The “Origin” of Geometry

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The German phenomenologist Edmund Husserl wrote a famous essay, “The Origin of Geometry” that called for a new kind of “historical” research, to recover the “original” meaning of geometry, to the man, whoever he was, who first invented it.

It seems to me not that hard to imagine the origin of geometry. Once upon a time, even twice or several times, someone first noticed some simple facts. For example, when one stick lies across another stick, there are four spaces that you can see. You can see that they are equal in pairs, opposite to opposite. It happened something like this, perhaps at some campfire, 20 or 30,000 years ago.

In the dead ashes are lying two sticks, one across the other.

Ancient #1: Look at that! Do you see that?

Ancient #2: What? See what?

1: Those two sticks. How they cross—see, they make four spaces. Two big ones, two little ones. The big ones are across from each other on the sides, and the little ones are across from each other, one the top and bottom.

2: So what? (Kicks one of the stocks.) Now what happened to your four spaces?

1: You changed them around. Now the top and bottom are bigger, and the ones on the side are littler. There are still four spaces. And you still have little facing little, big facing big.

2: Yes, that’s the way it is now.

1: Turn them any way you like, you always get four spaces, and they are equal in opposite pairs.

2: I don’t believe it.

1: How can’t you believe it? Can’t you see it?

2: Just watch now. I turn the top stick, little by little. The top and bottom spaces get smaller, the side spaces get bigger.

1: All right.

2: What if I stop now? Where are your big and little spaces now, Mr. Smart Aleck Wise Guy?

1: You stopped before they could switch around. Before the little spaces became bigger than the big ones and the big spaces became smaller than the little ones.

2: Yes, that’s what I did. That shows you’re way off, you’re screwed up.

1: When the sticks cross now, they make four equal spaces. That’s a special interesting way to make two sticks cross. I like that. You did something good.

2: Let’s go chase a rabbit and eat it.

Something like this must have happened more than once. Someone noticed something interesting about a couple of sticks, or bits of straw, or crossed fingers. Something that has to be so, whether you want it or not. An invariant. A geometric fact.

After a while, a name is invented for those four spaces where the sticks cross. "Angles." Two crossed sticks, or fingers, or forearms, make four angles. The opposite angles are equal, two by two. Some word is invented for the general case. "Line."

The little anecdote becomes profound, if we raise it to the level of ontology. Was it always true that two lines intersecting in a plane divide the plane into four regions, pairwise alternately congruent? Is that a fact about the Universe? Was it true before there were any fireplaces or Ancient People, before there were thoughts at all?

This revives an argument I have had with two friend-opponents, both of whom happen to be named "Martin". I take the liberty of representing them in an interlocutor named "Merton." Merton is wrong, but he is very smart. I will do full justice to his intelligence, his persistence, and his commitment to Platonism. (My arguments with the two Martins were about the facts of arithmetic rather than the facts of geometry. The basic issues are the same.)

Merton: Of course that simple theorem is true. It's a true fact about lines in the plane. You recognize that it's true, and then you see how to prove it. It was true before you noticed it, and it will be true after there's no one around to notice it or prove it.

Reuben: Let's look at the sticks in the fireplace, or, if you're more comfortable, let's look at the diagram in my geometry book. What do you see?

M: What you just explained with your simple-minded story. Two intersecting lines divide the plane into four parts, etc., blah blah blah.

R: Is that really what you see? Isn't that really your interpretation of what you see?

M: What do you mean, really? Really what?

R: Do you even really see a plane?

M: Well, if you want to be literalistic, of course I see a piece of paper, a page in a book.

R: Yes. And there's a diagram on the page. Describe it.

M: Two lines intersecting at an angle.

R: Lines? Really lines?

M: Well, no, of course they're only segments of lines. That's all you can draw on the page.

R: Then how can it be that you see the whole plane being divided into four parts?

M: What I mean is, the diagram is meant to represent the two infinite lines, which extend without limit, even though they can't be drawn on the page.

R: That's right. I agree. That's what the diagram is meant to represent. We understand the diagram, and what it's meant to represent. However, I know some hardheaded people who like to give sensible people like you and me a hard time. One of them is a promising graduate student, Melissa. I know what she would say about this.

