this we first find the solution of the Laplace equations (5) with the boundary conditions (6):

$$\phi_{1,2}(\mathbf{r},z)=\int\phi_{1,2k}(0)\exp(\pm kz+i\mathbf{kr})d^2k, \qquad (15)$$

where

$$\phi_{1,2k}(0)=\frac{1}{(2\pi)^2}\int\phi_{1,2}(\mathbf{r},0)\exp(-i\mathbf{kr})d^2\mathbf{r},$$

$$\mathbf{k}=(k_x,k_y),$$

and afterwards use the conditions (8) and (11) at the $z=\eta$ boundary to determine the required dependence. However, even in the first step it is clear that explicit expressions for $V_n$ and $\Psi_2$ can be found only by expanding all quantities in (8) and (11) in series in the parameter $|\nabla\eta|$, the small angle of inclination of the surface. As a result we expand the Hamiltonian in powers of the canonical variables. One sees easily that the Hamiltonian (14) is quadratic in $\Psi$.

## 3. THE KELVIN–HELMHOLTZ INSTABILITY

We discuss how one can use the Hamiltonian approach to solve the linear problem of the stability of a plane boundary.

In the linear approximation in $\Psi_2$ and $V_n$ one finds easily from (8) and (11)

$$\Psi_2=-\frac{1}{1+\varepsilon}\{\Psi+\hat{k}^{-1}(\mathbf{V}\nabla\eta)\},$$

$$V_n\approx\frac{\partial\phi_1}{\partial z}\Bigg|_{z=0}=\frac{k\Psi}{1+\varepsilon}-\frac{\varepsilon}{1+\varepsilon}(\mathbf{V}\nabla\eta).$$

Here $\hat{k}$ is a two-dimensional integral operator; its Fourier transform is equal to $k=\sqrt{k_x^2+k_y^2}$. After we substitute these expressions into (14) and change to the $k$ representation, the quadratic Hamiltonian $H^{(2)}$ takes the form

$$H^{(2)}=\frac{1}{2}\int\left\{\frac{k|\Psi_k|^2}{1+\varepsilon}+[(1-\varepsilon)g+\alpha k^2]\left|\eta_k\right|^2\right\}d^2k+\int\frac{\varepsilon}{1+\varepsilon}i(\mathbf{kV})\Psi_k\eta_{-k}d^2k-\frac{1}{2}\int\frac{\varepsilon}{(1+\varepsilon)k}(\mathbf{kV})^2|\eta_k|^2d^2k. \qquad (16)$$

When there is no wind, the first term in $H^{(2)}$ gives gravitational–capillary waves with the dispersion law

$$\omega_k^2=\frac{k}{1+\varepsilon}[(1-\varepsilon)g+\alpha k^2]. \qquad (17)$$

The second integral in the Hamiltonian (16) is a conserved quantity,

$$\frac{\varepsilon}{1+\varepsilon}(\mathbf{VP}),$$

where **P** is the total momentum of the system. This term can be removed by changing to a comoving system of coordinates, moving with the velocity

$$\mathbf{U}=\varepsilon\,\frac{\mathbf{V}}{1+\varepsilon}.$$

Everywhere in what follows we shall therefore write $H^{(2)}$ in the form

$$H^{(2)}=\frac{1}{2}\int\left[k\left|\Psi_k\right|^2+\frac{1}{k}\left(\omega_k^2-\frac{\varepsilon(\mathbf{kV})^2}{1+\varepsilon}\right)\left|\eta_k\right|^2\right]d^2k. \tag{18}$$

Hence it follows that a plane boundary becomes unstable under the condition

$$-\Gamma_k^2=\omega_k^2-\frac{\varepsilon(\mathbf{kV})^2}{1+\varepsilon}<0. \tag{19}$$

This instability is called the Kelvin–Helmholtz instability and its growth rate is equal to $\Gamma_k$.

In what follows we shall assume that the parameter $\varepsilon$ is small and this considerably simplifies the whole subsequent analysis. It is clear from (19) that the Kelvin–Helmholtz instability occurs for velocities $V$ larger than the critical one,

$$V_{\rm cr}=\frac{1}{\sqrt{\varepsilon}}\min\frac{\omega_k}{k}=\sqrt{\frac{2g}{\varepsilon k_0}},$$

where $k_0=\sqrt{g/\alpha}$ is the value of the wavenumber of the neutral perturbation corresponding to the instability threshold. If one is just above criticality,

$$\delta=\frac{V^2-V_{\rm cr}^2}{V_{\rm cr}^2}\ll 1$$

the instability is quasimonochromatic in character. Wave excitation occurs in a small neighborhood of $\mathbf{k}=\mathbf{k}_0$ of width $\Delta k\sim k\sqrt{\delta}$ (here the vector $\mathbf{k}_0$ is parallel to the wind velocity and magnitude is equal to $k_0$). In this case the growth rate $\Gamma_k$ can be written in the form

$$\Gamma_k^2=\omega_0^2\left[\delta-\frac{1}{2}\frac{q_x^2}{k_0^2}-\frac{q_y^2}{k_0^2}\right], \tag{20}$$

where

$$\mathbf{k}=\mathbf{k}_0+\mathbf{q},\quad |q|\ll k_0,\quad V=(V,0,0),\quad \omega_0^2=2gk_0.$$

