The birth of random evolutions

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“Random evolutions” are stochastic linear dynamical systems. “Random” means, not just random inputs or initial conditions, but random media—a random process in the equation of state.

The theory was born in Albuquerque in the late 60’s, flourished and matured in the 70’s, sprouted a robust daughter in Kiev in the 80’s, and is today a tool or method, applicable in a variety of “real-world” ventures.

A year or two after I came to the University of New Mexico, we hired a young probabilist, Richard Griego. Griego took his Ph. D. in Urbana and was well versed in stochastic processes a la Doob. He showed me something I, a p.d.e. specialist, had never heard of at N.Y.U. or Stanford--Brownian motion can solve Laplace's equation or the heat equation! We wrote this up in a popular article for the Scientific American.

Probabilistic methods worked for p.d.e.’s of the parabolic or elliptic types. But I was a student of Peter Lax. I had learned to concentrate on hyperbolic equations, like the wave equation.

“Can you do it for the hyperbolic case?” I asked.

So far as Richard had heard, it seemed you couldn’t. There was a vague impression that Doob had even proved it couldn’t be done.

We organized a little seminar, with Bert Koopmans, a statistician, and Nathaniel Friedman, a young ergodicist. (Nat is now a sculptor, and organizer of mathematical art, in Albany, N.Y.) We also had the pleasure of interest on the part of Einar Hille, who had come to New Mexico after retiring from Yale. By good luck, Richard discovered in Nat's bookcase a copy of Mark Kac's <u>Magnolia Petroleum Company Lectures in Pure and Applied Science</u>. These <u>Lectures</u> were not in the University of New Mexico's library, nor in many other libraries. Most of their contents had been published elsewhere, but one section had never appeared in a journal. Kac considered a particle moving on a line at speed <u>c</u>, taking discrete steps of equal size, and undergoing "collisions" (reversals of direction) at random times, according to a Poisson process of intensity <u>a</u>. He showed that the expected position of the particle satisfies either of two difference equations, according to its initial direction. With correct scaling followed by a passage to the limit, the difference equations become a pair of first-order p. d. e.'s. Differentiating these and adding them yields the "telegraph equation":

$$d/dt(du/dt) + a\ du/dt = c\ d/dx(c\ du/dx))$$

This is an equation of <u>hyperbolic</u> type. (If you drop the lower-order term, it's just the one-dimensional wave equation.)

A probabilistic solution of a hyperbolic equation!

I was fascinated by Kac's little-known feat. Surely the telegraph equation couldn't be the only hyperbolic equation with a probabilistic meaning! Equally intriguing: if we can construct a probabilistic solution, we can exploit it. The central limit theorem was beckoning! If we could prove a central limit theorem for Kac's model, we'd have a limit theorem about p.d.e.'s.

As I contemplated the steps by which Kac constructed his probabilistic solution of the telegraph equation, I realized that one could replace the group of translations at speed c by any group of operators!

Suppose A is the generator of a group of linear operators, acting on a linear space B. Instead of translations moving randomly to the left and right at speed c, substitute time evolutions according to generators A and -A. In place of a particle whose position at time t is the sum of random translations, we get a random element of B, the result of successive evolutions "forward" (generator A) and "backward" (generator -A.) What is the expected value of this B-valued random process? Direct differentiation showed that in place of the classical telegraph equation, this expectation satisfies an operator differential equation. This can be obtained by simply substituting, in the classical telegraph equation, the abstract generator A for c d/dx, the generator of translation at speed c:

$$d/dt(du/dt) + a\, du/dt = A^2 u$$

We called this the "abstract telegraph equation."

For two speeds c and -c, switching at random times according to a Poisson process is equivalent to defining the process as a two-state Markov chain. But clearly one could let the particle move at any n speeds, with the switch between speeds governed by an n-state Markov chain. In the abstract case, instead of switching among random speeds, you could switch among n semi-group generators A_j operating on a given linear space B. The result would be a random process on B, which would in fact be a random product of random solution operators, each generated by its randomly chosen generator. Each solution operator

exp((t sub k)(A sub k))

has a time coefficient t sub k equal to the occupation time in the k'th state assumed by the n-state Markov chain. At first we called this random product of solution operators a "random semigroup." Peter Lax pointed out that it was not really a semigroup; he suggested the name, "random evolution."

If the generators A sub k don't commute, Richard found that this random product should be written "backwards", last operator first; if we define the random evolution this way, u(t), the expected value of the random evolution, satisfies a simple ordinary differential equation:

du/dt = Z u + Q u

u(t) is an n-tuple of elements from the linear space B, indexed according to each of n possible initial modes of evolution. Z is a diagonal matrix of the operators A_j. Q is a real matrix, the transition matrix of the Markov chain which controls the switching among the A_j.

