A quick, slick way to Heron's area formula--but what about a tetrahedron?

Many people know Heron's formula for the area of a triangle:

$$A = \text{sgrt } s(s-a)(s-b)(s-c),$$

where a,b,c are the lengths of the sides, and the "semiperimeter"

$$s = a+b+c/2.$$

Not everybody knows the proof. In fact, the classical proof, attributed to Hero of Alexandria (possibly invented by Archimedes) is too long and tricky to be taught in the usual geometry class. It can easily be derived from the Law of Cosines, but since it isn't a trig formula, it's usually omitted from a course in trigonometry.

Now here is a 2-line derivation usingelementary algebra! I noticed it a few months ago, and it was in the November, 2002 issue of Focus (publication of the Mathematical Association of America (USA)). Here it is:

First eliminate the superfluous letter s from the formula, to get

$$4A = \text{sqrt } \{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)\}$$

Now, the area A "vanishes" (equals 0) in case the triangle is "degenerate"--if

$$a = b + c, \text{ or } b = a + c, \text{ or } c = a + b.$$

that is, it vanishes if a equals either

$$(b + c) \quad \text{or } (c - b) \quad \text{or} \quad (b - c).$$

By the Factor Theorem, then, if A is a polynomial it must be a product of

$(- a + b + c)(a - b + c)(a + b - c)$ times something else. Such a polynomial would have to be of degree at least 3! But then, since the area is a homogeneous quadratic function of the lengths of the sides, it can't be a polynomial! The next simplest homogeneous quadratic function is the square root of a quartic. To get a quartic, we need one more linear factor, and by symmetry that can only be a constant multiple of $a + b + c$. We're done, that"s Heron! (Well, we still need the constant. Plug in $a = b = c = 1$, plug in the area of the unit equilateral triangle, and the constant falls out.)

BUT THEN a reader wrote to the editor of Focus; all this has been done before, in a note in the College Mathamatics Journal, 1987. (I was not upset. Many other great discoveries in mathematics were found independently several times.)

Besides, this is all very nice, but it’s not a proof! We made an undesirable assumption--that the area IS the square root of some quartic polynomial).

However, once the formula is in hand it is simple to verify it by Cartesian coordinates. To do so, it is best to multiply out the four factors, and then verify that 16 A^2 equals the resulting product. There is even a short cut to avoid any tedious algebra. Notice that the product is homogeneous symmetric of degree 4, and even in each letter a,b or c. But a symmetric homogeneous 4th-degree even polynomial in a,b,c can only be a linear combination of the sum of 4th powers and the sum of products of squares. Now plugging in values for a,b,c in TWO special triangles of known area, you get the unknown coefficients--respectively, -1 and +2.

Now you can verify the correctness of this multiplied out formula for $16A^2$, by putting the origin at one vertex, the point ($x = 1$, $y = 0$) at a second vertex, and then independently choose the scale of the y-axis so that the third vertex is on the horizontal line $y = 1$. The area is 1/2, and the algebra to verify the formula is not bad.

Now the inexperienced student might say, "Well done! We have a neat derivation, two formulas, and an easy proof!"

The experienced student says, "What next? Where do we go from here?"

Several roads are open. Spherical triangles, non-Euclidean triangles, Brahmagupta's formula for a cyclic quadrilateral. I chose the most natural and interesting (in my opinion.) If a formula is good in two dimensions, it would be better still in three!

In other words, find a formula for the volume of a tetrahedron analogous to Heron's formula for the area of a triangle.

Now, a tetrahedron has 4 faces and 6 edges. Do we want a formula in terms of faces, edges, or both? A tetrahedron can be described (parametrized) in several ways, but they all require 6 parameters. (For example, various choices of 3 numbers can specify a triangle to serve as the "base;" then 3 more numbers are needed to locate the 4th vertex, say by distances from the 3 base vertices.)

This shows that the areas of the 4 faces are not enough information to determine the tetrahedron. However, there are exactly 6 sides, so I hurried to find a formula for the volume as a symmetric function of the six side lengths. After many wasted hours, I realized that such a thing was impossible! Because there is a simple example of six numbers which can be the side lengths of two different tetrahedra.