Near the instability threshold it is thus natural to change to envelopes:

$$\Psi(\mathbf{r},t)=\Psi_1(\mathbf{r},t)\exp(i\mathbf{k}_0\mathbf{r})+\Psi_1^*(\mathbf{r},t)\exp(-i\mathbf{k}_0\mathbf{r})$$

$$\eta(\mathbf{r},t)=\eta_1(\mathbf{r},t)\exp(i\mathbf{k}_0\mathbf{r})+\eta_1^*(\mathbf{r},t)\exp(-i\mathbf{k}_0\mathbf{r}), \tag{21}$$

where $\Psi_1$ and $\eta_1$ are slowly varying functions of $\mathbf{r}$,

$$\Psi_1(\mathbf{r},t)=\frac{1}{(2\pi)^2}\int\Psi(\mathbf{k}_0+\mathbf{q})\exp(i\mathbf{qr})d^2q,$$

$$\eta_0(\mathbf{r},t)=\frac{1}{(2\pi)^2}\int\eta(\mathbf{k}_0+\mathbf{q})\exp(i\mathbf{qr})d^2q.$$

Substituting (21) into the Hamiltonian (18) and carrying out the necessary averaging we find

$$H^{(2)}=\frac{1}{2}\int\left\{k_0\left|\Psi_1\right|^2-\frac{\omega_0^2}{k_0}\left[\delta\left|\eta_1\right|^2-\frac{1}{k_0^2}\left(\frac{|\eta_{1x}|^2}{2}+\left|\eta_{1y}\right|^2\right)\right]\right\}d^2r. \tag{22}$$

After a simple scaling,

$$\omega_0 t\rightarrow t,\quad \sqrt{2}k_0x\rightarrow x,\quad k_0y\rightarrow y$$

the equation of motion for the amplitude $\eta_1$ corresponding to this Hamiltonian takes the form of the linear Klein–Gordon equation:

$$\eta_{1tt}-\nabla^2\eta_1=\delta\eta_1. \tag{23}$$

This equation changes as $\delta$, the amount by which threshold is exceeded, increases. For an arbitrary value of $\delta$ the equation for $\eta_1(\mathbf{r},t)$ becomes an integral equation:

$$\frac{\partial^2\eta_1}{\partial t^2}=[\hat{\omega}_k^2+\varepsilon(\mathbf{V}\nabla)^2]\eta_1,$$

where we can use Eq. (17) to express the operator $\hat{\omega}_k^2$ in terms of the operator $\hat{k}$.

Concluding this section we discuss a number of physical consequences connected with the existence of the small parameter $\varepsilon$, the ratio of the densities of the upper and the lower media. We have already discussed one of these effects in the Introduction: the threshold value of the velocity is larger by a factor $1/\sqrt{\varepsilon}$ than the minimum phase velocity of the surface waves. Another consequence which is very important for the nonlinear analysis can be obtained from a comparative estimate of $\Psi_1$ and $\Psi_2$ in the linear stage of the instability (indeed, it is sufficient to consider the case where one is just above criticality):

$$\begin{pmatrix}\Psi_1\\ \Psi_2\\ \omega_0\eta/k_0\end{pmatrix}\sim\begin{pmatrix}\sqrt{\delta}-i\sqrt{\varepsilon}\\ -i/\sqrt{\varepsilon}\\ 1\end{pmatrix}. \tag{24}$$

This estimate shows that the fluctuations in the velocity of the upper component are at least $1/\sqrt{\varepsilon}$ times larger than the velocity of the lower component. Physically this situation is very transparent: the upper fluid being considerably lighter, it rapidly follows all changes connected with the motions of the lower fluid which are slow because of its higher density and, hence, its larger inertia. One understands easily that the relation between the velocity fluctuations will be maintained both when one goes further beyond criticality and also in the nonlinear stage of this instability. In particular, this means that to leading order in $\varepsilon$ the velocity component of the upper fluid will be equal to zero:

$$\frac{\partial\phi_2}{\partial n}\sqrt{1+(\nabla\eta)^2}=(\mathbf{V}\nabla)\eta. \tag{25}$$

The difference between the left-hand and the right-hand sides of this equation gives a boundary condition in the next order—the equality (8) of the normal components of the velocities.

The approximate boundary condition (25) together with the Laplace equation makes it possible to express the potential $\Psi_2$ solely in terms of $\eta$. It will be shown in the next section that the nonlinearity which then arises on the same level as the nonlinearity due to capillarity is the main one in the nonlinear stage of the Kelvin–Helmholtz instability.

