2 For $v \in R$ there is a short exact sequence

$$0 \to R \xrightarrow{\alpha} R \to R/vR \to 0$$

where $\alpha(v') = vv'$ for $v' \in R$, which is a free presentation of R/vR. For any R module B, $\operatorname{Hom}(R,B) \approx B$ and the homomorphism $\operatorname{Hom}(\alpha,1)\colon \operatorname{Hom}(R,B) \to \operatorname{Hom}(R,B)$ corresponds to $\alpha^*\colon B \to B$, where $\alpha^*(b) = vb$. Hence there is an isomorphism $\operatorname{coker} \operatorname{Hom}(\alpha,1) \approx B/vB$, and we have proved

$$\operatorname{Ext}(R/vR,B) \approx B/vB \approx (R/vR) \otimes B$$

Since Hom commutes with finite direct sums, it follows that for any finitely generated torsion module A there is an isomorphism (nonfunctorial)

$$\operatorname{Ext}(A,B) \approx A \otimes B$$

because such a module A is a finite direct sum of cyclic modules (by theorem 4.14 in the Introduction).

An *extension of B by A* is a short exact sequence

$$0 \to B \to E \to A \to 0$$

With a suitable definition of equivalence of extensions (by a commutative diagram), of the sum of two extensions, and of the product of an extension by an element of R, there is obtained a module whose elements are equivalence classes of extensions of B by A. This module is isomorphic to $\operatorname{Ext}(A,B)$. In fact, given an extension $0 \to B \to E \to A \to 0$ and a free presentation of A, $0 \to C_1 \to C_0 \to A \to 0$, there is, by theorem 5.2.1, a commutative diagram

$$\begin{array}{ccccccc} 0 \to & C_1 & \to & C_0 & \searrow & & \\ & \varphi_1 \downarrow & & \varphi_0 \downarrow & & A \to 0 & \\ 0 \to & B & \to & E & \nearrow & & \end{array}$$

uniquely determined up to chain homotopy. Then $\varphi_1 \in \operatorname{Hom}(C_1,B)$ is unique up to $\operatorname{im}[\operatorname{Hom}(C_0,B) \to \operatorname{Hom}(C_1,B)]$, and so determines an element of $\operatorname{Ext}(A,B)$. This function from extensions of B by A to $\operatorname{Ext}(A,B)$ induces an isomorphism of the module of equivalence classes of extensions with $\operatorname{Ext}(A,B)$.

Given a graded module $C = \{C_q\}$, there is a graded module $\operatorname{Ext}(C,B) = \{[\operatorname{Ext}(C,B)]^q = \operatorname{Ext}(C_q,B)\}$. If C is a chain complex, $\operatorname{Ext}(C,B)$ is a cochain complex with

$$\delta^q = \operatorname{Ext}(\partial_{q+1},1)\colon \operatorname{Ext}(C_q,B) \to \operatorname{Ext}(C_{q+1},B)$$

A homomorphism

$$h\colon H^q(C;G) \to \operatorname{Hom}(H_q(C;G'), G \otimes G')$$

natural in C and G is defined by

$$(h\{f\})\{\Sigma\, c_i \otimes g_i'\} = \Sigma\, f(c_i) \otimes g_i'$$

for $\{f\} \in H^q(C;G)$ and $\{\Sigma\, c_i \otimes g_i'\} \in H_q(C;G')$ [after verification that $\Sigma\, f(c_i) \otimes g_i'$ is independent of the choice of f in its cohomology class and $\Sigma\, c_i \otimes g_i'$ in its homology class]. For $u \in H^p(C;G)$ and $z \in H_q(C;G')$ we define $\langle u,z\rangle \in G \otimes G'$ to be 0 if $p \neq q$ and to be $h(u)(z)$ if $p = q$. In this notation

$$\langle \{f\}, \{\Sigma\, c_i \otimes g_i'\}\rangle = \Sigma\, \langle f,c_i\rangle \otimes g_i'$$

The homomorphism h enters in the following *universal-coefficient theorem for cohomology.*

3 THEOREM *Given a chain complex C and module G such that* $\operatorname{Ext}(C,G)$ *is an acyclic cochain complex, there is a functorial short exact sequence*

$$0 \to \operatorname{Ext}(H_{q-1}(C),G) \to H^q(C,G) \xrightarrow{h} \operatorname{Hom}(H_q(C),G) \to 0$$

and this sequence is split.