In the classical case, the n semigroups are translations in R^3. Their generators, the diagonal elements of Z, are first-order differential operators in the spatial variables. The equations are a hyperbolic system of first-order differential equations, coupled through Qu. So we called the general case, with abstract generators $\underline{A}_j$, an "abstract hyperbolic system."

In the special case of only two semigroups, each the negative of the other, with a symmetric mechanism of reversing direction, the abstract system of two equations is equivalent to the abstract telegraph equation written above.

For this special case we proved a limit theorem. In probability language it's called a "central limit theorem", and in differential equations language it's a "singular perturbation theorem." (When a lower-order term in a differential equation has a small coefficient, it's called a regular perturbation. When a leading-order term has a small coefficient, it's called a singular perturbation.)

In the telegraph equation, the two leading terms have second order in $\underline{x}$ and in $\underline{t}$. We can't throw away u_{xx}, for we'd be left with an ill- posed problem. But if we throw away u_{tt}, we get the heat equation: a well-behaved equation which has a probabilistic solution--Brownian motion. If we can make our random linear process approach Brownian motion, surely its expected value--which satisfies the telegraph equation--will go to Brownian motions' expected value--which satisfies the heat equation. We'll have a probabilistic proof of a singular perturbation theorem for the heat and telegraph equations.

How to put a small coefficient in front of the second timederivative, to make our process look like Brownian motion? It's not hard to guess that we must speed up both the linear motion (translation) and the switching, to make the collisions frequent. The expected time between collisions is the reciprocal of _a,_ so to make the time between collisions small, multiply _a_ by a large number _R_. To speed up the linear motion, multiply the speed _c_ by another large number _S_. Then if you divide the equation by S^2 and choose

$$R = (cS)^2/a$$

you get the telegraph equation (classical or abstract!), with u_{tt} divided by S^2. The mean free path is the speed _Sc_ times the expected time between collisions, 1/_Ra_. This reduces to 1/_Sc_, going to zero as _R_ and _S_ go to infinity. Easy to guess now that as _R_ and _S_ go to infinity, the expected value u(t) goes to a solution of the heat equation. Since the fundamental solution of the heat equation is the distribution function of the normal random variable, it's no longer a surprise that this singular perturbation theorem for a p.d.e. can be proved by means of the central limit theorem of probability.

What happens in the abstract telegraph equation, where _c d/dx_ is replaced by _A_? Then, of course, the solution converges to the solution of an "abstract heat equation,"

$$du/dt = A^2u.$$

Prof. Hille published an announcement of our results in the _Proceedings_ of the National Academy of Science. Then we submitted the complete paper to the _Journal of Functional Analysis_. We were surprised and disappointed when it was rejected. However, the letter of rejection was twelve pages long, and contained sensible suggestions for improvement. The referee’s references to “the book” left no doubt that he was William Feller. We took advantage of his advice, and the improved paper appeared in the A.M.S Transactions. I still regret that I never met Feller in person.

By another piece of luck I stumbled across the M.I.T. thesis of Mark Pinsky. Pinsky had extended Kac's work on the telegraph equation. Instead of Kac's two velocities c and $-c$, Pinsky allowed n arbitrary velocities, switching according to an ergodic n-state Markov chain. More important, Pinsky proved a central limit theorem for this n-state process. The identity of the limiting equation in this generality is much less obvious than in the telegraph equation that Griego and I had treated.

Mark spent two weeks in Albuquerque the following summer. My daughter baby-sat so that he and Joanna could sight-see. Mark and I undertook to extend his central limit theorem to abstract semigroups--or, what is the same thing, to extend the central limit theorem Griego and I had proved for the abstract telegraph equation to an arbitrary abstract hyperbolic system.

We assumed that the generators were mutually commutative. This was an undesirable restriction, but it was a significant step forward. It included the Kac and Griego-Hersh papers, with switching between cA and $-cA$, and Pinsky's thesis, with switching between first-order spatial differential operators with constant coefficients.

We showed that in the limit of an appropriately scaled small parameter, all components of the solutions of the commutative abstract hyperbolic system converge to solutions of an abstract heat equation:

$$du/dt = Hu.$$

H is a certain quadratic expression in the n generators of the constituent semi-groups. In the classical case (Pinsky's thesis), these generators are constant-coefficient first-order spatial differential operators, and H is a second-order elliptic operator in R^3 with constant coefficients.

Pinsky found an abstract characterization of random evolutions in terms of multiplicative functionals, which he presented in a book (104, 107).

