



## Exactly integrable dynamics of interface between ideal fluid and light viscous fluid

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### Abstract

It is shown that dynamics of the interface between ideal fluid and light viscous fluid is exactly integrable in the approximation of small surface slopes for two-dimensional flow. Stokes flow of viscous fluid provides a relation between normal velocity and pressure at interface. Surface elevation and velocity potential of ideal fluid are determined from two complex Burgers equations corresponding to analytical continuation of velocity potential at the interface into upper and lower complex half planes, respectively. The interface loses its smoothness if complex singularities (poles) reach the interface.

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Dynamics of an interface between two incompressible fluids is an important fundamental problem which has numerous applications ranging from interaction between sea and atmosphere to flow through porous media and superfluids. If one neglects gravity and surface tension, that problem can be effectively solved in some particular cases in two dimensions with the use of complex variables. Integrable cases include Stokes

flow of viscous fluid with free surface [1], dynamics of free surface of ideal fluid with infinite depth [2] and finite depth [3], dynamics of an interface between two ideal fluids [4], ideal fluid pushed through viscous fluid in a narrow gap between two parallel plates (Hele–Shaw flow) [5–7].

Here a new integrable case is found which corresponds to two-dimensional motion of the interface between heavy ideal fluid and light viscous fluid in absence of gravity and capillary forces. The interface position is given by  $z = \eta(x, t)$ , where the first, heavier fluid (indicated by index 1) with the density  $\rho_1$

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occupies the region  $-\infty < z < \eta(x, t)$  and the second, lighter fluid (index 2) with the density  $\rho_2$  occupies the region  $\eta(x, t) < z < \infty$ .

Suppose that the kinematic viscosity of the fluid 2,  $\nu_2$ , is very large so that fluid's 2 flow has small Reynolds numbers and, neglecting inertial effect in the Navier–Stokes equation, one arrives to the Stokes flow equation [8]:

$$\nu_2 \nabla^2 \mathbf{v}_2 - \frac{1}{\rho_2} \nabla p_2 = 0, \quad \nabla \cdot \mathbf{v}_2 = 0, \quad (1)$$

where  $\mathbf{v}_2 \equiv (v_{2,x}, v_{2,z})$  is the velocity of the fluid 2,  $\nabla = (\partial_x, \partial_z)$ , and  $p_2$  is the fluid's 2 pressure (similar physical quantities for the fluid 1 have index 1 below). Additional assumption necessary for applicability of Eq. (1) is a small density ratio,

$$\rho_2/\rho_1 \ll 1, \quad \rho_1 \equiv 1, \quad (2)$$

which ensure that the fluid 2 responds very fast to perturbations of the interface as inertia of the fluid 2 is very small compare with fluid's 1 inertia while time dependent perturbations of the fluid 2 decay very fast due to large viscosity  $\nu_2$ . According to Eq. (1), the response of the fluid 2 to motion of the interface is static. For any given normal velocity of the interface,  $v_n$ , Eq. (1) allows to determine the pressure  $p_2|_{z=\eta}$  at the interface. In other words, the fluid 2 adiabatically follows the slow motion of the heavy fluid 1 and Reynolds number of the fluid 2 remains small at all time.

The velocity of the potential motion of the ideal fluid 1,  $\mathbf{v}_1 = \nabla\phi$ , can be found from solution of the Laplace equation,  $\nabla^2\phi = 0$ , which is a consequence of the incompressibility condition,  $\nabla \cdot \mathbf{v}_1 = 0$ , for potential flow. Boundary conditions at infinity are decaying,  $|\mathbf{v}_1|, p_1 \rightarrow 0$  for  $z \rightarrow -\infty$ ,  $|\mathbf{v}_2|, p_2 \rightarrow 0$  for  $z \rightarrow +\infty$ .

Motion of the interface is determined from the kinematic boundary condition of continuity of normal component of fluid velocity across the interface:

$$v_n \equiv v_{1n}|_{z=\eta} = v_{2n}|_{z=\eta} = \partial_t \eta [1 + (\partial_x \eta)^2]^{-1/2}, \quad (3)$$

where  $v_{1(2)n} = \mathbf{n} \cdot \mathbf{v}_{1(2)}$  and  $\mathbf{n} = (-\partial_x \eta, 1)[1 + (\partial_x \eta)^2]^{-1/2}$  is the interface normal vector.