## 4. NONLINEAR INTERACTION

We turn to the analysis of the nonlinear interaction. It is clear from expression (14) for the Hamiltonian that the nonlinearity come from three terms—the first, the second, and the third one. In order to find explicit expressions of the first two terms in terms of $\Psi$ and $\eta$ we must express the normal velocity component $V_n$ and the potential $\Psi_2$ as series in the parameter $k\eta$.

It follows from the estimate (24) that if one is above criticality, so that $\delta > \varepsilon$ holds, the potential $\Psi$, apart from small terms of the order of $\varepsilon$, $\delta$, or $k\eta$, is the same as the value of the potential $\Psi_1$ of the lower fluid at the boundary. This means that the ratio of the cubic term to the quadratic one, $(1/2)\int k|\Psi_k|^2 d^2k$, will be of order $k\eta$. Accordingly, the fourth-order terms will contain an additional small factor $k\eta$ compared to the cubic one, and so on.

Similar relations between the cubic and the quadratic terms arise for the potential $\Psi_2$ when one solves the Laplace equation (5) with the approximate boundary conditions (25) and, hence, for the second term in the Hamiltonian (14). These evaluations are valid for $\delta \ll \varepsilon$. Hence, it is clear that the ratio of the terms in each order from the first and the second terms will be on the order of the ratios of their quadratic terms, i.e.,

$$\int k|\Psi_k|^2 d^2k : \int \frac{\varepsilon(\mathbf{kV})^2}{k}|\eta_k|^2 d^2k \approx \frac{\Gamma'^2_{k_0}}{\omega'^2_{k_0}} \sim \delta. \qquad (26)$$

In the region of small $\delta$ the main contribution to the nonlinear interaction will thus be connected with the renormalized velocity. In that case the interaction Hamiltonian has the form

$$H_{\text{int}} = \frac{\varepsilon}{2}\int \delta\Psi_2(\mathbf{V}_{\text{cr}}\nabla)\eta d^2r + \int \alpha\left(\sqrt{1+(\nabla\eta)^2} - 1 - \frac{(\nabla\eta)^2}{2}\right) d^2r, \qquad (27)$$

where $\delta\Psi_2$ is the correction to $\Psi_2$ which is nonlinear in the parameter $k\eta$.

It is important to note that by virtue of the boundary conditions (25) this interaction Hamiltonian is a functional depending solely on $\eta$. One sees easily that the contribution of the kinetic energy to $H_{\text{int}}$ will be proportional to $\varepsilon V^2$. It will be shown in what follows that just above criticality this term gives a renormalization of the instability growth rate.

To find the $\eta$-dependence of $\Psi_2$ we introduce an operator $\hat{L}$ defined by

$$\Psi_2 = \phi_2(r,\eta) = \hat{L}\phi_2(r,0) = \left(1 + \eta\frac{\partial}{\partial z} + \frac{1}{2}\eta^2\frac{\partial^2}{\partial z^2} + \dots\right)\phi_2(r,0). \qquad (28)$$

It is clear that for harmonic functions which vanish as $z \to \infty$ the shift operator $\hat{L}$ can be expressed in terms of the integral operator $\hat{k}$:

$$\hat{L} = 1 - \eta\hat{k} + \frac{1}{2}\eta^2\hat{k}^2 - \dots \qquad (29)$$

Analogous to the definition (28) we have for $\partial\phi_2/\partial z|_{z=\eta}$

$$\left.\frac{\partial\phi_2}{\partial z}\right|_{z=\eta} = \hat{L}[-\hat{k}\phi_2(0)].$$

Hence we easily find the solution of Eq. (25) for $\Psi_2$:

$$\Psi_2 = \hat{L}[-\hat{L}\hat{k} - (\nabla\eta\hat{L}\nabla)]^{-1}(\mathbf{V}\nabla)\eta.$$

Expanding $\Psi_2$ in the parameter $k\eta$ and substituting it into (27) for $H_{\text{int}}$ we get for the terms which are of third and fourth order in the amplitude

$$H^{(3)} = \frac{\varepsilon}{2}\int (\mathbf{V}\nabla)\eta\{\eta - \hat{k}^{-1}\eta\hat{k} + \hat{k}^{-1}(\nabla\eta,\nabla)\hat{k}^{-1}\} \times (\mathbf{V}\nabla)\eta d^2r, \qquad (30)$$

$$\begin{aligned} H^{(4)} = \frac{\varepsilon}{2}\int (\mathbf{V}\nabla)\eta\{&\eta^2\hat{k} - (\nabla\eta^2,\nabla)\hat{k}^{-1} - \hat{k}^{-1}\eta\hat{k}\eta\hat{k} \\ &+ \hat{k}^{-1}\eta\hat{k}(\nabla\eta,\nabla)\hat{k}^{-1} + \hat{k}^{-1}(\nabla\eta,\nabla)\hat{k}^{-1}\eta\hat{k} \\ &- \hat{k}^{-1}(\nabla\eta,\nabla)\hat{k}^{-1}(\nabla\eta,\nabla)\hat{k}^{-1}\}(\mathbf{V}\nabla)\eta d^2r \\ &- \frac{1}{8}\int \alpha(\nabla\eta)^4 d^2r. \end{aligned} \qquad (31)$$