After the first Griego-Hersh paper was published, and again after my paper with Pinsky was published, I received letters from Tom Kurtz. Kurtz had his own abstract-space version of singular perturbation theory. He used it to re-prove both the abstract telegraph equation limit theorem (Griego-Hersh) and the commutative abstract hyperbolic systems limit theorem (Hersh-Pinsky>. (He did the same thing to the non-commutative limit theorem I obtained later with George Papanicolaou.) It seemed that he was determined to find non-probabilistic proofs for all our probabilistic theoreMS(76). But as his career developed he became committed to probabilistic problems and methods. He obtained first-order limit theorems (75) (laws of large numbers) for random evolutions, in addition to second-order theorems (central limit theorems) of the type which Griego and I had concentrated on. Particularly interesting were his limit theorems for sequences of semigroups of nonlinear operators.

The following year I had a sabbatical at my alma mater, the Courant Institute. I attended a probability seminar led by Monroe Donsker. There George Papanicolaou was reporting on recent work (72) of Khas'minski. George suggested we work together. Before I returned to New Mexico we succeeded in proving limit theorems for random evolutions made of non-commuting semi-groups. These proofs were more probabilistic than my earlier ones. Now we assumed that the process by which the semigroups switch has a state whose recurrence time has finite mean and variance. (This is automatically satisfied in case of an ergodic finite-state Markov chain.) Under this hypothesis, the evolution is a random product of random factors, each of which starts and ends at the distinguished ergodic state. These random factors are independent and identically distributed. We were able to calculate their common expected value, estimate its dependence on the small parameter, and find what happens when the parameter vanishes. At one point in our work, we were stuck. A few words from S.R.S. Varadhan were sufficient to point toward the solution. As in the commutative case, all components of the "abstract hyperbolic system" which is satisfied by the expected values of the random process converge to solutions of a single "abstract heat equation," with a single "abstract second-order elliptic" operator on the right-hand side. This operator is again given by a quadratic expression in the generators of the constituent semigroups. But these do not commute, so the quadratic expression is defined by non-commutative algebra. It is given explicitly in our paper. George went on to write several more papers applying random evolutions to physical problems.

A specific, concrete example of non-commuting group generators is a collection of first-order differential operators in R^3 with position-dependent coefficients. Each of these generates its group of translations along its family of curved trajectories in R^3. The limiting equation is again second-order parabolic, but now with variable (position-dependent) coefficients.

In translation semi-groups, whether with constant or variable coefficients, the small parameter has a very simple physical interpretation. Just as in the case of the telegraph equation discussed above, it is proportional to the <u>mean free path</u>--the average distance traveled between collisions. (A simple derivation of this fact is given in Hersh-Pinsky (55).) Vanishing of the small parameter--the mean free path--means a drastic transformation of the physical process, from discrete particles in collision to smooth motion in a continuum. Our convergence theorem says that in the limit of small mean free path, the expected motion of a Newtonian particle goes over to diffusion.

It may seem surprising that the limit depends only on shrinking the mean free path, not on increasing the density of particles. But physically the vanishing of the mean free path comes about by increasing the particle density. Increasing the density of particles results in a passage to the diffusion limit by making the mean free path go to zero.

While in New York I spoke at Mark Kac's seminar at Rockefeller University. This was the first time I had met Kac. Fortunately, I was able to see him several more times before he died. During the seminar he listened closely, and challenged me twice. "Is the square root of the Laplacian operator really the generator of a group"? Was I right to call my formula for the limiting operator "explicit"? Both times I explained, and he yielded.

In the telegraph equation the term <u>a du/dt</u> is what makes energy dissipate, so I called this term the "parabolic part" of the equation. This term carries the stochastic part of Kac's solution of the telegraph equation--switching according to a Poisson process with intensity <u>a</u>. If <u>a</u> is 0, there is no switching, the "parabolic term" drops out, and the equation reduces to the one-dimensional wave equation. <u>The presence of the "parabolic term", a du/dt, is what makes possible a probabilistic solution of this hyperbolic equation.</u> Kac leaned forward and exclaimed, "That's right!" A treasured and memorable moment for me.

Later Richard Griego reminded me that in (61) Kac wrote of Norbert Wiener: "What is really surprising is that he was apparently unaware of the intimate connections between his measure and his own work on potential theory. It is now almost universally known that the generalized Wiener-Perron capacitory potential of a closed set _F_ (in Euclidean space of dimension 3 or higher) at a point _p_ is the Wiener measure of the set of paths which originate from _p_ and which at some time hit _F_. Moreover, Wiener's famous criterion for regularity of boundary points has a most appealing interpretation in terms of his measure. The surprise is heightened if one recalls that Wiener's work on potential theory was almost simultaneous with that on Brownian motion."

Griego reminded me of this quotation because of the instructive fact that what Kac said of Wiener with regard to potential theory, he could have said of himself with regard to random evolutions. Kac was a great advocate of function space integrals as a tool for solving p.d.e's--the Feynman-Kac formula (61). And Kac found a probabilistic solution of the telegraph equation
(62). But he didn't connect his solution of the telegraph equation with his function-space integrals. Had he done so, his use of an approximation by a finite difference equation would have become unnecessary, and he could have done the whole random evolution thing by himself--if he had cared for abstract spaces and operators!