PROOF We first consider the case in which C is a free chain complex. There is then a short exact sequence of chain complexes

$$0 \to Z \to C \to B \to 0$$

where $Z_q = Z_q(C)$ and $B_q = B_{q-1}(C)$. This sequence is split because B is free, and by theorem 5.4.8, there is an exact cohomology sequence

$$\cdots \to H^{q-1}(Z;G) \xrightarrow{\delta^*} H^q(B;G) \to H^q(C;G) \to H^q(Z;G) \xrightarrow{\delta^*} H^{q+1}(B;G) \to \cdots$$

Since Z and B have trivial boundary operators, $H^q(Z;G) = \operatorname{Hom}(Z_q(C),G)$ and $H^q(B;G) = \operatorname{Hom}(B_{q-1}(C),G)$. Furthermore, the homomorphism

$$\delta^*\colon H^q(Z;G) \to H^{q+1}(B;G)$$

equals $\operatorname{Hom}(\gamma_q,1)\colon \operatorname{Hom}(Z_q(C),G) \to \operatorname{Hom}(B_q(C),G)$, where $\gamma_q\colon B_q(C) \subset Z_q(C)$. Hence there is a functorial short exact sequence

$$0 \to \operatorname{coker}[\operatorname{Hom}(\gamma_{q-1},1)] \to H^q(C;G) \to \ker[\operatorname{Hom}(\gamma_q,1)] \to 0$$

To interpret the modules in the above sequence we have the short exact sequence

$$0 \to B_q(C) \xrightarrow{\gamma_q} Z_q(C) \to H_q(C) \to 0$$

which is a free presentation of $H_q(C)$. By the characteristic property of Ext, there is an exact sequence

$$0 \to \operatorname{Hom}(H_q(C),G) \to \operatorname{Hom}(Z_q(C),G) \xrightarrow{\operatorname{Hom}(\gamma_q,1)} \operatorname{Hom}(B_q(C),G) \to \operatorname{Ext}(H_q(C),G) \to 0$$

Therefore, $\ker[\operatorname{Hom}(\gamma_q,1)] \approx \operatorname{Hom}(H_q(C),G)$ and $\operatorname{coker}[\operatorname{Hom}(\gamma_q,1)] \approx \operatorname{Ext}(H_q(C),G)$. Substituting these into the short exact sequence containing $H^q(C;G)$ yields the desired short exact sequence

$$0 \to \operatorname{Ext}(H_{q-1}(C),G) \to H^q(C;G) \to \operatorname{Hom}(H_q(C),G) \to 0$$

with the homomorphism $H^q(C;G) \to \operatorname{Hom}(H_q(C),G)$ easily verified to equal h.

This sequence is functorial and is split (because the sequence of chain complexes

$$0 \to Z \to C \to B \to 0$$

is split).

For arbitrary C such that Ext (C,G) is acyclic, the result follows by using a free approximation to C (as in the proof of theorem 5.2.14) to reduce it to the case of a free complex. ∎

It follows from theorem 3 that if X is a path-connected topological space, then $H^0(X;R)$ is a cyclic R module generated by 1 [or, in other words, the augmentation map is an isomorphism $\eta\colon R \approx H^0(X;R)$]. From theorems 3 and 5.4.10, it follows that for any X, $H^0(X;G)$ is isomorphic to the direct product of as many copies of G as path components of X.

4 COROLLARY *If (X,A) is a topological pair such that $H_q(X,A;R)$ is finitely generated for all q, then the free submodules of $H^q(X,A;R)$ and $H_q(X,A;R)$ are isomorphic and the torsion submodules of $H^q(X,A;R)$ and $H_{q-1}(X,A;R)$ are isomorphic.*

PROOF Let $H_q(X,A;R) = F_q \oplus T_q$, where F_q is free and T_q is the torsion module of H_q. Then

$$\operatorname{Hom}(H_q(X,A;R),R) \approx \operatorname{Hom}(F_q,R) \oplus \operatorname{Hom}(T_q,R) \approx F_q$$

and by example 2,

$$\operatorname{Ext}(H_q(X,A;R),R) \approx \operatorname{Ext}(F_q,R) \oplus \operatorname{Ext}(T_q,R) \approx T_q$$

The result follows from theorem 3. ∎

For many purposes it would be more useful to have a formula expressing $H^*(C;G)$ in terms of $H^*(C;R)$. Such a formula can be proved in the case of C or G finitely generated. We begin by establishing some properties of finitely generated modules.

Let $\mu\colon \operatorname{Hom}(A,G) \otimes G' \to \operatorname{Hom}(A, G \otimes G')$ be the functorial homomorphism defined by $\mu(f \otimes g')(a) = f(a) \otimes g'$ for $f \in \operatorname{Hom}(A,G)$, $g' \in G'$, and $a \in A$.