It is well known that the nonlinear interaction leads to the generation of harmonics. If the system is close to the instability threshold multiples appear in addition to the basic spatial harmonic close to $\mathbf{k} = \mathbf{k}_0$. This means that we must write $\Psi(\mathbf{r},t)$ and $\eta(\mathbf{r},t)$ not in the form (21), but in the form

$$\begin{aligned} \Psi(\mathbf{r},t) = &\Psi_0 + \Psi_1(\mathbf{r},t)\exp(i\mathbf{k}_0\mathbf{r}) + \Psi_1^*(\mathbf{r},t) \\ &\times\exp(-i\mathbf{k}_0\mathbf{r}) + \Psi_2(\mathbf{r},t)\exp(2i\mathbf{k}_0\mathbf{r}) + \Psi_2^*(\mathbf{r},t) \\ &\times\exp(-2i\mathbf{k}_0\mathbf{r}) + \dots, \\ \eta(\mathbf{r},t) = &\eta_0 + \eta_1(\mathbf{r},t)\exp(i\mathbf{k}_0\mathbf{r}) + \eta_1^*(\mathbf{r},t) \\ &\times\exp(-i\mathbf{k}_0\mathbf{r}) + \eta_2(\mathbf{r},t)\exp(2i\mathbf{k}_0\mathbf{r}) + \eta_2^*(\mathbf{r},t) \\ &\times\exp(-2i\mathbf{k}_0\mathbf{r}) + \dots\,. \end{aligned} \qquad (32)$$

The amplitudes of the harmonics $\Psi_i(\mathbf{r},t)$ and $\eta_i(\mathbf{r},t)$ are assumed here, as before, to be slowly varying compared to the exponentials. Their distributions in $k$-space will have widths on the order of the width of the main harmonic. The ratio of the amplitudes of the zeroth and the second harmonics to the amplitude of the first harmonic will be a small quantity in the case of a weak nonlinearity, i.e., at small angles. In that case their effect in the next order in the amplitude of the scillations for the main, first, harmonic leads to a renormalization of the linear growth rate. It is clear that this renormalization will be proportional to $|\eta_1|^2$.

In evaluating the contribution from these processes it is convenient to change in the Hamiltonian (30) to the $k$-representation. As a result $H^{(3)}$ takes the form

$$H^{(3)}=\frac{1}{3}\int U_{k_1k_2k_3}\eta_{k_1}\eta_{k_2}\eta_{k_3}\delta(k_1+k_2+k_3)$$

$$\times\frac{d^2k_1d^2k_2d^2k_3}{2\pi},$$

where

$$U_{k_1k_2k_3}=-\frac{\varepsilon}{2}\{(\mathbf{k}_1\mathbf{V})(\mathbf{k}_2\mathbf{V})C_{12}+(\mathbf{k}_2\mathbf{V})(\mathbf{k}_3\mathbf{V})C_{23}$$

$$+(\mathbf{k}_3\mathbf{V})(\mathbf{k}_1\mathbf{V})C_{31}\},$$

$$C_{ij}=1+\frac{(\mathbf{k}_i\mathbf{k}_j)}{k_ik_j}.$$

Using (32) we can write down the equations of motion for the separate spatial harmonics. In particular, for the first harmonic we have

$$\frac{\partial^2\eta_1(q)}{\partial t^2}=\Gamma^2_{k_0+q}\eta_1(q)$$

$$-k_0\int[U_{-k_0,2k_0,-k_0}\eta_2(q_1)\eta_{-1}(q_2)$$

$$+U_{-k_0,k_0,q_2}\eta_1(q_1)\eta_0(q_2)\delta(q-q_1-q_2)]$$

$$\times\frac{d^2q_1d^2q_2}{2\pi}. \quad (33)$$

We can write the equation for the second harmonic as follows:

$$\frac{\partial^2\eta_2(q_1)}{\partial t^2}=\Gamma^2_{2k_0}\eta_2(q_1)-2k_0\int U_{-2k_0,k_0,k_0}\delta(q_1-q_2$$

$$-q_3)\eta_1(q_2)\eta_1(q_3)\frac{d^2q_2d^2q_3}{2\pi}.$$

Neglecting the time derivatives in this equation and assuming that for $\delta\ll 1$ we have $\Gamma^2_{2k_0}\approx-\omega_0^2$ we find

