Collapse and stable self-trapping for Bose-Einstein condensates with $1/r^b$-type attractive interatomic interaction potential

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We consider dynamics of Bose-Einstein condensates with long-range attractive interaction proportional to $1/r^b$ and arbitrary angular dependence. It is shown exactly that collapse of a Bose-Einstein condensate without contact interactions is possible only for $b \geqslant 2$. The case $b = 2$ is critical and requires the number of particles to exceed a critical value to allow collapse. The critical collapse in that case is a strong one, trapping into the collapsing region a finite number of particles. The case $b > 2$ is supercritical with an expected weak collapse which traps a rapidly decreasing number of particles during approach to collapse. For $b < 2$ a singularity at $r = 0$ is not strong enough to allow collapse but an attractive $1/r^b$ interaction admits stable self-trapping (a stable three-dimensional soliton solution) even in the absence of an external trapping potential.

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I. INTRODUCTION

The dynamics of Bose-Einstein condensates (BECs) with short-range s-wave interaction has been the subject of extensive research in recent years [1–3]. Condensates with a positive scattering length have a repulsive (defocusing) nonlinearity which stabilizes the condensate with the help of external trap. Condensates with a negative scattering length have an attractive (focusing) nonlinearity which formally admits solitons. However, without a trap these solitons are unstable and their perturbation leads either to the collapse of the condensate [1–4] or to condensate expansion. An external trap prevents expansion of the condensate and makes solitons metastable for a sufficiently small number of atoms. Otherwise, for a larger number of atoms, the focusing nonlinearity results in the collapse of solitons. The effect of a long-range dipolar interaction on BECs was first studied theoretically [5–9] and more recently observed experimentally [10–12] (see also [13,14] for reviews). In particular, collapse of a BEC with dominant dipole-dipole forces predicted based on an approximate variational estimate [7] and obtained based on an exact analysis [8] was recently observed experimentally [15].

Here we look for the possibility of collapse of a BEC due to a long-range attraction versus formation of a stable self-trapped condensate for a general type of long-range interaction,

$$V(\mathbf{r}) = \frac{f(\mathbf{n})}{r^b}, \quad b > 0, \quad \mathbf{n} \equiv \frac{\mathbf{r}}{r}, \quad r \equiv |\mathbf{r}|, \qquad (1)$$

where $f(\mathbf{n})$ is an arbitrary bounded function, $|f(\mathbf{n})| < \infty$, and $\mathbf{r} = (x_1, x_2, x_3)$. We do not require $f(\mathbf{n})$ to be sign-definite. By attractive interaction we mean that $f(\mathbf{n})$ is negative at least for some nonzero range of angles so that one can choose a wave function to provide a negative contribution to the energy functional.

Possible experimental realizations of (1) are numerous. For example, recent experimental advances allow us to study the interaction of ultracold Rydberg atoms with principal quantum number of about 100 (see, e.g., [16–20]). These interactions between atoms in highly excited Rydberg levels are long range and dominated by dipole-dipole–type forces. The strength of the interaction between Rb atoms is about 10^{12} times stronger (at typical distances of $\sim 10\ \mu$m) than that between Rb atoms in the ground state (see, e.g., [19] for a review). The strength and angular dependence of the interaction between Rydberg atoms can be tuned in a wide range [19,21]. For example, the spatial dependence for Rb with principal quantum number $n \simeq 100$ can be $\propto 1/r^3$ for $r \lesssim 9.5\ \mu$m and $\propto 1/r^6$ (van der Waals character) for $r \gtrsim 9.5\ \mu$m [19]. Another alternative is to admix Rydberg atoms with the ground-state atoms, creating an effective long-range interaction potential [20]. The short-range s-wave scattering interaction is limited to a much smaller distance ($\sim$few nanmeters) so that the range of dominance of the long-range interaction potential is quite high. The radiative lifetime of the Rydberg atoms scales as n^3 and for large n that time is in the millisecond range [19]. If we compare that time scale with the collapse time (0.1 ms) of a BEC with a dipole-dipole interaction [15] one can conclude that observing BEC collapse with Rydberg atoms is feasible. Another possible form of long-range attractive interaction is a gravity-like $1/r$ potential, which is proposed to be realized in a system of atoms with laser-induced dipoles such that an arrangement of several laser fields causes cancellation of anisotropic terms [22]. Terms proportional to $1/r^2$ are also possible [22].

The paper is organized as follows. In Sec. II we use a nonlocal Gross-Pitaevskii equation for the mean-field BEC dynamics with the long-range interaction potential to derive the virial theorem for the mean square radius of the condensate wave function. In Sec. III we show exact the possibility of the BEC collapse for $2 \leqslant b \leqslant 3$ and find the sufficient collapse conditions. In Sec. IV we prove that collapse is impossible for $b < 2$. In Sec. V we show that the ground-state soliton is nonlinearly stable for $b < 2$. In Sec. VI we introduce strong and weak regimes of collapsing solutions and find that collapse for $b = 2$ is in the strong one while for $b > 2$ the weak regime is expected. In Sec. VII we summarize the main results.