A dynamic boundary condition is a continuity of stress tensor,  $\sigma_{1(2),jm} = -p_{1(2)}\delta_{jm} + \sigma'_{1(2),jm}$ ,  $\sigma'_{1(2),jm} \equiv \rho_{1(2)}\nu_{1(2)}(\frac{\partial v_{1(2),m}}{\partial x_j} + \frac{\partial v_{1(2),j}}{\partial x_m})$ ,  $x_1 \equiv x$ ,  $x_2 \equiv z$ , across the interface:  $n_j \sigma_{1,jm}|_{z=\eta} = n_j \sigma_{2,jm}|_{z=\eta}$

(repetition of indexes  $j, m$  means summation from 1 to 2), which gives two scalar dynamic boundary conditions:

$$p_1|_{z=\eta} = p_2|_{z=\eta} + n_m n_j \sigma'_{2,jm}|_{z=\eta},$$

$$l_m n_j \sigma'_{2,jm}|_{z=\eta} = 0, \quad (4)$$

where the absence of viscous stress in the ideal fluid 1,  $\nu_1 = 0$ , is used,  $n_m, l_m$  are components of the interface normal vector,  $\mathbf{n}$ , and the interface tangential vector,  $\mathbf{l} = (1, \partial_x \eta)[1 + (\partial_x \eta)^2]^{-1/2}$ . The pressure  $p_1$  of the fluid 1 at the interface can be determined from a nonstationary Bernoulli equation,  $[\partial_t \phi + \frac{1}{2}(\nabla\phi)^2 + \frac{p_1}{\rho_1}]|_{z=\eta} = 0$ .

To obtain a closed expression for interface dynamics in terms of fluid's 1 variables only, one can first find an expression for the pressure at the interface through the normal velocity  $v_n$ .

It follows from Eq. (1) that  $\nabla^2 p_2 = 0$  and the Fourier transform over  $x$  allows to write the solution of the Laplace equation with the decaying boundary condition at  $x \rightarrow \infty$  as  $p_{2k}(z) = p_{2k}(0) \exp(-|k|z) \equiv \int dx p_2(x, z) \exp(-ikx)$ .

To determine  $v_2|_{z=\eta}$  one can introduce a shift operator,  $\hat{L}_2$ , defined from series expansion:  $\mathbf{v}_2(x, z)|_{z=\eta} \equiv \hat{L}_2 \mathbf{v}_2(x, 0) = [(1 + \eta \partial_z + \frac{1}{2} \eta^2 \partial_z^2 + \dots) \mathbf{v}_2(x, z)]|_{z=0}$  and use Eq. (1) to find  $v_{2,xk}(z) = [c_k - ikz \frac{p_{2k}(0)}{2\rho_2 \nu_2 |k|}] \times \exp(-|k|z)$ ,  $v_{2,zk}(z) = [i \text{sign}(k) c_k + (|k|z + 1) \times \frac{p_{2k}(0)}{2\rho_2 \nu_2 |k|}] \exp(-|k|z)$ , where  $v_{2,xk}(z)$ ,  $v_{2,zk}(z)$  are the Fourier transform over  $x$  of the components of the velocity  $\mathbf{v}_2$  and functions  $c_k$ ,  $p_{2k}(0)$  should be determined from the dynamic boundary conditions (4).

Operator  $\hat{L}_2$  can be expressed, using Eq. (1), in terms of the operator  $\hat{k}$ :  $\hat{L}_2 = 1 - \eta \hat{k} + \frac{1}{2} \eta^2 \hat{k}^2 + \dots$ , where the integral operator  $\hat{k}$  is an inverse Fourier transform of  $|k|$  and is given by

$$\hat{k} = -\frac{\partial}{\partial x} \hat{H}. \quad (5)$$

Here  $\hat{H} f(x) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{+\infty} \frac{f(x')}{x'-x} dx'$  is the Hilbert transform and PV means Cauchy principal value of integral.  $\hat{H}$  can be also interpreted as a Fourier transform of  $i \text{sign}(k)$ .