$$\eta_2(q_1)=-\frac{2k_0}{\omega_0^2}\int U_{-2k_0,k_0,k_0}\eta_1(q_2)\eta_1(q_3)$$

$$\times\delta(q_1-q_2-q_3)\frac{d^2q_2d^2q_3}{2\pi}.$$

Substituting this expression into (34) and assuming $U_{-2k_0,k_0,k_0}=-\varepsilon(\mathbf{k}_0\mathbf{V})^2$, we find that the contribution to the averaged Hamiltonian $\bar{H}$ from the interaction processes of the first and second harmonics can be expressed in the form of an integral:

$$\bar{H}_{2k_0}=-\int\omega_0^2 2k_0|\eta_1(\mathbf{r},t)|^4d^2r. \quad (34)$$

Similarly we look for the contribution from the zeroth harmonic:

$$\frac{\partial^2\eta_0(q_3)}{\partial t^2}=-\omega^2_{q_3}\eta_0(q_3)-|q_3|\int U_{q_3,k_0,-k_0}\delta(q_3-q_1-q_2)$$

$$\times\eta_1(q_1)\eta_{-1}(q_2)\frac{d^2q_1d^2q_2}{2\pi}. \quad (35)$$

In this equation one can approximately put $\omega^2_{q_3}=g|q_3|$. The asymptotic form of the matrix element as $q\to 0$, is

$$U_{k_0,-k_0-q,q}\simeq-\varepsilon(\mathbf{k}_0\mathbf{V})(\mathbf{q}\mathbf{V})\frac{(\mathbf{k}_0\mathbf{q})}{|k_0||q|}\to 0, \quad (36)$$

i.e., the matrix element tends to zero like $q$. The left-hand side of Eq. (35) can be estimated as $\omega_0^2\delta\eta_0$. It is small compared to the linear term on the right-hand side, which is of the order of $\omega_0^2\sqrt{\delta}\eta_0$. Taking into account the behavior of the matrix element (36) we get as a result a contribution from the zeroth harmonic which is small in the parameter $\delta$ compared to the contribution from the second harmonic.

We now find the contribution from the 2→2 processes themselves. To do this we substitute Eq. (21) for $\eta$ into the Hamiltonian $H^{(4)}$ given by (31) and average. As a result we get

$$\bar{H}^{(4)}=\frac{5}{8}\omega_0^2k_0\int|\eta_1|^4d^2r. \quad (37)$$

Collecting all terms, (22), (34), and (37), together we are led to the following averaged expression for the Hamiltonian:

$$\bar{H}=\int\left\{k_0\left|\Psi\right|^2-\frac{\omega_0^2}{k_0}\left(\delta\left|\eta\right|^2-\frac{1}{2k_0^2}\left|\eta_x\right|^2\right.\right.$$

$$\left.\left.-\frac{1}{k_0^2}\left|\eta_y\right|^2\right)\right\}d^2r-\frac{11}{8}\omega_0^2k_0\int|\eta|^4d^2r. \quad (38)$$

Here and everywhere in what follows we drop the index of the envelopes.

After scaling,

$$\omega_0t\to t,\quad \sqrt{2}k_0x\to x,\quad k_0y\to y,\quad \sqrt{\frac{11}{4}}k_0\eta\to\eta$$

we can rewrite the Hamiltonian (38) in the form

$$H=\int\left(\left|\eta_t\right|^2-\delta\left|\eta\right|^2+\left|\nabla\eta\right|^2-\frac{1}{2}\left|\eta\right|^4\right)d^2r. \quad (39)$$

Accordingly, the Hamiltonian of the linear Klein–Gordon equation (22) acquires a nonlinear correction:

$$\eta_{tt}=\delta\eta+\Delta\eta+|\eta|^2\eta. \quad (40)$$

The main feature of this equation, like that of the Hamiltonian (39) which generates it, is the fact that the nonlinearity in the first nonvanishing order does not lead to stabilization of the instability. One can consider the nonlinearity in Eq. (40) as a correction to the critical velocity $V_{cr}$, reducing its magnitude. It is clear that the nonlinearity amplifies the instability.

It is important to note also that Eq. (40) describes the behavior of the system both for positive $\delta$, corresponding to an instability, and also for $\delta<0$, when there is no instability. Although the system is stable for $V<V_{cr}$ and small $\delta$ in the linear approximation, for finite amplitudes an instability is possible due to the nonlinearity. The excitation of oscillations in that case will have a stable character, and so the amplitude becomes infinite after a finite time. This statement,

almost self-evident for uniform (coordinate independent) solutions requires a special discussion in the case of initial conditions localized in some region.