II. VIRIAL THEOREM FOR THE NONLOCAL GROSS-PITAEVSKII EQUATION

The mean-field BEC dynamics is governed by a nonlocal Gross-Pitaevskii equation (NGPE)

$$i\hbar\frac{\partial\Psi(\mathbf{r})}{\partial t}=\Big[-\frac{\hbar^2}{2m}\nabla^2+\frac{1}{2}m\omega_0^2\big(\gamma_1^2x_1^2+\gamma_2^2x_2^2+\gamma_3^2x_3^2\big)+g|\Psi(\mathbf{r})|^2+\int d^3\mathbf{r}'V(\mathbf{r}-\mathbf{r}')|\Psi(\mathbf{r}')|^2\Big]\Psi(\mathbf{r}),\quad(2)$$

where Ψ is the condensate wave function, the contact interaction is proportional to $g=4\pi\hbar^2a/m$, a is the s-wave scattering length, m is the atomic mass, ω_0 is the external trap frequency, γ_1,γ_2, and γ_3 are the anisotropy factors of the trap, and the wave function is normalized to the number of atoms, $\int|\Psi|^2d^3\mathbf{r}=N$. The contact interaction term can be also included in the potential $V(\mathbf{r})$ as $\frac{g}{2}\delta(\mathbf{r})$ but we have not done that because we focus here on the effect of the long-rage potential (1). If $V(\mathbf{r})\equiv 0$ then a standard Gross-Pitaevski equation (GPE) [1] is recovered.

The NGPE (2) can be written through variation $i\hbar\frac{\partial\Psi}{\partial t}=\frac{\delta E}{\delta\Psi^*}$ of the energy functional

$$E=E_K+E_P+E_{\rm NL}+E_R,\quad(3)$$

which is an integral of motion: $\frac{dE}{dt}=0$ and

$$E_K=\int\frac{\hbar^2}{2m}|\nabla\Psi|^2d^3\mathbf{r},\quad E_{\rm NL}=\frac{g}{2}\int|\Psi|^4d^3\mathbf{r},$$
$$E_P=\int\frac{1}{2}m\omega_0^2\big(\gamma_1^2x_1^2+\gamma_2^2x_2^2+\gamma_3^2x_3^2\big)|\Psi|^2d^3\mathbf{r},\quad(4)$$
$$E_R=\frac{1}{2}\int|\Psi(\mathbf{r})|^2V(\mathbf{r}-\mathbf{r}')|\Psi(\mathbf{r}')|^2d^3\mathbf{r}d^3\mathbf{r}'.$$

We consider the time evolution of the mean square radius of the wave function, $\langle r^2\rangle\equiv\int r^2|\Psi|^2d^3\mathbf{r}/N$. Using (2), integrating by parts, and taking into account vanishing boundary conditions at infinity one obtains

$$\partial_t\langle r^2\rangle=\frac{\hbar}{2mN}\int 2ix_j(\Psi\partial_{x_j}\Psi^*-\Psi^*\partial_{x_j}\Psi)d^3\mathbf{r},\quad(5)$$

where $\partial_t\equiv\frac{\partial}{\partial t}$, $\partial_{x_j}\equiv\frac{\partial}{\partial x_j}$ and repeated index j means summation over all space coordinates, $j=1,\ldots,3$. After a second differentiation over t, one gets [8]

$$\partial_t^2\langle r^2\rangle=\frac{1}{2mN}\Big[8E_K-8E_P+12E_{\rm NL}-2\int|\Psi(\mathbf{r})|^2|\Psi(\mathbf{r}')|^2\times\big(x_j\partial_{x_j}+x_j'\partial_{x_j'}\big)V(\mathbf{r}-\mathbf{r}')d^3\mathbf{r}\Big],\quad(6)$$

which is called by a virial theorem [8] similar to the GPE [23–28].

It follows from (1) that $(x_j\partial_{x_j}+x_j'\partial_{x_j'})V(\mathbf{r}-\mathbf{r}')=-bV(\mathbf{r}-\mathbf{r}')$ and using (3) we rewrite (6) as

$$\partial_t^2\langle r^2\rangle=\frac{1}{2mN}[4bE+(8-4b)E_K-(8+4b)E_P+(12-4b)E_{\rm NL}].\quad(7)$$

Here the nonlocal nonlinear term E_R was absorbed into E in comparison with (6).

Catastrophic collapse of a BEC in terms of the NGPE means a singularity formation, $\max|\Psi|\to\infty$, in a finite time. Because of conservation of N, the typical size of an atomic cloud near a singularity must vanish. The virial theorem (7) describes collapse when the positive-definite quantity $\langle r^2\rangle$ becomes negative in finite time, implying $\max|\Psi|\to\infty$ before $\langle r^2\rangle$ turns negative. The kinetic energy E_K diverges to infinity at the collapse time. Then the potential energy must also diverge to ensure conservation of the energy functional E. But the divergence of the potential energy implies that $\max|\Psi|\to\infty$ because of the conservation of N. Another way to see divergence to infinity of E_K is from the uncertainty relation $E_K\geqslant\frac{\hbar^2}{2m}(9/4)N/\langle r^2\rangle$ [see [25,28] as well as Eq. (8)] for $\langle r^2\rangle\to 0$. Generally, $\langle r^2\rangle$ may not vanish at collapse (e.g., if there are nonzero values of $|\Psi|$ away from the collapse center) but E_K diverges to infinity at the collapse time because of $\max|\Psi|\to\infty$. We use in the following divergence to infinity of E_K as a necessary and sufficient condition of collapse formation while vanishing of $\langle r^2\rangle$ is only a sufficient condition for collapse.