In a similar way one can show that

$$[\partial_x \mathbf{v}_2(x, z)]|_{z=\eta} = \hat{L}_2 \partial_x \mathbf{v}_2(x, 0),$$

$$\begin{aligned} & [\partial_z v_{2,x}(x, z)]|_{z=\eta} \\ &= -\hat{L}_2 \hat{k} v_{2,x}(x, 0) - \frac{1}{2v_2 \rho_2} \hat{L}_2 \hat{k}^{-1} \partial_x p_2(x, 0), \end{aligned}$$

$$p_2(x, \eta) = \hat{L}_2 p_2(x, 0)$$

and using kinematic (3) and dynamic (4) boundary conditions one can find  $p_1(x, \eta)$  as a linear functional of  $v_n$ . That linear functional can be expressed in a form of powers series with respect to small parameter  $|\partial_x \eta|$ , which has a meaning of typical slope of the interface inclination relative to the interface undisturbed (plane) position.

At leading order approximation over small parameter  $|\partial_x \eta|$  one gets:  $p_1(x, \eta) = p_2(x, 0)$ ,  $v_n = v_{2,z}(x, \eta)$ , and, respectively, response of pressure to normal velocity is given by

$$p_1|_{z=\eta} = 2\rho_2 v_2 \hat{k} v_n. \tag{6}$$

In other words, Eq. (6) determines a static response of the fluid 2 to the motion of the interface.

Eq. (6) together with the kinematic boundary condition (3) and the Laplace equation for the velocity potential  $\phi$  completely defines the potential motion of the fluid 1.

Following Zakharov [9], one can introduce the surface variable  $\psi(x) \equiv \phi(x, \eta)$ , which is the value of the velocity potential,  $\phi(x, z)$ , at the interface. Kinematic boundary condition (3) can be written at leading order over small parameter  $|\partial_x \eta|$  as

$$\partial_t \eta = -\hat{H} v, \tag{7}$$

where a new function,  $v = \partial_x \psi$ , is introduced which has a meaning of the tangent velocity of the fluid 1 at the interface.

Similar to the shift operator  $\hat{L}_2$ , one can define a shift operator,  $\hat{L}_1 = 1 + \eta \hat{k} + \frac{1}{2} \eta^2 \hat{k}^2 + \dots$ , which corresponds to the harmonic function  $\phi$  with vanishing boundary condition  $\phi \rightarrow 0$  for  $z \rightarrow -\infty$ . A Fourier transform of  $\phi$ ,  $\phi_k(z) = \phi_k(0) \exp(|k|z)$ , allows to find the components of fluid velocity at the interface:  $(\partial_x \phi)|_{z=\eta} = \hat{L}_1 \partial_x \phi(x, 0) = \hat{L}_1 \partial_x \hat{L}_1^{-1} \psi$ ,  $(\partial_z \phi)|_{z=\eta} = \hat{L}_1 \hat{k} \hat{L}_1^{-1} \psi$  through surface variables  $\eta$ ,  $\psi$ . Time derivative  $\phi_t$  in the nonstationary Bernoulli can be found from  $\partial_t \psi = \partial_t \phi|_{z=\eta} + \partial_t \eta \partial_z \phi|_{z=\eta}$  and one gets at leading order approximation over  $|\partial_x \eta|$ :

$$\frac{\partial v}{\partial t} - \frac{1}{2} \partial_x [(\hat{H} v)^2 - v^2] = 2v_2 \rho_2 \partial_x^2 v. \tag{8}$$

Note that Eq. (8) does not include variable  $\eta$  which is a peculiar property of lowest perturbation order over  $|\partial_x \eta|$ .

Because the surface tension and gravity are neglected here, the total energy of two fluid equals to total kinetic energy,  $K$ .  $K$  decays,  $\frac{dK}{dt} \simeq -\frac{v_2 \rho_2}{\pi} \int dk \times |k|^3 |\psi_k|^2 < 0$ , due to dissipation in the fluid 2. If the fluid 2 is absent, which corresponds to  $\rho_2 = 0$ , then  $K$  is conserved,  $\frac{dK}{dt} = 0$ , and the motion of the fluid 1 can be expressed in the standard Hamiltonian form [2,9]:  $\frac{\partial \eta}{\partial t} = \frac{\delta K}{\delta \psi}$ ,  $\frac{\partial \psi}{\partial t} = -\frac{\delta K}{\delta \eta}$ .