## 5. CRITERION FOR WAVE COLLAPSE

To find the conditions for a stable excitation we must study the solutions of Eq. (40). The simplest of them is the spatially uniform solution $\eta=\eta(t)$. In that case (40) is a Newtonian equation for the motion of a particle in a spherically symmetric potential. Putting $\eta(t)=\mathrm{Re}\, e^{i\phi}$ we get the equation

$$\ddot{R}=-\frac{\partial U_{\mathrm{eff}}}{\partial R}, \tag{41}$$

where the effective potential is defined by the expression

$$U_{\mathrm{eff}}=-\frac{\delta}{2}R^2-\frac{R^4}{4}+\frac{M^2}{2R^2}, \tag{42}$$

and $M=R_0^2\dot{\phi}=\mathrm{const}$ is the angular momentum with $R_0=R(t=0)$. Equation (41) has an energy integral

$$E=\frac{\dot{R}^2}{2}+U_{\mathrm{eff}},$$

whence we see easily for what values of $E$ and $M$ it is possible for $R$ to become infinite. In that case the time $t_0$ for reaching the singularity will be determined by the integral

$$t_0=\int_{R_0}^{\infty}\frac{dR}{\sqrt{2(E-U_{\mathrm{eff}})}}.$$

In the vicinity of $t=t_0$ the function $R(t)$ has the asymptotic form

$$R(t)\simeq\frac{\sqrt{2}}{t_0-t}. \tag{43}$$

[For $\delta=M=0$ this is an exact solution of (41).] Apart from the spatially uniform solution considered above, near the singularity when the nonlinearity is much larger than the pumping ($|\eta|^2\gg\delta$) Eq. (40) admits a self-similar substitution

$$\eta=\frac{1}{t_0-t}g\left(\frac{r}{t_0-t}\right). \tag{44}$$

We can find $g(\xi)$ for axially symmetric solutions by solving the ordinary differential equation

$$(1-\xi^2)g''+\frac{1-4\xi^2}{\xi}g'+g(|g|^2-2)=0.$$

After substituting $g=\mathrm{Re}^{i\varphi}$ we can write this equation in the form of the set

$$(1-\xi^2)(R''-\varphi'^2R)+\frac{1-4\xi^2}{\xi}R'+R(R^2-2)=0,$$

$$(1-\xi^2)(2\varphi'R'+\varphi''R)+\frac{1-4\xi^2}{\xi}\varphi'R=0.$$

The second equation of this set can be integrated and as a result we have

$$\varphi'=\frac{C}{R^2\xi|1-\xi^2|^{3/2}}.$$

The requirement that $\varphi$ be finite at the origin means that we have $C=0$. Therefore we find $\varphi=\mathrm{const}$ and $R$ satisfies the equation

$$R''+\frac{1-4\xi^2}{\xi(1-\xi^2)}R'+R\frac{R^2-2}{1-\xi^2}=0,\quad R'(0)=0. \tag{45}$$

The solution of Eq. (45) for $\xi\to\infty$ has the asymptotic form

$$R\simeq c_1\xi^{-2}+c_2\xi^{-1}.$$

It can be normalized ($\int R^2d^2\xi<\infty$) for $c_2=0$. All integrals in expression (39) for $H$ turn out to be also finite in that case. This means that the total Hamiltonian in this solution is equal to zero (otherwise it would depend on the time). The square of the norm $\int R^2d^2\xi$ is for this solution also independent of the time (under the condition $c_2=0$), which is incorrect in the general case. This self-similar solution can therefore not be considered as the asymptotic form for the initial condition of the general situation. We can determine what solution actually develops by solving Eq. (40) numerically. We must add to what we have said that, apart from this, another equally important problem remains open—the problem of the stability of the solutions considered above. However, it is clear physically that if the initial condition for Eq. (40) has a sufficiently extended plateau the solution in the region of the plateau will be nearly spatially uniform. We shall show in what follows that the integral criteria for a wave collapse for Eq. (40) confirm this consideration. Moreover, one can explain that these criteria are only weakly sensitive to a change in the dimensionality as, for instance, is the case for the nonlinear Schrödinger equation.[13,14]

We consider the time evolution of the quantity $B=\int|\eta|^2d^2r>0$. Equation (40) makes it possible to write

$$\frac{d^2B}{dt^2}=\int\{2|\eta_t|^2+\eta_{tt}\eta^*+\eta_{tt}^*\eta\}d^2r=-4H$$
$$+\int\{6|\eta_t|^2+2|\nabla\eta|^2+2|\sqrt{\delta}||\eta|^2\}d^2r.$$

Multiplying both sides of this equation by $B$ and using the Cauchy–Bunyakov inequality we have the following inequality:

$$B_{tt}B-\tfrac{3}{2}B_t^2\geqslant-4HB+2\delta B^2. \tag{46}$$

This form of the majorizing inequality is the most general one known in the literature for obtaining a criterion for collapse.[15] A particular case of (46) is the inequality

$$B_{tt}B-\tfrac{3}{2}B_t^2\geqslant 0, \tag{47}$$

obtained from (46) for $H<0$ and $\delta>0$. An inequality of the form (47) was first used in Ref. 16 and later in a paper by Kalantarov and Ladyzhenskaya[17] for obtaining an integral criterion for collapse in the one-dimensional Boussinesq equation. In the present paper we analyze the general inequality (46) and find sufficient criteria for collapse.