The NGPE is not applicable near a singularity and other physical mechanisms, such as inelastic two- and three-body collisions, are important and these can cause a loss of atoms from the condensate [1]. In addition, the multipole expansion used for derivation of the dipole-dipole–type potential is not applicable for very short distances (about a few Bohr radii). However, as previously explained, the NGPE with the potential (1) is a good approximation for a wide range of typical interatomic distances ranging from a fraction of nanometer to $\sim$10 μm.

III. SUFFICIENT COLLAPSE CONDITIONS FOR $2\leqslant b\leqslant 3$

Consider the case $2\leqslant b\leqslant 3$. Then one immediately obtains from Eq. (5) for $g\leqslant 0$ that $\partial_t^2\langle r^2\rangle\leqslant\frac{2bE}{mN}$. Integrating that differential inequality over time we get that $\langle r^2\rangle\leqslant\frac{bE}{mN}t^2+\partial_t\langle r^2\rangle|_{t=0}t+\langle r^2\rangle|_{t=0}$. If $E<0$ we conclude that $\langle r^2\rangle\to 0$ for large enough t, which provides a sufficient criterion of collapse of the BEC. The condition $E<0$ is sufficient but not necessary for collapse. We now use generalized uncertainty relations between E_K, N, $\langle r^2\rangle$, and $\partial_t\langle r^2\rangle$ [25,28] to obtain a much stricter condition of collapse. For the reader's convenience we repeat the derivation of Refs. [8,25] to show that these uncertainty relations result from the Cauchy-Schwarz inequality and Eq. (5) with use of integration by parts ($\Psi\equiv Re^{i\phi}$, $R=|\Psi|$) as follows:

$$E_K=\frac{\hbar^2}{2m}\int[(\nabla R)^2+(\nabla\phi)^2R^2]d^3\mathbf{r},$$
$$\frac{2mN}{\hbar}|\partial_t\langle r^2\rangle|=4\Big|\int x_j\partial_{x_j}\phi R^2d^3\mathbf{r}\Big|\leqslant 4\Big(N\langle r^2\rangle\int(\nabla\phi)^2R^2d^3\mathbf{r}\Big)^{1/2},\quad(8)$$
$$N=-\frac{2}{3}\int x_jR\partial_{x_j}Rd^3\mathbf{r}\leqslant\frac{2}{3}\Big(N\langle r^2\rangle\int(\nabla R)^2d^3\mathbf{r}\Big)^{1/2}.$$

Using Eqs. (7) and (8) one can obtain a basic differential inequality:

$$\partial_t^2\langle r^2\rangle \leqslant \frac{1}{2mN}\left[4bE-(b-2)\frac{\hbar^2}{2m}\left(\frac{9N}{\langle r^2\rangle}+\frac{m^2N(\partial_t\langle r^2\rangle)^2}{\hbar^2\langle r^2\rangle}\right)-(4+2b)m\omega_0^2NF(\gamma)\langle r^2\rangle\right], \tag{9}$$

where $F(\gamma)\equiv\min(\gamma_1^2,\gamma_2^2,\gamma_3^2)$ results from the estimate of the upper bound of the term proportional to E_P in Eq. (5). The change of variable $\langle r^2\rangle = B^{4/(b+2)}/N$ gives the following differential inequality:

$$\partial_t^2 B \leqslant \frac{b+2}{2m}\left[bEB^{\frac{b-2}{b+2}}-(b-2)\frac{\hbar^2}{8m}\frac{9N^2}{B^{\frac{6-b}{b+2}}}-\frac{b+2}{2}m\omega_0^2F(\gamma)B\right], \tag{10}$$

which can be rewritten as

$$B_{tt} = -\frac{\partial U(B)}{\partial B} - q^2(t), \tag{11}$$

where

$$U(B) = -\frac{(b+2)^2}{4m}EB^{\frac{2b}{b+2}} + \frac{\hbar^2 9(b+2)^2N^2}{32m^2}B^{\frac{2b-4}{b+2}} + \frac{(b+2)^2}{8}\omega_0^2F(\gamma)B^2, \tag{12}$$

and $q^2(t)$ is some unknown nonnegative function of time. Equation (11) has a simple mechanical analogy [25] with the motion of a "particle" with coordinate B under the influence of the potential force $-\frac{\partial U(B)}{\partial B}$ in addition to the force $-q^2(t)$. Due to the influence of the nonpotential force $-q^2(t)$ the total energy $\mathcal{E}$ of the "particle" is time dependent: $\mathcal{E}(t) = \frac{B_t^2}{2} + U(B)$. Collapse certainly occurs if the "particle" reaches the origin $B = 0$. It is clear that if the particle were to reach the origin without the influence of the force $-q^2(t)$ then it would reach the origin even faster under the additional influence of this nonpositive force. Therefore one can consider in the following the particle dynamics without the influence of the nonconservative force $-q^2(t)$ to prove sufficient collapse conditions.