M: Do you? And what is that?

R: She'd say she sees a kind of cross, which could be shrunk or extended, but would never divide the plane into separate regions. It just makes a hole in the middle, you can always go around it.

M: Oh. Well, if you want to be difficult, you can always misinterpret anything.

R: Right. So you and I agree that Melissa is just being a pest. We understand what the diagram is meant to represent, an infinite plane in which two infinite lines intersect, thereby dividing that infinite plane into four wedges, which are two-by-two congruent.

M: Right. Of course, it's obvious.

R: And this fact on which we agree is a fact about what?

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M: About this figure, this diagram.

R: No, not really. Melissa has you on that.

M: OK. It's a fact about the correct and intended interpretation of the diagram.

R: Now, an intended interpretation, as you well put it, is what kind of a thing?

M: What do you mean? Well, it's abstract, it's not physical.

R: Isn't intending and interpreting a kind of activity that is engaged in by creatures like you and me?

M: Oh, I see what you're getting at. You think you're leading me into a trap where I'll be forced to admit that mathematical facts like this one are referring to shared thinking or understanding of human beings, rather than to objective properties of entities that exist independently of human thought.

R: I wouldn't call it a trap. Just try to think about what we've been saying, and what it means in terms of what some philosophers like to call "ontology". The question of what exists, or, if you prefer, what we're talking about when we talk about something or anything.

M: No, it's a trap, you know exactly what you're doing and where all this has been leading.

R: Don't upset yourself, Merton. This should just be a friendly chat about topics of mutual interest.

M: Well, I really don't have a lot of time for this kind of thing. I should be getting back to work.

So I imagine the conversation going. But I may be unfair. I may be wrong about Merton's willingness to face up to the meaning of mathematical talk. Let's scratch out the last bit of dialogue, and continue in a serious and thoughtful way.

M: It doesn't make a difference how we talk or what words we might use. Of course there is an intention and an understanding involved in looking at this diagram. There is an objective reality which cannot be diagramed directly, but only suggested by incomplete but correctly understood diagrams. Such is the case here.

R: That's interesting. We see two line segments crossing, and we know that we could interpret them as parts of two infinite lines. Is that right?

M: Of course it is. You know it is.

R: Just as we know that the lines are not supposed to have any thickness, even though in the actual drawing we couldn't see them unless they had a positive thickness?

M: That's more or less the same issue.

R: And the segments are supposed to be perfectly straight, even though if we were fussy enough with super-accurate measurements we might detect a little curvature?

M: Yes, yes, right, right.

R: If what we see were really a small part of a much bigger picture, it could be that in the bigger picture it turned out that the two segments we see are parts of curved lines, lines that might even intersect again some place far away?

M: Why would you want to suppose anything as weird as that?

R: I'm trying to clarify and examine the way we have this correct understanding. And are you sure it is correct? You know, the Earth isn't really flat.

M: Yes, so I've heard.

R: So—

M: Don't say it! I know what you're trying to do. You're trying to tell me that if we actually did extend these two lines as physical marks here on the surface of this earth, they would intersect again on the opposite side. Right?

R: Well, I might have been thinking that way. But since you anticipated me, I can be even more ridiculous. How do I know that these two line segments are actually, as you now propose, arcs of great circles? Why not small pieces of almost anything? If I am sure they can be extended indefinitely, how do I know *how* they should be extended? There's no limit to the complicated and peculiar ways they could be extended, here on the surface of this earth, or even in that infinite plane which we both understand was intended to be the meaning of this diagram.

M: Well, what is the point of creating confusion? The meaning of the diagram is perfectly clear.

R: I agree. What makes it so clear? Not that there's only one way to interpret it. Melissa's way would really be the way the diagram was intended, if it was in a complex-variable or a topology text, illustrating a punctured plane. Of course, you have to know whether you're looking at a topology book or a book on plane Euclidean geometry. Even a book on classical plane protective geometry would imply a different interpretation of the diagram. The diagram by itself doesn't force one correct interpretation on us. Yet of course you're right, there is one correct interpretation, and I suppose hardly any geometry teacher or student fails to make that interpretation. That's what we do. So do I, and so do you, Merton.

