Workshop on positivity

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Positivity preservers: theory and applications mini-course

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1. Entrywise positive maps

Fix a domain $I \subset \mathbb{C}$ and integers $m,\, n \geq 1$. Let $\mathcal{P}_n(I)$ denote the set of $n \times n$ Hermitian positive semidefinite matrices with all entries in I.

A function $f : I \to \mathbb{C}$ acts *entrywise* on a matrix

$$A = (a_{jk})_{1 \leq j \leq m,\ 1 \leq k \leq n} \in I^{m \times n}$$

by setting

$$f[A] := (f(a_{jk}))_{1 \leq j \leq m,\ 1 \leq k \leq n} \in \mathbb{C}^{m \times n}.$$

Below, we allow the dimensions m and n to vary, while keeping the uniform notation $f[-]$.

We also let $\mathbf{1}_{m \times n}$ denote the $m \times n$ matrix with each entry equal to one. Note that $\mathbf{1}_{n \times n} \in \mathcal{P}_n(\mathbb{R})$.

The main goal of the mini-course is to provide answers to the following question in various settings:

Which functions preserve positive semidefiniteness when applied entrywise to a class of positive matrices?

In other words, when is it true that $f[A] \in \mathcal{P}_n$ for all matrices $A \in \mathcal{S}$ for a given set $\mathcal{S} \subseteq \mathcal{P}_n$?

Remark 1.1. More generally, if $\phi : A \to B$ is a map between two C^*-algebras, then ϕ induces *amplified* maps $\phi^{(k)} : \mathcal{M}_k(A) \to \mathcal{M}_k(B)$, where

$$\phi^{(k)}((a_{ij})_{i,j=1}^n) = (\phi(a_{ij}))_{i,j=1}^n.$$

We say that ϕ is *k-positive* if the map $\phi^{(k)}$ is positive, and that ϕ is *completely positive* if ϕ is k-positive for all k. In this mini-course, we focus on the case where $\phi : \mathbb{R} \to \mathbb{R}$, but where ϕ is an arbitrary function (not necessarily linear).

1.1. The 2×2 case

Obtaining a useful description of the entrywise positive maps on $\mathcal{P}_n$ for a fixed value of n is a difficult problem that remains open. The 2×2 case is easy to settle though.

Theorem 1.2 (Vasudeva [12])**.** *Given a function $f : (0,\infty) \to \mathbb{R}$, the entrywise map $f[-]$ preserves positivity on $\mathcal{P}_2\big((0,\infty)\big)$ if and only f is non-negative, non-decreasing, and multiplicatively mid-convex:*

$$f\big(\sqrt{xy}\big)^2 \leq f(x)f(y) \qquad \text{for all } x, y > 0. \tag{1.1}$$

In particular, f is either identically zero or never zero on $(0,\infty)$, and f is also continuous.

Last modified: May 23, 2019. A large part of the material in these notes comes from the survey "A panorama of positivity" by Belton, Guillot, Khare, and Putinar arXiv:1812.05482 and from lecture notes from A. Khare http://www.math.iisc.ac.in/ khare/teaching/Math341-notes.pdf. Please consult these references for more details.

Proof. Suppose f preserves positivity on K_2. Then clearly f is non-negative and for $0 < x < y < \infty$, the matrix

$$A = \begin{pmatrix} y & x \\ x & y \end{pmatrix}$$

is positive semidefinite. It follows immediately that $0 \leq \det f[A] = f(y)^2 - f(x)^2$ and so f is non-decreasing. Similarly, the matrix

$$B = \begin{pmatrix} x & \sqrt{xy} \\ \sqrt{xy} & y \end{pmatrix}$$

is positive semidefinite, and so $0 \leq \det f[B] = f(x)f(y) - f(\sqrt{xy})^2$ from which Equation (3.2) follows.