It follows from Eq. (12) that the potential $U(B)$ is a monotonic function for $E \leqslant 3\hbar\omega_0N[(b^2-4)F(\gamma)]^{1/2}/(2b) \equiv E_{\text{critical}}$ (see curve 1 in Fig. 1) while for $E > E_{\text{critical}}$ the potential $U(B)$ has a barrier at $B_m^{4/(b+2)} = b[E-(E^2-E_{\text{critical}}^2)^{1/2}]/[(b+2)m\omega_0^2F(\gamma)]$ with particle energy $\mathcal{E}_m = U(B_m)$ at the top (see curve 2 in Fig. 1). One can separate the sufficient collapse condition into three different cases:

(a) for $E \leqslant E_{\text{critical}}$, the particle reaches the origin in a finite time irrespective of the initial value of $B|_{t=0}$;

(b) for $E > E_{\text{critical}}$ and $\mathcal{E}(0) > \mathcal{E}_m$, the particle is able to overcome the barrier and thus it always falls to the origin in a finite time irrespective of the initial value of $B|_{t=0}$;

(c) for $E > E_{\text{critical}}$ and $\mathcal{E}(0) < \mathcal{E}_m$, the particle is not able to overcome the barrier and thus it falls to the origin in a finite time only if $B|_{t=0} < B_m$.

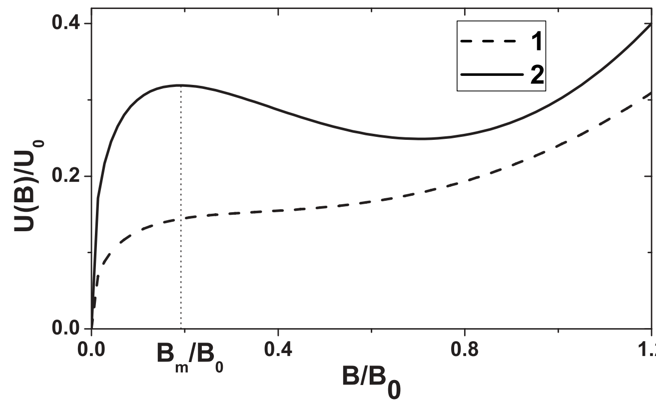

FIG. 1. Typical behavior of the potential $U(B)$ from Eq. (12) for $E \leqslant E_{\text{critical}}$ (curve 1) and $E > E_{\text{critical}}$ (curve 2). $U_0 = (N\hbar/m)^{\frac{b+2}{2}}\omega_0^{\frac{2-b}{2}}$ and $B_0 = (N\hbar/m\omega_0)^{\frac{b+2}{4}}$.

It is important to stress that we have proven here analytically only sufficient collapse conditions. Generally, even if none of these three conditions are satisfied one cannot exclude collapse formation for some particular values of the initial conditions of Eq. (2). Generally, it is determined by the nonpotential force $-q^2(t)$. The inequality (9) reduces to equality for $\gamma_1^2 = \gamma_2^2 = \gamma_3^2$, $g = 0$ and a Gaussian initial condition with $\psi|_{t=0} = \frac{N^{1/2}}{\pi^{3/4}r_0^{3/2}}e^{-r^2/(2r_0^2)}$. For that initial condition

$$E = \frac{3\hbar^2}{4m}\frac{N}{r_0^2} + \pi^{-1/2}fN^2r_0^{-b}\Gamma(3/2-b/2) + \frac{3}{4}m\omega_0^2r_0^2N \tag{13}$$

provided $f(\mathbf{n}) = \text{constant} \equiv f$.

Assume that the trap contribution to E is negligible (i.e., we set $\omega_0 \to 0$) then $E < 0$ in (13) either if the constant r_0 is small and $b > 2$ or if $b = 2$ and $N > N_c^{(\text{var})}$, where

$$N_c^{(\text{var})} = -3\hbar^2/(4mf). \tag{14}$$

This means that for $2 \leqslant b \leqslant 3$ we can easily have the simplest sufficient collapse condition $E < 0$ satisfied for the long-distance potential alone (for $g = 0$); that is, the potential alone can result in collapse of the BEC. $N_c^{(\text{var})}$ in (14) is the variational estimate for the critical number of particles, N_c, for $b = 2$. If $N < N_c$ then collapse is impossible for any initial conditions (and for any trap) as shown in the following. N_c is independent of the trap and it is an analog of the critical particle number for the standard two-dimensional (2D) GPE with contact interactions only.

For $2 < b < 3$ we can also introduce another critical value of particles, $N_{\text{c,trap}}$, from the condition that for $N > N_{\text{c,trap}}$ the energy E does not have minimum as a function of system parameters for fixed N. We find from Eq. (13) that E does not have a minimum for any r_0 if $N > N_{\text{c,trap}}^{(\text{var})}$, where

$$N_{\text{c,trap}}^{(\text{var})} = -\frac{\hbar^{5/2}}{m^{3/2}\omega_0^{1/2}f}\frac{6(b-2)^{\frac{b-2}{4}}\sqrt{\pi}}{b(b+2)^{\frac{b+2}{4}}\Gamma(3/2-b/2)} \tag{15}$$

