9.3 Notes

In this chapter, it would be best if we make a list of the variables and the initial conditions. Another thing is that when we solve for the initial condition and find the necessary variables, we will have to solve for the proportionality constant.

Example 1

In a certain chemical reaction, a substance is converted into another substance at a rate proportional to the square fo the amount of the first substance present at any time t. Initially ($t = 0$), 50 grams of the first substance was present; 1 hr later, only 10 grams of it remained. Find an expression that gives the amount of the first substance present ay any time t. What is the amount present after 2 hrs.

Listing out the things we need to know.

Let $A =$ first substance.
@ $t = 0$, 50 grams of the first substance is left
@ $t = 1$, 10 grams is left. Rate is $\frac{dA}{dt}$
Substance at a rate proportional to the square fo the amount of the first substance present at any time t is written; kA^2. Where k is the proportionality constant.

So what we have when we put everything together,

$$\frac{dA}{dt} = kA^2$$

Moving the dt to the other side and integrate

$$\begin{aligned} \int \frac{1}{A^2} dA &= \int k dt \\ -\frac{1}{A} &= kt + C \\ A &= -\frac{1}{kt + C} \end{aligned}$$

Now putting the initial conditions, and remember that $A(t) = 50$,

$$\begin{aligned} A(0) &= -\frac{1}{C} \\ 50 &= -\frac{1}{C} \\ \text{so we have} \quad C &= -\frac{1}{50} \end{aligned}$$

When we put the equation together we have this,

$$A(t) = -\frac{1}{kt - \frac{1}{50}}$$

When we do the algebra, we get this,

$$A(t) = -\frac{50}{50kt - 1}$$

Now we have to solve for k, the proportionality constant. By doing this we put in the the second condition, $A(1) = 10$ grams

$$A(1) = -\frac{50}{50k - 1}$$
$$10 = -\frac{50}{50k - 1}$$
$$1 = -\frac{5}{50k - 1}$$
$$50k - 1 = -5$$
$$k = -\frac{2}{25}$$

Putting this into the general equation and simplifying, we get;

$$A(t) = \frac{50}{4t + 1}$$

Example 2
Newton's law of cooling states that the rate at which the temperature of an object changes is directly proportional to the difference between the temperature of the object and that of the surrounding medium. A cup of coffee is prepared with boiling water($212°F$) and left to cool on the counter in a room where the temperature is $72°F$. If the temperature of the coffee is $140°F$ after 2 min, determine when the coffee will be cool enough to drink (say,$110°F$).

Let A = the air temp, B = temp of the coffee, and t is time.
Initial condition is $B(0) = 212$, $A = 72$, $B(2) = 140$, and find the time when $B(t) = 110$.
The rate is $\frac{dB}{dt}$.
The part that is written: directly proportional to the difference between the temperature of the object and that of the surrounding medium is $k(A - B)$, where k is the proportionality constant.

$$\frac{dB}{dt} = k(A - B)$$

Using separation of variables we get the next line

$$\int \frac{dB}{A - B} = k \int dt$$

You can u-sub on the left of the equation

$$u = A - B$$
$$du = -dB$$
$$-du = dB$$

We then have

$$\ln(A - B) = -kt - C$$

Remember that when we get rid of the natural log, we use e. Solving for B, we get;

$$B(t) = A - De^{kt}$$

Where $D = e^{-C}$. Using the initial condition is $B(0) = 212$ and $A = 72$ to solve for D, we get $D = -140$. When we put in D becareful of the sign change.

$$B(t) = 72 + 140e^{kt}$$

Now we want to solve for k, the proportionality constant using B(2)

$$B(t) = 72 + 140e^{kt}$$
$$B(2) = 72 + 140e^{2k}$$
$$68 = 140e^{2k}$$
$$\frac{17}{35} = e^{2k}$$

Simplifying and using the natural log to solve for k, we get,

$$k = \frac{1}{2}\ln(\frac{17}{35})$$

Putting this into the equation, we now get,

$$B(t) = 72 + 140e^{\frac{1}{2}\ln(\frac{17}{35})t}$$

For the final part finding the time it takes the coffee to cool down to 110 degrees F.

$$110 = 72 + 140e^{\frac{1}{2}\ln(\frac{17}{35})t}$$

Solving for t we get

$$t = \frac{2\ln(\frac{19}{70})}{\ln(\frac{17}{35})} \text{ min.}$$

A few things we have to remember from college algebra when using natural log.
1. If the fraction in the natural log is $0 < x < 1$ then the calculated solution is a negative number.
2. If the fraction in the natural log is $x > 1$, then the calculated solution is a positive number.
For our case, the numerator and denominator is both a negative number so the overall result is a positive number, where time has to be positive. You can leave the solutions this way, because it cannot be simplified for it is not in a compound fraction form.

Example 3
The rate at which a rumor spreads through an Alpine village of 400 residents is jointly proportional to the number of residents who have heard it and the number who have not. Initially, 10 residents heard the rumor, but 2 days later, this number had increased to 80. Find the number of people who will have heard the rumor after 1 week.

We will let R =rumor, t = time in days.
The initial condition is $R(0) = 10$, and $R(2) = 80$, and we have to find $t = 7$.
The rate $=\frac{dR}{dt}$ and the part where it is written: jointly proportional to the number of residents who have heard it and the number who have not is, $kR(400 - R)$

$$\frac{dR}{dt} = kR(400 - R)$$
$$\int \frac{dR}{R(400 - R)} = k \int dt$$

We now have to use partial fraction decomposition to help us solve this integral. The set up is:

$$\frac{1}{R(400 - R)} = \frac{A}{R} + \frac{B}{400 - R}$$

Where we have to solve for A and B. Where we have $A = B$ when we solve it and $A = \frac{1}{400}$. Therefore the integral above now becomes,

$$\frac{1}{R(400 - R)} = \frac{1}{400}\left(\frac{1}{R} + \frac{1}{400 - R}\right)$$

So now the integral will become,

$$\frac{1}{400} \int \left(\frac{1}{R} + \frac{1}{400 - R}\right) dR = kt + C$$
$$\ln R - \ln(400 - R) = 400kt + 400C$$

On the left side of the equation we can use the natural log properties
$\ln(AB) = \ln A + \ln B$
$\ln \frac{A}{B} = \ln A - \ln B$

$$\ln \frac{R}{40 - R} = 400kt + 400C$$
$$\frac{R}{40 - R} = e^{400kt + 400C}$$

Where $D = e^{400C}$

$$\frac{R}{40 - R} = De^{400kt}$$

Have to solve for R

$$R(t) = \frac{400e^{400kt}}{1 + De^{400kt}}$$

Using the initial condition, $R(0) = 10$, and to find k use $R(2) = 80$.

$$R(0) = \frac{400}{1 + D}$$

And $D = \frac{1}{39}$
So the equation becomes

$$R(t) = \frac{400e^{400kt}}{39 + e^{400kt}}$$
$$R(2) = \frac{400e^{800k}}{39 + e^{800k}}$$
$$80(39 + e^{800k}) = 400e^{800k}$$
$$k = \frac{1}{800} \ln\left(\frac{317}{20}\right)$$

so now the equation becomes,

$$R(t) = \frac{400e^{\frac{1}{2}\ln\left(\frac{317}{20}\right)t}}{39 + e^{\frac{1}{2}\ln\left(\frac{317}{20}\right)t}}$$

$$R(7) = \frac{400e^{\frac{7}{2}\ln\left(\frac{317}{20}\right)}}{39 + e^{\frac{7}{2}\ln\left(\frac{317}{20}\right)}}$$