It is convenient to rewrite inequality (46), giving it the form of Newton's second law by introducing a new quantity $A$ instead of $B$: $B=A^{-2}$. As a result we get for $A$ the inequality

$$A_{tt}\leqslant-\frac{\partial V(A)}{\partial A}, \tag{48}$$

where $A$ has the meaning of a coordinate and the quantity

$$V(A)=-\frac{HA^4}{2}+\delta\frac{A^2}{2}$$

that of the potential energy of a "particle" (cf. (41)).

If the norm of $B$ becomes infinite after a finite time this will mean that the solution of Eq. (41) is no longer smooth and a singularity arises in it no later than when $B$ becomes an infinite quantity. The appearance of the singularity means for the quantity $A$ that the "particle" would reach the origin ($A=0$) after a finite time. If the "velocity" $A_t$ is negative one can integrate inequality (48) once:

$$E(t)=\frac{A_t^2}{2}+V(A)\geqslant E(0). \tag{49}$$

The sign of the inequality means that the "particle" on its motion to the center acquires energy. One easily finds from this equation all possible cases when the "particle" reaches the point $A=0$. Collapse occurs

1) for $H<0$ and $\delta<0$, if $E(0)>0$ and $A_t<0$;
2) for $H<0$ and $\delta>0$, if $A_t<0$;
3) for $H>0$ and $\delta>0$, if $A_t<0$, $A^2(0)<\delta/2H$ and $E(0)<\delta^2/8H$ or if $A_t<0$ and $E(0)>\delta^2/8H$.

In all these cases one can give an estimate for the upper limit of the time $t_0$ for collapse using the integral

$$t_0\leqslant\int_0^{A(0)}\frac{dA}{\sqrt{2[E(0)-V(A)]}}.$$

Let us note that the sign $\neq$ in Eq. (48) means that, except of the potential force, $\partial V(A)/\partial A$, an additional negative force directed towards the $A=0$, acts on the "particle." Therefore, the condition $A_t(0)<0$ in the first and second cases is not a must. Obviously in these cases the particle, after reflection, will reach point $A=0$.

Near the singularity the norm $B=\int|\eta|^2d\mathbf{r}$ becomes infinite as

$$\int|\eta|^2d\mathbf{r}\geqslant\frac{C}{(t_0-t)^2}. \tag{50}$$

This estimate follows from the fact that the "particle" near the origin, $A=0$, has a well defined velocity. Therefore as $t\rightarrow t_0$ the quantity $A$ vanishes at least linearly:

$$A\leqslant C_1(t_0-t),$$

and this gives the estimate (50) for $B$.

It is important to note that the estimate (50) is independent of the dimensionality of the space. It is clear from a comparison with (43) that it corresponds to an increase in the amplitude for spatially uniform solutions according to the law $\eta\propto(t_0-t)^{-1}$ for an almost unchanged or even an increasing region of the collapsing solution. Compression of the distribution during the collapse is unlikely. If compression is possible, it is considerably weaker than the self-similar law (44).

## 6. CONCLUDING REMARKS

We have thus shown that the generation of waves by a wind due to the Kelvin–Helmholtz instability has an explosive nature in the nonlinear stage. This growth occurs up to angles of order unity when Eq. (40) loses its applicability. Note that Eq. (40) was obtained assuming that the value of $k\eta$ was small. The cancellation of the fourth-order terms which were taken into account in (40) is therefore possible only when one takes into account the main nonlinearities which automatically gives an angle $k\eta\sim1$. For such angles capillarity will play a decisive role. It is natural to assume that this explosive growth is stopped by the collapse of the waves, the formation of a drop, and the appearance of foam on the crests of the waves. We must note that this is in accordance with satellite and airplane data. According to Refs. 6–10 one observes for wind speeds close to the critical velocity (1) a steep increase of the fraction of the sea surface occupied by foam.