is the variational estimate for $N_{\text{c,trap}}$. This means that any soliton-type solution is impossible for $N > N_{\text{c,trap}}$ and collapse inevitably occurs. The critical number of particles, $N_{\text{c,trap}}$, is defined for $2 < b < 3$ and is determined by the trap (since without the trap particles could spread unboundedly, preventing collapse for the wide class of initial conditions, while the trap blocks that scenario and eventually results in collapse for $N > N_{c,\mathrm{trap}}$). $N_{c,\mathrm{trap}}$ is the analog of the critical number of particles, $N_{\mathrm{cr,GPE}} \equiv \kappa \frac{a_{\mathrm{ho}}}{|a|}$, $\kappa \simeq 0.5$, $a_{\mathrm{ho}} = (\hbar/m\omega_0)^{1/2}$ in the standard three-dimensional (3D) GPE [29,30]. Both $N_{c,\mathrm{trap}}^{(\mathrm{var})}$ and $N_{\mathrm{cr,GPE}}$ are undefined without the trap because formally both the NGPE for $2 < b < 3$ and the standard 3D GPE can have collapse for an arbitrarily small number of particles for appropriately chosen initial conditions (and of course these equations are based on the mean-field approximation so for small $N \sim 1$ these equations will not be applicable).

For $2 < b < 3$ and $N < N_c^{(\mathrm{var})}$ the energy E (13) has a local minimum for a finite value of r_0 while $E \to -\infty$ for $r_0 \to 0$. This means that depending on initial conditions (i.e. the initial value of r_0) the BEC either collapses in a finite time or stabilizes on a soliton solution. That soliton solution is however metastable because of the finite probability of tunneling of the condensate to small values of r_0.

The $b = 3$ case is special because convergence of the integral E_R at small distances requires that angular integration (integration over a sphere of radius one) of $f(\mathbf{n})$ gives zero: $\int f(\mathbf{n})d\mathbf{n} = 0$. A particular example for $b = 3$ was considered in Ref. [8] for the case of the dipole-dipole interaction potential with all dipoles oriented in one fixed direction. In that case indeed $\int f(\mathbf{n})d\mathbf{n} = 0$. Also in that case E_{NL} vanishes from (7), which allows collapse even for $g > 0$.

If either $b = 3$ and $\int f(\mathbf{n})d\mathbf{n} \neq 0$ or $b > 3$ then it is necessary to introduce a cutoff at a small distance r_c (typically at few Bohr radii) and the potential would lose a general form (1). Also for $b > 3$ the integral $\int_{|\mathbf{r}|>r_c} V(\mathbf{r})d^3\mathbf{r}$ is finite so generally we have a very similar situation to a standard δ-correlated potential [1]. Thus $b = 3$ is a border between short-range potentials (for $b > 3$) and long-range potentials (for $b \leqslant 3$) in three dimensions.

IV. NONEXISTENCE OF COLLAPSE FOR $b < 2$

Now we prove that for $b < 2$ collapse is impossible for $g = 0$ because the singularity of (1) is not strong enough. We use the inequality $\int \frac{|\Psi(\mathbf{r})|^2}{|\mathbf{r}-\mathbf{r}'|^2} d^3\mathbf{r} \leqslant 4 \int |\nabla\Psi(\mathbf{r})|^2 d^3\mathbf{r}$ [31], which holds for any $\mathbf{r}'$. We generalize that inequality using Hölder's inequality (assuming $b < 2$) as follows:

$$\begin{aligned}\int \frac{|\Psi(\mathbf{r})|^2}{|\mathbf{r}-\mathbf{r}'|^b} d^3\mathbf{r} &= \int |\Psi(\mathbf{r})|^{2-b} \frac{|\Psi(\mathbf{r})|^b}{|\mathbf{r}-\mathbf{r}'|^b} d^3\mathbf{r} \\ &\leqslant \left[\int (|\Psi(\mathbf{r})|^{2-b})^{\frac{1}{1-b/2}} d^3\mathbf{r}\right]^{1-\frac{b}{2}} \\ &\quad \times \left[\int \left(\frac{|\Psi(\mathbf{r})|^b}{|\mathbf{r}-\mathbf{r}'|^b}\right)^{\frac{2}{b}}\right]^{\frac{b}{2}} d^3\mathbf{r} \\ &\leqslant 2^b N^{1-\frac{b}{2}} \left(\frac{2m}{\hbar^2} E_K\right)^{\frac{b}{2}}. \end{aligned} \tag{16}$$

Using now boundness of $f: f(\mathbf{n}) \leqslant f_m \equiv \max_{\mathbf{n}} |f(\mathbf{n})|$ in (1) and inequality (16) we obtain a bound for E_R in (4),

$$E_R \geqslant -f_m 2^{b-1} N^{2-\frac{b}{2}} \left(\frac{2m}{\hbar^2} E_K\right)^{\frac{b}{2}}, \tag{17}$$

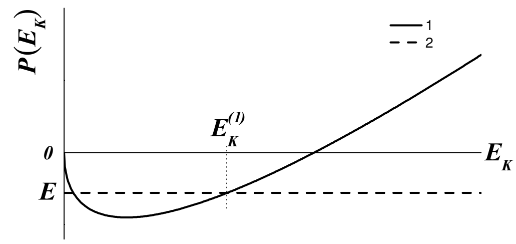

FIG. 2. Schematic of the function $P(E_K)$ defined in (18) (solid curve). It is seen that the equation $P(E_K) = E$ (with E shown by the dashed line) has either one of two roots for $E_K > 0$ depending on sign of E. $E_K^{(1)}$ designates the largest of these roots.