Equations, similar to (7), (8) can be derived for three-dimensional motion also with the main difference that the operator  $\hat{k}$  in three dimensions is not given by (5) but determined from the Fourier transform of  $|k|$  over two horizontal coordinates. Subsequent analysis is, however, restricted to two-dimensional fluid motion only.

The real function  $v(x)$  can be uniquely represented as a sum of two complex functions  $v^{(+)}$  and  $v^{(-)}$ ,  $v = [v^{(+)} + v^{(-)}]/2$ , which can be analytically continued from real axis  $x$  into upper and lower complex half-planes, respectively. The Hilbert transform acts on these functions as  $\hat{H} v^{(+)} = i v^{(+)}$ ,  $\hat{H} v^{(-)} = -i v^{(-)}$  and Eq. (8) splits into two decoupled complex Burgers equations for  $v^{(+)}$  and  $v^{(-)}$ :

$$\frac{\partial v^{(\pm)}}{\partial t} + v^{(\pm)} \partial_x v^{(\pm)} = \tilde{\nu} \partial_x^2 v^{(\pm)}, \tag{9}$$

where an effective viscosity,  $\tilde{\nu} = 2v_2 \rho_2$  is introduced to make connection with the standard definition of real Burgers equation [10]. Similar reduction of integro-differential equation (like Eq. (8)) to complex Burgers equation was done in Ref. [11].

If the fluid 2 is absent,  $\tilde{\nu} = 0$ , complex Burgers equations (9) are reduced to inviscid Burgers equations (the Hopf equations) which were derived for ideal fluid with free surface in Ref. [2] (note that definition of  $v^{(\pm)}$  in this Letter differs from similar definition in Ref. [2] by a factor 1/2). While viscosity  $v_2$  is large enough to make sure that Reynolds number in the fluid 2,  $\mathbf{R}_2$ , is small,  $\mathbf{R}_2 \sim v_n / (v_2 k) \ll 1$  ( $k$  is a typical wave vector of surface perturbation) but effective viscosity  $\tilde{\nu}$  can be small provided  $\rho_2 \ll \mathbf{R}_2 \ll 1$  so that Reynolds number,  $\mathbf{R}$ , in complex Burgers equation (9) is large,  $\mathbf{R} \sim \mathbf{R}_2 / \rho_2 \gg 1$ .

Complex Burgers equation is transformed into the complex heat equation  $\frac{\partial u^{(\pm)}}{\partial t} = \tilde{\nu} \partial_x^2 u^{(\pm)}$  via the Cole–

Hopf transform [10]:  $v^{(\pm)} = -2\tilde{\nu} \frac{\partial_x u^{(\pm)}}{u^{(\pm)}}$ . Solution of the heat equation with initial data

$$u^{(\pm)}(x, t)|_{t=0} \equiv u_0^{(\pm)}(x),$$

$$u^{(\pm)}(x, t) = (4\pi\tilde{\nu}t)^{-1/2} \times \int_{-\infty}^{\infty} dx' \exp\left[-\frac{(x-x')^2}{4\tilde{\nu}t}\right] u_0^{(\pm)}(x'),$$

is an analytic function in complex  $x$  plane for any  $t > 0$  because integral of right-hand side (rhs) of this equation over any closed contour in complex  $x$  plane is zero (Morera's theorem). Then, according to the Cole–Hopf transform, solution of the complex Burgers equation can have pole singularities corresponding to zeros of  $u^{(\pm)}(x, t)$ . Number of zeros,  $n(\gamma)$ , of  $u^{(\pm)}(x, t)$  (each zero is calculated according to its order) inside any simple closed contour  $\gamma$  equals to  $\frac{1}{2\pi i} \int_{\gamma} dx \partial_x u^{(\pm)}(x, t)/u^{(\pm)}(x, t)$ . Integration of Eq. (9) over  $\gamma$  allows to conclude that  $n(\gamma)$  is conserved as a function of time provided zeros do not cross  $\gamma$ . Thus number of zeros in entire complex plane can only change in time because zero can be created or annihilated at complex infinity,  $x = \infty$ , provided  $u^{(\pm)}(x, t)$  has an essential singularity at complex infinity.