Conversely, suppose f is non-negative, non-decreasing, and multiplicatively mid-convex. Let $a, b, c \in (0, \infty)$ and assume

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathcal{P}_2.$$

Then $b^2 \leq ac$ and so $b \leq \sqrt{ac}$. Since f is non-decreasing and multiplicatively mid-convex, we conclude that

$$f(b)^2 \leq f(\sqrt{ac})^2 \leq f(a)f(c).$$

We conclude that $f[M] \in \mathcal{P}_2$.

Next, suppose $f(x) = 0$ for some $x > 0$. We claim that $f \equiv 0$ on $I(0, \infty)$. To see the claim, first define $x_0 := \sup\{x > 0 \ : \ f(x) = 0\}$. For any $x \in (0, x_0)$, there exists $x_1 \in (x, x_0)$ such that $f(x_1) = 0$ by definition of x_0. But then $f(x) = 0$ since f is non-decreasing. Thus, f vanishes on $(0, x_0) \cap I$. We now produce a contradiction if $x_0 < \infty$. Indeed if $x_0 < y < \infty$, then choose any $x_1 \in (x_0^2/y, x_0)$. Thus, $\sqrt{x_1 y} \in (x_0, y) \subset I$, so by (3.2),

$$f(\sqrt{x_1 y})^2 \leq f(x_1)f(y) = 0.$$

This contradicts the definition of x_0, and proves the claim.

Finally, define $g(x) := \ln f(e^x)$. It is clear that g is nondecreasing and mid(point)-convex on $\mathbb{R}$, whenever f satisfies (3.2). Hence by [10, Theorem VII.C], g is necessarily continuous (and hence convex) on $\mathbb{R}$. We conclude that f is continuous on $(0, \infty)$. $\square$

Having characterized the functions f that preserve positivity on $\mathcal{P}_2$, can we find some examples of functions that do so on $\mathcal{P}_n$? The Schur product provide many more examples.

Definition 1.3. Let $A = (a_{ij}), B = (b_{ij})$ be two $n \times m$ matrices. The *Hadamard product* (or Schur product, or entrywise product) of A and B, denoted $A \circ B$ is the $n \times m$ matrix obtained by multiplying the matrices entry-by-entry, i.e.,

$$A \circ B = (a_{ij} b_{ij}).$$

An important property of the Hadamard product is that it preserves positive semidefiniteness.

Theorem 1.4 (Schur product theorem [11])**.** *Let $A, B \in \mathcal{P}_n$. Then $A \circ B \in \mathcal{P}_n$.*

Proof. Let $A = \sum_{i=1}^n \lambda_i u_i u_i^T$ and $B = \sum_{j=1}^n \mu_j v_j v_j^T$ be eigen-decompositions of A and B respectively. Then

$$A \circ B = \sum_{i,j=1}^n \lambda_i \mu_j (u_i u_i^T) \circ (v_j v_j^T) = \sum_{i,j=1}^n \lambda_i \mu_j (u_i \circ v_j)(u_i \circ v_j)^T \in \mathcal{P}_n,$$

where we used the fact that for any $x, y \in \mathbb{R}^n$,

$$(xx^T) \circ (yy^T) = (x_i x_j y_i y_j)_{i,j=1}^n = (x \circ y)(x \circ y)^T.$$

$\square$

As a consequence of the Schur product theorem, for any $A \in \mathcal{P}_n$, we have $A \circ A =: A^{\circ 2} \in \mathcal{P}_n$, $A \circ A \circ A =: A^{\circ 3}$, etc.. More generally, taking positive linear combinations and limits, we obtain the following result.

Corollary 1.5. *Let $f(z) = \sum_{i=0}^{\infty} c_i z^i$ with $c_i \geq 0$. Assume the series is convergent on the entries of $A \in \mathcal{P}_n$. Then $f[A] \in \mathcal{P}_n$.*

1.2. Necessary conditions: the Horn–Loewner theorem

Obviously, if $f : \mathbb{R} \to \mathbb{R}$ preserves positivity on $\mathcal{P}_n(\mathbb{R})$, then $f(x) \geq 0$ for all $x \geq 0$ (since psd matrices have non-negative diagonal entries). A much stronger necessary condition from R. Horn (and attributed to C. Loewner).