which gives a respective bound of E in (4) (where recall that we assume $g = 0$):

$$E \geqslant E_K - f_m 2^{b-1} N^{2-\frac{b}{2}} \left(\frac{2m}{\hbar^2} E_K\right)^{\frac{b}{2}} \equiv P(E_K). \tag{18}$$

A function $P(E_K)$ in (18) has a minimum for $E_K = E_K^{(0)} \equiv 2^{-2}[f_m b]^{2/(2-b)} N^{\frac{4-b}{2-b}} (2m/\hbar^2)^{b/(2-b)}$, resulting in a lower bound

$$E \geqslant -\frac{2-b}{b} 2^{-2} [f_m b]^{\frac{2}{2-b}} N^{\frac{4-b}{2-b}} \left(\frac{2m}{\hbar^2}\right)^{\frac{b}{2-b}}. \tag{19}$$

Boundness of the energy functional E from below ensures that collapse is impossible for $b < 2$. To prove that, we show boundness of E_K while collapse requires $E_K \to \infty$. We choose any value of E which satisfies (19). Figure 2 shows schematically the function $P(E_K)$ from (18).

Inequality (18) requires that $E_K \leqslant E_K^{(1)}(E)$, where $E_K^{(1)}(E)$ is the largest root of the equation $P(E_K) = E$. This proves that E_K is bounded for fixed N, which completes the proof of the absence of collapse for $b < 2$. A particular version of that result for $b = 1$ and $f(\mathbf{n}) =$ constant was first obtained in Ref. [32]. The nonexistence of collapse for the nonsingular potential $V(\mathbf{r})$ was shown previously based on the approximate analysis in Ref. [33]. The proof of the nonexistence of collapse for the particular example of nonsingular potentials with a positive-definite bounded Fourier transform was given in Ref. [34]. These results can be easily generalized for any bounded potential similar to the analysis here. Thus collapse can occur for a singular potential only and the singularity should be strong enough (i.e., $b \geqslant 2$).

V. NONLINEAR STABILITY OF THE GROUND-STATE SOLITON FOR $b < 2$

We now look for a soliton solution of the NGPE (2) as $\Psi(\mathbf{r},t) = A(\mathbf{r})e^{-i\mu t/\hbar}$, where μ is the chemical potential. In that case the NGPE (2) reduces to a time-independent equation

$$\left[-\mu - \frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_0^2(\gamma_1^2 x_1^2 + \gamma_2^2 x_2^2 + \gamma_3^2 x_3^2) + \int d^3\mathbf{r}' V(\mathbf{r}-\mathbf{r}')A(\mathbf{r}')^2\right]A(\mathbf{r}) = 0, \tag{20}$$

where we again assume $g = 0$ although generalization to the $g \neq 0$ case is straightforward. Equation (20) is the stationary point of the energy functional E for a fixed number of particles:

$\delta(E-\mu N)=0$. Multiplying Eq. (20) by A and $x_j\partial_{x_j}A$ and integrating by parts one obtains using (1) and (3) that

$$E_{K,s}=-\mu N_s\frac{b}{4-b}+E_{P,s},\quad E_{R,s}=\mu N_s\frac{2}{4-b},$$
$$E_s=-\mu N_s\frac{b-2}{4-b}+2E_{P,s}, \tag{21}$$

where subscript "s" means values of all integrals are taken on the soliton solution. Especially simple and interesting is the case of self-trapping ($\omega_0=0$) when the condensate is in steady state without any external trap. All integrals in that case depend on the number of particles, N_s, only.

Assume radial symmetry $f(\mathbf{n})=\text{constant}<0$ in (1). A ground-state soliton is determined from a condition that $A(\mathbf{r})$ never crosses zero [35,36]. To prove the ground-state soliton stability we show that it realizes a minimum of the Hamiltonian for a fixed N_s. One can make inequality (16) sharper by minimizing a functional $\mathcal{F}(\Psi)\equiv N^{1-\frac{b}{2}}(\frac{2m}{\hbar^2}E_K)^{\frac{b}{2}}/\int\frac{|\Psi(\mathbf{r})|^2}{|\mathbf{r}-\mathbf{r}'|^b}d^3\mathbf{r}$. That minimum is achieved at one of the stationary points $\frac{\delta\mathcal{F}}{\delta\Psi^*}=0$ and after simple rescaling one can see that these points correspond to soliton solutions of the time-independent NGPE (20). Among these stationary points the minimum is achieved at the ground-state soliton $\Psi_{\text{s,ground}}$. It gives a bound $\mathcal{F}(\Psi)\geqslant\min\mathcal{F}(\Psi)=\mathcal{F}(\Psi_{\text{s,ground}})$ which is sharper than the inequality (16). Following an analysis similar to Eqs. (17)–(19) we obtain that for any Ψ

$$E\geqslant\min E=E_{\text{s,ground}}, \tag{22}$$

that is, the ground-state soliton solution attains the minimum of E for fixed N. This proves exactly the stability of the soliton for $f(\mathbf{n})=\text{constant}$. Similar ideas were used in a nonlinear Schrödinger equation (NLS) which is a GPE with $\omega_0=0$ [35,36]. The ground-state soliton was also found numerically for $b=1$ [37].