First, take the rectangular tetrahedron with vertices at the origin and at (1,0,0), (0,1,0), and (0,0,1). The three faces in the coordinate planes all have area 1/2. The fourth face is equilateral, with edges all equal to sqrt 2, so this tetrahedron has 6 edges: {1, 1, 1, sqrt2, sqrt2, sqrt2} and volume 1/6. (V = 1/3 h A). On the other hand, there is another tetrahedron with base triangle {1,1,1} and slant heights {sqrt2,sqrt2,sqrt2}. It is a nice exercise in trigonometry to find its altitudes; then its volume turns out to be sqrt5/12! So the six edge lengths aren't enough to determine the volume; it also matters how they are connected. (In my article in Focus I gave a wrong answer! It was corrected by Peter Ross of Santa Clara U niversity.)

At this point I called for help from my mentor, Peter Lax. He came through. He e-mailed me. "There are 20 triplets of the 6 edge lengths; call them type A if they bound a face, type B if they share a vertex, and type C otherwise. (There are 4 each of type A and B, 12 of type C.) Square each product of type C and add them up (getting a certain homogeneous 6th-degree polynomial in 6 letters.) Then subtract the squares of the four products that bound the four faces. Then group the 6 edges into 3 pairs of opposite edges. For each pair, form the product of a square of one edge by the 4th power of the other, and subtract these twelve terms from the previously obtained result of subtraction. Take the square root, divide by 2--that's it! The volume."

I was stunned by this information. Certainly, I didn't doubt that it was correct. In fact, I was about to present it in a lecture when it occurred to me to check it with a simple special case.

It's wrong!

But the same Peter Ross wrote that on page 13 of George Polya's Patterns of Plausible Inference, (volume 2 of Mathematics and Plausible Reasoning) there is a formula of the kind I had been seeking! A little patience showed that Lax's formula and Polya's were almost the same. Only Polya says, divide by 12, not 2. Slight correction.

So I had sweated and strained to search for something that was already in the literature. But this was worse, because Polya's book was actually on my bookshelf.

Howvever, Polya doesn't give either a proof or a derivation. He just throws the formula at you, and says, "Check it in the following simple special cases." I threw in the towel, as they say in the boxing game, and asked Peter Lax for his derivation.

It is really very simple. But it requires a difficult, unpleasant first step. You(temporarily) give up symmetry, in order to eventually get some symmetry back.

The well known formula for the volume of a tetrahedron is 1/6 of the determinant of the 3-by-3 matrix whose rows(or columns) are three vectors that generate or determine the tetrahedron. (Peter had forgotten that little 1/6.) How to get from this to a symmetric expression in all six sides?

There are two little tricks. The first is to multiply the mentioned matrix by its transpose. The determinant of the product is the product of the determinants, which are equal, so it's the square of the volume (multiplied by 6^2.)

Now, the elements of this product matrix are the 9 inner products of the three basis vectors. The product of each vector by itself is just the square of its length.So all we have to do is get rid of the inner products of unequal vectors, and bring in the lengths of the other 3 edges. But if a and be are two of the basis edges, the 3d edge of that triangle is one of the edges of the opposite face of the tetrahedron, and it is given by the vector $a - b$. Thesquare of the length of $a - b$ is $a^2 + b^2 + 2\ a.b$, so $a.b$ can be eliminated in terms of a^2, b^2, and $(a-b)^2$. After that, says Peter, it's just multiplying, rearranging terms, and so on. Not that much work.

NOW IT'S YOUR TURN!

What can you say about the volume of a 4-simplex, as the square root of an 8th-degree polynomial in its 10 edges? Notice that whereas each edge of a 3-simplex (tetrahedron) has exactly one opposite edge, this is no longer true for a 4-simplex.

(Richard Askey wrote to Focus that the physicist Regge once complained that he couldn't get Mathematica to give him such a 4-d volume formula.)