Theorem 1.6 (Horn [6])**.** *Let $f : (0, \infty) \to \mathbb{R}$ be continuous. Fix a positive integer n and suppose $f[-]$ preserves positivity on $\mathcal{P}_n\big((0, \infty)\big)$. Then $f \in C^{n-3}((0, \infty))$,*

$$f^{(k)}(x) \geq 0 \qquad \text{whenever } x \in (0, \infty) \text{ and } 0 \leq k \leq n-3,$$

and $f^{(n-3)}$ is a convex non-decreasing function on $(0, \infty)$. Furthermore, if $f \in C^{n-1}\big((0, \infty)\big)$, then $f^{(k)}(x) \geq 0$ whenever $x \in (0, \infty)$ and $0 \leq k \leq n-1$.

The proof of Theorem 1.6 is based on a clever determinant calculation. The key expression involves Vandermonde determinants. Recall that the *Vandermonde determinant* $V(\mathbf{u})$ associated to a vector $\mathbf{u} = (u_1, \ldots, u_n)^T$ is given by

$$V(\mathbf{u}) := \prod_{1 \leq j < k \leq n} (u_k - u_j) = \det \begin{pmatrix} 1 & u_1 & \cdots & u_1^{n-1} \\ 1 & u_2 & \cdots & u_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & u_n & \cdots & u_n^{n-1} \end{pmatrix}, \qquad \text{if } n > 1. \tag{1.2}$$

and $V(\mathbf{u}) = 1$ if $n = 1$.

Proposition 1.7. *Fix an integer $n > 0$ and define $N := \binom{n}{2}$. Suppose $a \in \mathbb{R}$ and let a function $f : (a - \epsilon, a + \epsilon) \to \mathbb{R}$ be N-times differentiable for some fixed $\epsilon > 0$. For fixed vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, define $\Delta : (-\epsilon', \epsilon') \to \mathbb{R}$ via:*

$$\Delta(t) := \det f[a\mathbf{1}_{n \times n} + t\mathbf{u}\mathbf{v}^T],$$

for a sufficiently small $\epsilon' \in (0, \epsilon)$. Then $\Delta(0) = \Delta'(0) = \cdots = \Delta^{(N-1)}(0) = 0$, and

$$\Delta^{(N)}(0) = \binom{N}{0, 1, \ldots, n-1} V(\mathbf{u}) V(\mathbf{v}) \prod_{k=0}^{n-1} f_\epsilon^{(k)}(a), \tag{1.3}$$

where the first factor on the right is a multinomial coefficient.

Before proving Proposition 1.7, we will show how it implies Theorem 1.6.

Proof of Theorem 1.6. The main idea of the proof is to use Equation (1.3) with $\mathbf{u} = \mathbf{v}$, and the fact that

$$\lim_{t\to 0^+} \frac{\Delta(t)}{t^N} = \frac{\Delta^{(N)}(0)}{N!} \geq 0$$

to obtain information on the derivatives of f when they exist. That at least $n-3$ derivatives must exist is non-trivial though, and requires other arguments.

Case 1. Smooth case. Let us first consider the case where f is smooth. We proceed by induction. The result is obvious when $n = 1$. Let us assume it holds for matrices of dimension up to $n-1$. Hence, $f, f', \dots, f^{(n-2)}$ are non-negative. For $\epsilon > 0$, define,

$$f_\epsilon(x) := f(x) + \epsilon x^n.$$

Observe that f_ϵ preserves positivity on $\mathcal{P}_n$ by the Schur product theorem. Let us fix $u_0 \in (0,1)$ and apply Proposition 1.7 to f_ϵ with the vectors $\mathbf{u} = \mathbf{v} = (1, u_0, u_0^2, \dots, u_0^{n-1}) \in \mathbb{R}^n$. By assumption, the function

$$\Delta(t) = \det f_\epsilon[a\mathbf{1}_{n\times n} + t\mathbf{u}\mathbf{u}^T]$$

satisfies $\Delta(t) \geq 0$ for all $t > 0$. Hence

$$0 \leq \lim_{t\to 0^+} \frac{\Delta(t)}{t^N}, \qquad \text{where } N = \binom{n}{2}.$$