For more general $f(\mathbf{n})\neq\text{constant}$, the minimum of E is still negative if $f(\mathbf{n})$ is negative for a nonzero range of values of $\mathbf{n}$. So in that case we expect that the ground-state soliton solution attains that minimum and, respectively, is stable. If $f(\mathbf{n})>0$ for any $\mathbf{n}$ then $\min E=0$. This corresponds to the unbounded spatial spreading of the NGPE solution for any initial conditions. Self-trapping is impossible in that case and solitons are possible for $\omega_0\neq 0$ only.

The case $b=2$ is on the boundary between the bounded and unbounded energy functionals, as can be seen from inequalities (18) and (22). If $N>N_{\text{s,ground}}$ then E is unbounded. If $N<N_{\text{s,ground}}$ then $E\geqslant E_{\text{s,ground}}=0$ as follows from (21) for $b=2$. Thus $N_{\text{s,ground}}$ is the critical number of particles for collapse: $N_c=N_{\text{s,ground}}$. This is the exact result and compares with the variational estimate (14). That critical particle number N_c is similar to to the critical particle number for the collapse of the standard 2D GPE (as well as similar to the critical power in nonlinear optics) [23]. As we have discussed here, it is important to distinguish N_c from the critical number of particles of the 3D GPE with $\omega_0\neq 0$ [1,2].

VI. WEAK AND STRONG REGIMES OF COLLAPSE

To qualitatively distinguish among regimes of collapse and solitons one can consider, in addition to the exact analysis just given, a scaling transformations $\Psi(\mathbf{r})\to L^{-3/2}\Psi(\mathbf{r}/L)$ [38] which conserves the number of particles. Under this transformation the energy functional E (for $\omega_0=0$) depends on the parameter L as follows:

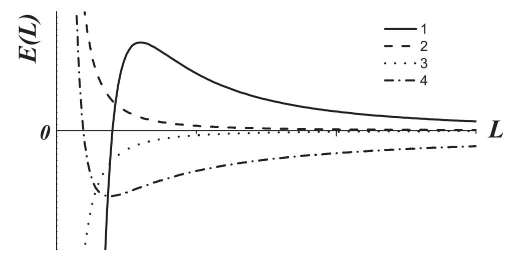

FIG. 3. Schematic of $E(L)$ from (23) for $b>2$ (curve 1), $b=2$ and $N<N_{\text{s,ground}}$ (curve 2), $b=2$ and $N>N_{\text{s,ground}}$ (curve 3), and $b<2$ (curve 4).

$$E(L)=L^{-2}E_K+L^{-b}E_R. \tag{23}$$

The virial theorem (7) and the relations (21) and (23) have striking similarities with the GPE if we replace b by the spatial dimension D in the GPE. That analogy suggests we refer to the case $b=2$ as the critical NGPE and to the case $b>2$ as the supercritical NGPE because cases $D=2$ and $D>2$ are called by critical and supercritical ones, respectively, for the standard GPE (NLS) [38]. Similarly, we refer to collapse for $b=2$ as a critical collapse and for $b>2$ as a supercritical collapse. Figure 3 shows a typical dependence of (23) on L for $b>2$, $b=2$, and $b<2$ with $E_R<0$ assumed. For $b>2$ there is a maximum of E (curve 1 in Fig. 3) corresponding to an unstable soliton. The solution of the NGPE either collapses or expands. For $b=2$ there is no extremum and collapse is impossible for $N<N_{\text{s,ground}}$ (curve 2 in Fig. 3) while the condensate can collapse for $N>N_{\text{s,ground}}$ (curve 3 in Fig. 3). The ground-state soliton corresponds to $N=N_{\text{s,ground}}$ and $E=0$, exactly located at the boundary between collapsing and noncollapsing regimes. For $b<2$ there is a minimum which corresponds to the stable ground-state soliton (curve 4 in Fig. 3).

Solutions of both the GPE and the NGPE with $\omega_0=0$ near collapse typically consist of a background of nearly linear waves and a central collapsing self-similar nonlinear core. The scaling (23) describes the dynamics of the core with time-dependent $L(t)$ such that $L(t)\to 0$ near collapse. Waves have negligible potential energy but carry a positive kinetic energy $E_{\text{waves}}\simeq E_{K,\text{waves}}$. The total energy $E=E_{\text{collapse}}+E_{\text{waves}}$ is constant, where E_{collapse} is the core energy.

It follows from (23) that for $b=2$ one can simultaneously allow conservation of N and E_{collapse} so that a negligible number of particles are emitted from the core. This scenario is called a strong collapse with the self-similar collapsing core centered at $\mathbf{r}=\mathbf{0}$ and approximated as

$$|\psi_{\text{c,strong}}(\mathbf{r},t)|\simeq\frac{1}{L(t)^{3/2}}\chi\left(\frac{\mathbf{r}}{L(t)}\right),\quad L(t)\to 0\quad\text{for}\quad t\to t_0, \tag{24}$$

where the function $\chi(\xi)$ with $\xi\equiv\mathbf{r}/L(t)$ describes the spatial structure of the collapsing solution and t_0 is the collapse time.