From physical point of view it is important that zeros of  $u^{(\pm)}(x, t)$  can reach real axis  $x = \text{Re}(x)$  which distinguishes the complex Burgers equation from the real Burgers equation. Solution of the real Burgers equation, which corresponds to Eq. (9) with  $v^{(\pm)}(x, t)|_{t=0} = \text{Re}[v^{(\pm)}(x, t)|_{t=0}]$ , has global existence (remains smooth for any time), while solution of the complex Burgers generally exists until some zero of  $u^{(\pm)}(x, t)$  hits real axis  $x$  for the first time.

To make connection with inviscid case [2] one can look at initial condition for  $v^{(+)}(x, 0)$  with one simple pole in the lower half-plane:

$$v^{(+)}(x, 0) = \frac{2A}{x + ia}, \quad \text{Re}(a) > 0. \quad (10)$$

Solution of the inviscid ( $\tilde{\nu} = 0$ ) Burgers equation with initial condition (10) gives [2]:  $v_{\text{inviscid}}^{(+)}(x, t) = \frac{4A}{x + ia + \sqrt{(x + ia)^2 - 8At}}$ , which has two moving branch points:  $x_{1,2} = -ia \pm 2\sqrt{2At}$ . One of these branch points reaches real axis in a finite time if either  $A < 0$  or  $\text{Re}(A) \neq 0$ . As the branch point touches the real

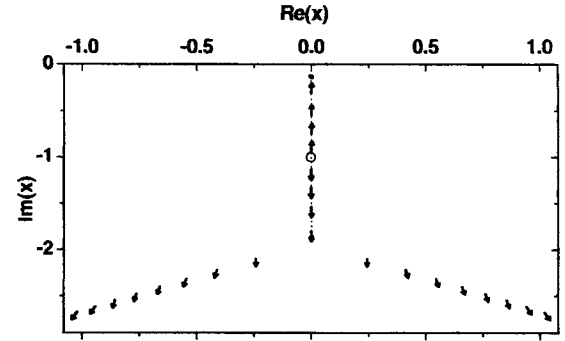


Fig. 1. Motion of poles of the complex velocity,  $v^{(+)}(x, t)$ , in complex  $x$  plane for  $A = -1/8$ ,  $\tilde{\mu} = 7.6$ ,  $\tilde{\nu} \simeq 0.0164$ ,  $a = 1$ ,  $t = 0.8$ . Arrows point out the position, direction and speed of moving poles. Uppermost arrow designates the pole which corresponds to zero of the Hermite function with the largest real part (that pole first reaches real axis for  $t_{\text{viscous}} \simeq 1.91$  producing singularity of the interface surface). Dotted line connects two branch points (filled circles) of inviscid solution. Upper branch point reaches real axis for  $t_{\text{inviscid}} = 1$  which corresponds to singularity in the solution of the inviscid Burgers equation. Empty circle designates the simple pole initial condition (10). Viscous solution becomes singular at later time compare to inviscid solution,  $t_{\text{viscous}} > t_{\text{inviscid}}$ .

axis, the inviscid solution is not unique any more and the interface loses its smoothness [2].

Consider now solution of the viscous Burgers equation (9) with nonzero effective viscosity  $\tilde{\nu}$  and with the simple pole conditions (10). Respectively, initial condition for the heat equation, is given by  $u_0^{(+)} = (x + ia)^{-A/\tilde{\nu}}$  and has branch point at  $x = -ia$ . Solution of the heat equation gives  $u^{(+)}(x, t) = \exp(i\frac{\pi\tilde{\mu}}{2}) H_{\tilde{\mu}}(-\frac{i}{2\sqrt{\tilde{\nu}t}}[x + ia])$ , where  $\tilde{\mu} \equiv -A/\tilde{\nu}$  and  $H_{\mu}(z)$  is the Hermite function defined as  $H_{\mu}(z) = \frac{2^{\mu+1}}{\sqrt{\pi}} e^{z^2} \int_0^{\infty} dy e^{-y^2} y^{\mu} \cos(2zy - \frac{\pi\mu}{2})$ . Zeros of  $u^{(+)}(x, t)$  (and, equivalently, poles of  $v^{(+)}(x, t)$ ) move in complex  $x$  plane with time as (see Fig. 1)