When I returned to New Mexico after my sabbatical leave, I was fortunate again. My colleague Bob Cogburn (one of Michel Loeve's three Ph. D. students) became interested in random evolutions. With his mastery of probabilistic estimation, it was possible to carry all earlier asymptotic results on random evolutions to their natural generalization (15). The random process need not be finite or even discrete. The family of semigroups need not be finite or countable. The random process controlling the choice of semigroups need not be Markovian; it's sufficient if it satisfies a "mixing condition" (be almost independent when separated by long time intervals.) Shortly afterward, a comparable result was published by Papanicolaou and Varadhan (101). Under some extra hypotheses they also got a rate of convergence. Also around the same time Richard Ellis and Walter Rosenkrantz published a paper in which Kac's particle is restricted to a bounded interval, with appropriate boundary conditions (29-30). They obtained a central limit theorem for this case.

In 1972 I was asked by the Rocky Mountain Mathematics Consortium to organize a summer meeting on stochastic differential equations. This was part of a series of meetings which had been supported by a grant of $25,000 a year from the National Science Foundation. The year I became responsible, the N. S. F. changed its policy, and gave no money at all. We had planned to meet in Santa Fe, N. M. Fortunately, the Consortium included a Canadian school--the University of Alberta, in Edmonton. Jack Macki, Alberta's representative to the Consortium, convinced his university to contribute $10,000. There was also a small additional donation from the Provincial Government of Alberta. Because of this Canadian support, we had our meeting in Edmonton. Richard Griego took a lot of responsibility on this project.

The participants seemed glad to pay their own way to Edmonton. Mark Kac and Wendell Fleming were invited speakers. The proceedings became a special issue of the Rocky Mountain Mathematics Journal.

It's time to talk about some of our students. Mark Pinsky's student Bob Kertz in his thesis (74, 75) obtained a central limit theorem for random evolutions with discontinuities. Richard Griego's student Manuel Keepler wrote about duality and time reversals of random evolutions (70, 71). Keepler has been one of the more prolific of his generation. Donald Quiring, Bert Koopmans's student, wrote a thesis (107) on random evolutions controllled by a diffusion process. Griego's student John Hagood wrote (50) about the operator-valued Feynman-Kac formula.

Together with the Mexican probabilist Alberto Moncayo, Richard Griego wrote about constructing random evolutions by a "piecing-out" process (47). David Heath, in an (unfortunately} unpublished thesis (51), found a probabilistic model for the telegraph equation with space-dependent coefficients. Tom Kurtz's student Joe Watkins published several impressive papers (128--131) carrying the limit theorems of random evolutions farther than before.

In (46) Richard Griego and I used the random evolution model to get new results on the spectral theory of degenerate elliptic operators. Mark Pinsky developed a theory of stochastic processes on manifolds, some of which is expounded in his book (107). Bob Cogburn with his students systematically developed a theory of random processes in random environments (10--16). George Papanicolaou has published a great deal on wave propagation in random media and on homogenization (87--101). Fascinating applications of random evolutions to biology and engineering were contributed by Dunbar (26), Becus (2--5), and Iordache (64).

I felt that the joint paper with Cogburn had completed my romance with random evolutions, getting full generality in both the operators and the random switching. (Unlike some papers by later authors, the work reported here includes unbounded semigroups as well as bounded ones.) Unknown to me and my American collaborators, our work was noticed in Kiev. Vladimir Korolyuk, a distinguished probabilist and Academician, and his pupil Anatolyi Swishchuk, undertook a massive research into random evolutions controlled by semi-Markov processes. For this question they answered all the questions we had asked, and many others we had neglected.

Semi-Markov processes, like Markov processes, "have no memory." The mode of evolution you jump into after a "collision" depends only on what mode you are in at the time of the collision. The difference is that for Markov processes, the waiting time between collisions is exponentially distributed. In the semi-Markov theory, this restriction is relaxed. The waiting time can be prescribed according to the particular application in view. Swishchuk's two recent books (125, 126) give scores of references in physics, biology, and especially in modern finance. (Swishchuk is now at York University, in Toronto.)

I only learned about this work in 1992, years after it had been going on. "Out of the blue" I received an invitation to the Third (!!) conference on random evolutions, to be held on the shore of the Black Sea, in Katsively, where the Ukrainian Academy of Science has a beautiful resort.

I was able to take advantage of this invitation to visit Butznivets, the tiny Ukrainian village that my mother left in 1919. Of course I found no trace of her family, or of any of the Jewish families who used to live there.

References

I recommend the survey articles (120--122) by Swishchuk, and my own surveys (59) and (58). Chapter 12 of (32) by Ethier and Kurtz is a fine exposition of the subject, as is Mark Pinsky's outstanding book (109).

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