Equation (24) is applicable for $|\xi| < \xi_c$, where $\xi_c \gtrsim 1$. The number of particles, $N_{\text{collapse,strong}}$, in the collapsing solution is nearly constant provided ξ_c is nearly constant: $N_{\text{collapse,strong}} \simeq \int_{|\mathbf{r}|<\xi_c L(t)} |\psi_{\text{c,strong}}(\mathbf{r},t)|^2 d^3\mathbf{r} = \int_{|\xi|<\xi_c} \chi^2(\xi) d^3\xi \sim 1$. Thus the critical collapse is always a strong one with $N_{\text{collapse,strong}} \simeq N_c$. Substitution of (24) into the NGPE with $g = 0$ allows us to conclude that all terms are of the same order in powers of L (except for the trapping potential E_P, which is not important near collapse) if $L(t) \propto (t_0 - t)^{1/2}$. By analogy with the 2D GPE, which has a critical collapse [39–41], we also expect to observe logarithmic corrections to $(t_0 - t)^{1/2}$, which is a typical feature of critical collapses in many systems (see, e.g., [42]). If $N \gg N_c$ then multiple collapses will occur, each capturing about N_c particles, which is the analog of multiple filamentation turbulence and beam spray in nonlinear optics and laser-plasma ineractions [43,44]. The universality in the number of particles captured in each collapse, $N_{\text{collapse,strong}} \simeq N_c$, holds for the critical collapse only and does not hold for the supercritical case.

In the supercritical case $2 < b \leqslant 3$, the term $\propto L^{-b}$ in (23) dominates with $E_{\text{collapse}} \to -\infty$ as $L(t) \to 0$. Then the only way to ensure conservation of E is to assume a strong emission of linear waves (particles) from the collapsing core. Near the collapse time t_0 only a vanishing number of particles remains in the core (and of course all that is true until the NGPE losses its applicability), which is called a weak collapse [38]. Instead of the self-similar solution (24), the weak collapse is described by another type of self-similar solution:

$$|\psi_{\text{c,weak}}(\mathbf{r},t)| \simeq \frac{1}{L(t)^\alpha}\eta\left(\frac{\mathbf{r}}{L(t)}\right), \quad L(t) \to 0 \quad \text{for} \quad t \to t_0, \qquad \alpha = \frac{5-b}{2}, \tag{25}$$

where the function $\eta(\xi)$ with $\xi \equiv \mathbf{r}/L(t)$ describes the spatial structure of the collapsing solution and $\alpha = \frac{5-b}{2}$ is chosen from the condition that a substitution of (25) into the NGPE allows the same leading order in powers of L for linear (the kinetic energy) and nonlinear [the potential energy from $V(\mathbf{r})$] terms. Here we again neglect the trapping potential and assume $g = 0$. Assuming now that the left-hand-side of the NGPE is of the same order in power of $L(t)$ we obtain that $L(t) \propto (t_0 - t)^{1/2}$. Similar to (24), we assume that Eq. (25) is applicable for $|\xi| < \xi_c$, where $\xi_c \gtrsim 1$. The number of particles, $N_{\text{collapse,weak}}$, in the collapsing solution approaches zero near collapse: $N_{\text{collapse,weak}} \simeq \int_{|\mathbf{r}|<\xi_c L(t)} |\psi_{\text{c,weak}}(\mathbf{r},t)|^2 d^3\mathbf{r} = L(t)^{b-2} \int_{|\xi|<\xi_c} \eta^2(\xi) d^3\xi \sim L(t)^{b-2} \to 0$ for $t \to t_0$.

The solution of the NGPE in the form of the strong collapse (24) can also be considered for $2 < b \leqslant 3$, which results in the dominance of the nonlinear interaction and time-dependent terms in the NGPE over the kinetic energy term. It was shown that a similar solution for the supercritical GPE [38] is unstable and we expect that it might also be unstable for the NGPE. Thus supercritical collapse can be either weak or strong but a weak one appears to be more probable.

VII. CONCLUSION

We studied the collapse versus the long-time existence of Bose-Einstein condensates with a long-range attractive interaction proportional to $1/r^b$ and arbitrary angular dependence. We proved that collapse is impossible for $b < 2$. Instead, the attractive $1/r^b$ interaction allows stable self-trapping for $b < 2$ even in the absence of an external trapping potential (i.e., 3D stable solitons are possible). We showed that they are nonlinearly stable for the radially symmetric attracting interaction potential. For $b = 2$, collapse is strong with the critical number of particles, N_c, independent of the trapping potential, so we also refer to that collapse as a critical collapse. We showed that N_c is determined by the ground-state soliton solution. For $2 < b \leqslant 3$, we expect a weak collapsing solution but a strong collapse is also possible. In that case we define another critical number of particles, $N_{\text{c,trap}}$, which depends on the form of the trapping potential. For $N > N_{\text{c,trap}}$, collapse always occurs, while for $N < N_{\text{c,trap}}$, both collapsing and noncollapsing solutions are possible depending on the initial conditions. These results can be extended to the case of a general anharmonic trap, in which only conditions for collapse would be modified but no qualitative change of collapsing versus noncollapsing BEC dynamics would result.

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