We now discuss the role played by another mechanism for wave excitation by a wind which is connected with the Miles instability.[4] We noted already in the Introduction that the Kelvin–Helmholtz instability presupposes the existence of a tangential discontinuity. For real fluids (water and air) viscosity destroys the tangential discontinuity, forming a boundary layer near the surface. The flow in the boundary layer is shear flow. The instability predicted by Miles arises due to a resonance between a surface wave $\omega_k$ and the shear flow in the so-called coincidence layer $z=z^*$ where the phase velocity of the oscillations is comparable with the flow velocity:

$$\omega/k=V(z^*).$$

This instability and the beam instability in a plasma[18] have much in common. The Miles instability is the analog of the kinetic beam instability when a beam is considerably broadened in energy so that its growth rate is determined by the structure factor of the beam distribution. For sufficiently cold beams the instability has a hydrodynamic character and is independent of the details of the beam distribution function. The Kelvin–Helmholtz instability is the analog of the hydrodynamic beam instability. It is independent of the details of the transitional boundary layer and is determined only by the magnitude of the wind speed outside this layer. This problem goes far beyond the framework of the present paper. Here we note only that when waves are excited due to the Kelvin–Helmholtz instability one must at least satisfy the following condition: the wavelength of the excited oscillations must be long compared with the thickness of the boundary layer. In that case the flow outside the boundary layer can accurately be assumed to be irrotational, and this leads to the Kelvin–Helmholtz instability. For wind speeds around 6 m/s one can distinguish in the wave spectrum the gravitational and the capillary wavelength scales. Taking as a typical gravitational wavelength $\lambda\approx1$ m one can estimate the thickness $h$ of the boundary layer when a wind with a speed of 6 m/s blows on the hump. For this wavelength the thickness of the boundary layer is determined from the formula (see, e.g., Ref. 1)

$$h \sim \lambda/\sqrt{\mathrm{Re}},$$

where Re is the Reynolds number. Substituting real parameters with $\lambda=1$ m gives $h \simeq 0.16$ cm, which is small compared to the wavelength $\lambda_0=2\pi/k_0=1.7$ cm. This means that in this situation the Kelvin–Helmholtz instability will play an important role. If the thickness of the boundary layer is large compared to $\lambda_0$ the Miles instability will be the main one and, accordingly, for the nonlinear regime the Newell–Zakharov theory[5] is the main one. In a real situation apparently both mechanisms operate. They are united in the fact that both explain the strong increase of foam for the same velocities.

In conclusion the authors thank V. E. Zakharov for useful discussions and also R. E. Glasman, who kindly drew their attention to Refs. 6–10.

The present work was carried out with support from the Russian Fund for Fundamental Research (Grant No 94-01-00898) and from the Department of Marine Studies (ONR Contract No N00014-92-J-1343).

[1] L. D. Landau and E. M. Lifshitz, *Hydrodynamics*, Nauka, Moscow (1988) [English translation published by Pergamon].
[2] L. D. Landau, Zh. Éksp. Teor. Fiz. **11**, 592 (1941) [J. Phys. USSR **5**, 71 (1941)].
[3] J. W. Miles, J. Fluid Mech. **3**, 185 (1957); **6**, 568 (1959); **13**, 433 (1962); **56**, 427 (1993).
[4] A. C. Newell and V. E. Zakharov, Phys. Rev. Lett. **69**, 1149 (1992).
[5] V. E. Zakharov and N. N. Filonenko, Dokl. Akad. Nauk SSSR **170**, 1292 (1966) [Sov. Phys. Dokl. **11**, 881 (1967)]; Zh. Prikl. Mekh. Tekh. Fiz. No 5, 62 (1968).
[6] V. G. Bondur and E. A. Sharkov, Okeanologiya **6**, 22 (1982).
[7] R. L. Snyder and R. M. Kennedy, J. Phys. Oceanogr. **13**, 1482 (1983).
[8] R. E. Glasman and P. Weichman, J. Geophys. Res. **94** (C4), 4998 (1989).
[9] E. C. Mohahan, J. Geophys. Res. **95** (C2), 1767 (1990).
[10] V. E. Zakharov, in Proc. IUTAM Congress on Wave Breaking, Sydney, Australia, Springer Verlag, Berlin (1992).
[11] V. E. Zakharov, Zh. Prikl. Mekh. Tekh. Fiz. No 2, 8 (1968).
[12] E. A. Kuznetsov and M. D. Spektrov, Zh. Éksp. Teor. Fiz. **71**, 262 (1976) [Sov. Phys. JETP **44**, 136 (1976)].
[13] S. N. Vlasov, V. A. Petrishev, and V. I. Talanov, Izv. Vysshh. Uchebn. Zaved. Radiofiz. **13**, 908 (1970) [Radiophys. Quantum Electron. **13**, 716 (1973)].
[14] V. E. Zakharov, Zh. Éksp. Teor. Fiz. **62**, 1745 (1972) [Sov. Phys. JETP **35**, 908 (1972)].
[15] E. A. Kuznetsov, J. J. Rasmussen, K. Rypdal, and S. K. Turitsyn, Physica D (1995), in press.
[16] H. A. Levine, Am. Math. Soc. **192**, 1 (1974).
[17] V. Kalantarov and O. A. Ladyzhenskaya, Zapiski nauchn. seminara LOMI **69**, 77 (1977).
[18] A. N. Mikhaĭlovskiĭ, *Theory of Plasma Instabilities*, Atomizdat, Moscow (1969) [English translation published by Consultants Bureau].

Translated by D.ter Haar