M: You insist on subjectivizing and relativizing object reality. I suppose there's no way I can stop you from doing that. And why would I even try to stop you? Go right ahead, be as subjective as you like. The objective mathematical reality isn't damaged by that. Only anybody you manage to confuse with your pointless sophistry.

R: Try to keep calm, Merton. It's just a friendly chat. Do you remember the old junior high-school conundrum, "If a tree falls in the forest and nobody is there to hear it, was there a sound?"

M: Yes, I'm afraid I have to admit I do remember that one.

R: It's easy to answer, isn't it? Tell me if you agree with my answer.

M: Sure.

R: The only reason there is any difficulty is the failure to explain what you mean by "a sound". You might mean my subjective sensation of hearing anything—a noise, a voice, or whatever, and call that experience of mine or that sensation, "a sound."

M: Right, you could possibly do that.

R: On the other hand, if you have had high-school physics, you might remember that in physics sound refers to vibrations propagated as waves though various media, possible the atmosphere, or though a solid if you hear people talking on the other side of a closed door, and so on. That's also a way people use the word, "sound." Right?

M: Definitely right.

R: So the puzzle about the tree falling in the forest is no puzzle. There is no sound in the first sense of the word, there is a sound in the second sense.

M: A masterful exposition.

R: I know you're being sincere, not sarcastic, so thanks.

M: You're suggesting a similar ambiguity in the nature of mathematical reality.

R: No, you're suggesting it, and I'm denying it.

M: How's that? When did I do that?

R: You're aware that mathematical notions or conceptions exist in people's heads

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or their thoughts or their conversations or their writings. You haven't said so, but of course you know it is so.

M: How could I not know that?

R: Right. That corresponds to the notion of sound as subjective sensation.

M: I see where you're going.

R: And you're just saying that apart from that, there is the objective mathematical reality, just as there is the objective physical event that is also referred to as sound.

M: Yes, I would accept that analogy.

R: And I say it's a false analogy.

M: What's false?

R: Simply that there is no observable, identifiable, locatable, describably inhuman objective mathematical reality which you or anyone can show me, that would correspond to the sound waves of acoustics.

M: What do you mean? You know that it's objectively true that two intersecting lines in the plane cut it into four parts. You could even make a physical model of it, and experience the separation physically by putting up two very long intersecting fences. What do I have to do to make you see that that's a fact about lines in the plane, independent of whether anyone ever notices it or says it?

R: I fully agree. It is such a fact. But the lines in the plane are simply our mutually agreed on, intended and understood interpretations of this diagram. They're an idea that we understand very clearly, and can talk about with complete agreement. And that idea or conception indeed has objective properties, in the sense that many other clearly understood, agreed-on mutual ideas do have objective properties. Like the facts of law, of music, of literature, of proper behavior in public places in the U.S. in the year 2002. There are lots of facts about such things, and they are real facts, objectively and have objective properties. They are what we obviously see and experience them as—mutual understanding. There's no need and no basis in logic or science to imagine that these ideas and understanding can or did exist without us as a society, a culture, a profession, to understand them.

M: You mean that if somewhere else in the universe at some time in the very remote past or future, two lines intersect in the plane, they wouldn't make four pieces?

R: If people, or creatures much like people, had such thoughts, then and there, those thoughts would have the same consequences. That is a big "if". You can just as well imagine other cultural artifacts reoccurring in your science-fiction imagination. People can imagine as they please. But of course there couldn't *be* an infinite plane and two infinite lines in that plane, ever, anywhere, as far as our physical understanding permits. Such things are used in physics to state theories of mathematical physics. They are not phenomena, data, observations, or possible phenomena, data, or observations. Therefore to talk about what *would* happen if such phenomena were impossibly supposed to occur on Antares or Sirius, way back when the world was young, is vacuous. What happens when one stick lies on top of another is a *physical* regularity. We notice it, conceptualize it, and therefore create mathematics.

References

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