5 LEMMA *If A is a free module and G' is finitely generated, then for any module G, μ is an isomorphism.*

PROOF The result is trivially true if $G' = R$. Because the tensor product and Hom functors both commute with finite direct sums, it is also true if G' is a finitely generated free module. G' is assumed to be finitely generated, so there is a short exact sequence

$$0 \to \bar{\bar{G}} \to \bar{G} \to G' \to 0$$

where $\bar{G}$ (hence also $\bar{\bar{G}}$) is a finitely generated free module. There is a commutative diagram

$$\begin{array}{ccccccc}
\operatorname{Hom}(A,G) \otimes \bar{\bar{G}} & \to & \operatorname{Hom}(A,G) \otimes \bar{G} & \to & \operatorname{Hom}(A,G) \otimes G' & \to & 0 \\
\downarrow{\scriptstyle \bar{\bar{\mu}}} & & \downarrow{\scriptstyle \bar{\mu}} & & \downarrow{\scriptstyle \mu} & & \\
\operatorname{Hom}(A, G \otimes \bar{\bar{G}}) & \to & \operatorname{Hom}(A, G \otimes \bar{G}) & \to & \operatorname{Hom}(A, G \otimes G') & \to & 0
\end{array}$$

with exact rows (exactness follows from corollary 5.1.6 and, for the bottom row, from the fact that A is free). Because $\bar{\bar{\mu}}$ and $\bar{\mu}$ are isomorphisms, it follows from the five lemma that μ is also an isomorphism. ∎

There is also a functorial homomorphism

$$\mu\colon \operatorname{Hom}(A,G) \otimes \operatorname{Hom}(B,G') \to \operatorname{Hom}(A \otimes B, G \otimes G')$$

defined by $\mu(f \otimes f')(a \otimes b) = f(a) \otimes f'(b)$ for $f \in \operatorname{Hom}(A,G)$, $f' \in \operatorname{Hom}(B,G')$, $a \in A$, and $b \in B$. In case $B = R$, Hom $(B,G') \approx G'$, and μ corresponds to the homomorphism in lemma 5.

6 LEMMA *If B is a finitely generated free module, for arbitrary modules A and G, μ is an isomorphism*

$$\mu\colon \operatorname{Hom}(A,G) \otimes \operatorname{Hom}(B,R) \approx \operatorname{Hom}(A \otimes B, G)$$

PROOF The result is trivially true for $B = R$ and follows for a finite sum of copies of R because both sides commute with finite direct sums. ∎

7 COROLLARY *If A and B are free modules and either A and B or B and G' are finitely generated, μ is an isomorphism*

$$\mu\colon \operatorname{Hom}(A,G) \otimes \operatorname{Hom}(B,G') \approx \operatorname{Hom}(A \otimes B, G \otimes G')$$

PROOF Since A and B are free, so is $A \otimes B$. If A and B are finitely generated, so is $A \otimes B$, and there is a commutative diagram

$$\begin{array}{ccc}
[\operatorname{Hom}(R,G) \otimes \operatorname{Hom}(A,R)] \otimes [\operatorname{Hom}(R,G') \otimes \operatorname{Hom}(B,R)] & \xrightarrow{\bar{\mu}} & \operatorname{Hom}(R, G \otimes G') \otimes \operatorname{Hom}(A \otimes B, R) \\
\downarrow{\scriptstyle \mu \otimes \mu} & & \downarrow{\scriptstyle \mu} \\
\operatorname{Hom}(A,G) \otimes \operatorname{Hom}(B,G') & \xrightarrow{\mu} & \operatorname{Hom}(A \otimes B, G \otimes G')
\end{array}$$

in which $\bar{\mu}((f_1 \otimes f_2) \otimes (f_3 \otimes f_4)) = \mu(f_1 \otimes f_3) \otimes \mu(f_2 \otimes f_4)$. By lemma 6, $\bar{\mu}$ is an isomorphism and so are both vertical maps. Therefore the bottom map is also an isomorphism.

If B and G' are finitely generated, there is a commutative diagram

$$\begin{array}{ccc}
\operatorname{Hom}(A,G) \otimes \operatorname{Hom}(B,R) \otimes G' & \xrightarrow{1 \otimes \mu} & \operatorname{Hom}(A,G) \otimes \operatorname{Hom}(B,G') \\
\downarrow{\scriptstyle \mu \otimes 1} & & \downarrow{\scriptstyle \mu} \\
\operatorname{Hom}(A \otimes B, G) \otimes G' & \xrightarrow{\mu} & \operatorname{Hom}(A \otimes B, G \otimes G')
\end{array}$$

By lemma 5, both horizontal maps are isomorphisms, and by lemma 6, the left-hand vertical map is an isomorphism. Therefore the right-hand map is also an isomorphism. ∎