$$x_j(t) = i(2\sqrt{\tilde{\nu}t}z_j - a), \quad (11)$$

where  $z_1, z_2, \dots$ , are complex zeros of the Hermite function.

Consider a particular case,  $\tilde{\mu} = n$ ,  $n$  is a positive integer number. The Hermite function is reduced to the Hermite polynomial  $H_n(z)$  which has  $n$  zeros,  $z_1, z_2, \dots, z_n$  located at real axis  $z = \text{Re}(z)$ ,  $z_n$  corresponds to the largest zero. Location of real zeros of the Hermite function with real  $\tilde{\mu}$  is close to location of zeros of the Hermite polynomial with the closest integer

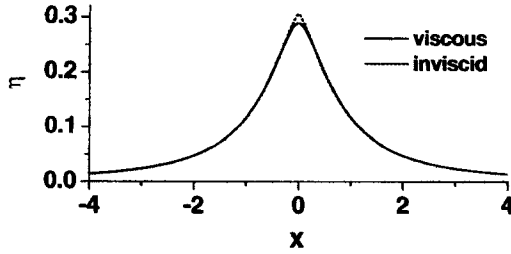


Fig. 2. The interface position,  $\eta(x, t)$ , according to solution of Eqs. (7), (8) with finite viscosity,  $\tilde{\nu} = 1/64$  (solid line) and zero viscosity,  $\tilde{\nu} = 0$  (dotted line) for  $A = -1/8$ ,  $a = 1$ ,  $t = 1$ . Viscous solution has 8 moving poles while inviscid solution is singular at  $x = 0$  ( $\partial_{xx}^2 \eta|_{x=0} \rightarrow -\infty$  as  $t \rightarrow t_{\text{inviscid}} = 1$ ). Both solutions are almost indistinguishable outside a small neighborhood around  $x = 0$ . As  $\tilde{\mu}$  increases, the viscous solution approaches inviscid.

$n$  to the given  $\tilde{\mu}$  while zeros with nonzero imaginary part (which corresponds to tails with nonzero real part in Fig. 1) disappear for  $\tilde{\mu} = n$ . Zeros of the Hermite polynomial are moving with time parallel to imaginary axis  $x = \text{Im}(x)$  in complex  $x$  plane according to (11) and the complex velocity  $v^{(+)}$  is described by set of moving poles:

$$v^{(+)} = -2\tilde{\nu} \sum_{j=1}^n \frac{1}{x - x_j(t)}. \quad (12)$$

$v^{(-)}$  is given by the same expression with  $x_j$  replaced by their conjugated values  $\bar{x}_j$ .

Eqs. (9) have also another wide class of solutions, “pole decomposition”, corresponding to Eq. (12) with  $\frac{dx_j}{dt} = -2\tilde{\nu} \sum_{l=1, l \neq j}^n \frac{1}{x_j - x_l}$ ,  $n$  is arbitrary positive integer [12,13]. Simple pole initial condition (10) with  $\tilde{\mu} = n$  is particular case for which  $x_j|_{t=0} = 0$  for any  $j$ .

As  $v(x, t)$  is known from solution of the heat equation and the Cole–Hopf transform one can find  $\eta(x, t)$  from Eq. (7). Interface dynamics is determined from the most rapid pole of  $v^{(\pm)}$  which first reaches real axis,  $x = \text{Re}(x)$ . E.g., for initial condition (10), the pole singularity of  $v^{(+)}$  first hits real axis,  $x = \text{Re}(x)$ , from below at time  $t_{\text{viscous}} = \frac{\text{Re}(a)^2}{4\tilde{\nu} \text{Re}(z_{\text{max}})^2}$ , where  $z_{\text{max}}$  is a complex zero of the Hermite function with the largest real part for given  $\tilde{\mu}$ . Simultaneously, the pole singularity of  $v^{(-)}$  first hits real axis from above at the same point. Fig. 2 shows  $\eta(x)$  at the time,  $t = t_{\text{inviscid}}$ , when singularity (branch point) of inviscid solution first reaches the interface breaking analyticity of in-

viscid solution. It is seen that viscous solution significantly deviates from inviscid one only in the narrow domain around  $x = 0$ .

Viscous solution remains analytic for  $t > t_{\text{inviscid}}$  until  $t < t_{\text{viscous}}$  ( $t_{\text{viscous}} \simeq 1.91$  for parameters in Fig. 1). However, for  $t \rightarrow t_{\text{viscous}}$ , surface elevation behaves as  $\eta \simeq (-a/2z_{\text{max}}^2) \log[x^2 + (2\sqrt{t\tilde{\nu}}z_{\text{max}} - a)^2]$  near  $x = 0$  (it is set here  $\text{Im}(a) = \text{Im}(A) = 0$ ) meaning that small slope approximation used for derivation of Eqs. (7), (8) is violated for  $t \rightarrow t_{\text{viscous}}$  and full hydrodynamic equations should be solved near singularity. One can find a range of applicability of Eqs. (7), (8) by looking at correction to these equations. E.g., the analysis for parameters of Fig. 1 shows that the correction is important for  $t \gtrsim 0.9t_{\text{viscous}}$  (for  $t = 0.9t_{\text{viscous}}$  correction to  $\eta|_{x=0}$  is about 30%). Detail consideration of that question is outside the scope of this Letter. Note that the question whether an actual singularity of the interface surface occurs in full hydrodynamic equations remains open.

To make connection with dynamics of ideal fluid with free surface (corresponds to the inviscid Burgers equations) [2] one can consider a limit  $\tilde{\nu} \rightarrow 0$  and, respectively,  $\tilde{\mu} \rightarrow \infty$ . It can be shown from the asymptotic analysis of the integral representation of the Hermite function that the largest zero,  $z_{\text{max}}$ , is given by  $z_{\text{max}} = 2^{1/2}\tilde{\mu}^{1/2} + O(\tilde{\mu}^{-1/6})$ . The leading order term,  $2^{1/2}\tilde{\mu}^{1/2}$ , exactly corresponds to the position of the upper branch point of inviscid solution (see Fig. 1) while term  $O(\tilde{\mu}^{-1/6}) < 0$  is responsible for the difference between  $t_{\text{inviscid}}$  and  $t_{\text{viscous}}$ . Even for moderately small  $\tilde{\nu}$  as in Fig. 1 that difference is numerically close to 1 because of small power  $\tilde{\mu}^{-1/6}$ .

It is easy to derive a wide class of initial conditions for which solution (7), (8) exists globally and the interface remains smooth at all times. E.g., one can take  $u^{(+)} = a_0 e^{ik_0 x - \tilde{\nu} k_0^2 t}$ ,  $k_0 = \text{Re}(k_0) > 0$  or any sum of imaginary exponent which ensures that there is no zeros at  $\text{Im}(x) = 0$ . However, if we suppose that there is a random force pumping of energy into system (or random initial condition) then one can expect that some trajectories with nonzero measure would have poles which reach real axis in a finite time.

In conclusion, one can mention possible physical applications. Eqs. (7), (8) describe a free surface dynamics of helium II with both normal ( $\nu_2 \neq 0$ ) and superfluid ( $\nu_1 = 0$ ) components. Derivation of these equations is slightly different from given in this Letter

because both fluids occupy the same volume but resulting equations are exactly the same as (7), (8). For classical fluids viscosity is nonzero but  $\nu_1$  can be neglected and the fluid 1 can be considered as ideal fluid provided the ratio of dynamic viscosities of two fluids is large,  $\nu_2\rho_2/(\nu_1\rho_1) \gg 1$ . E.g., that ratio is  $\sim 900$  for glycerin and mercury while ratio of their densities is  $\sim 0.09$  which makes them good candidates for experimental test of the analytical result of this Letter.

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