On the other hand, by Proposition 1.6 and de l'Hôpital's rule,

$$\lim_{t\to 0^+} \frac{\Delta(t)}{t^N} = \frac{\Delta^{(N)}(0)}{N!} = \frac{1}{N!}\binom{N}{0, 1, \dots, n-1} V(\mathbf{u})^2 \prod_{k=0}^{n-1} f_\epsilon^{(k)}(a).$$

Since $V(\mathbf{u})^2 > 0$, we conclude that

$$\prod_{k=0}^{n-1} f_\epsilon^{(k)}(a) \geq 0.$$

Now, $f_\epsilon^{(k)}(a) = f^{(k)}(a) + \epsilon n(n-1)\dots(n-k+1)a^{n-k}$. By the induction hypothesis, these derivatives are non-negative for $k = 0, 1, \dots, n-2$. We conclude that

$$f_\epsilon^{(n-1)}(a) = f^{(n-1)}(a) + \epsilon n! \geq 0 \qquad \forall \epsilon, a > 0.$$

Letting $\epsilon \to 0^+$, we conclude that $f^{(n-1)}$ is non-negative on $(0, \infty)$, as desired. This proves the result in the case where f admits at least $N = \binom{n}{2}$ derivatives.

Case 2. Non-smooth case. The non-smooth case is proved via a *mollifier* argument.

2a. Mollifiers. Let $\phi \in C^\infty(\mathbb{R})$ be a probability distribution with compact support in $(-1, 0)$. For $\delta > 0$, define

$$f_\delta(x) := \frac{1}{\delta}\int_{\mathbb{R}} f(x-u)\phi\left(\frac{u}{\delta}\right)\, du = \int_{-\delta}^{0} f(x-u)\phi\left(\frac{u}{\delta}\right)\, \frac{du}{\delta}.$$

One can show that the family of functions f_δ satisfies:

1. For any $\delta > 0$, the function f_δ is smooth on $(0, \infty)$.

2. $f_\delta \to f$ uniformly on every compact subset of $(0, \infty)$.

Observe that f_δ preserves positivity on $\mathcal{P}(0,\infty)$ since positive linear combinations and limits of psd matrices are psd. We therefore conclude by the smooth case that $f_\delta, f_\delta', \dots, f_\delta^{(n-1)}$ are all non-negative on $(0,\infty)$. Now, we can't simply let $\delta \to 0^+$ since f may not admit derivatives. Instead, we work with a discrete version of the derivative: divided differences.

2b. Divided differences.

Definition 1.8. Given $h > 0$ and a positive integer k, the k-th order *forward differences* with step size $h > 0$ are defined as follows:

$$(\Delta_h^0 f)(x) := f(x), \qquad (\Delta_h^k f)(x) = (\Delta_h^{k-1} f)(x+h) - (\Delta_h^{k-1} f)(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} f(x+jh).$$

Similarly, the k-th order *divided differences* with step size $h > 0$ are given by

$$(D_h^k f)(x) = \frac{1}{h^k} (\Delta_h^k f)(x).$$

2c. Mean-value theorem and Boas–Wieder. To conclude the proof, we need the following result.

Theorem 1.9. *Let $I \subseteq \mathbb{R}$ be a bounded interval and let $f : I \to \mathbb{R}$. Then*

1. *(Mean-value theorem for divided differences) If f is k times differentiable on I and $x, x+kh \in I$ for some $h > 0$, then there exists $y \in (x, x+kh)$ such that*

$$(D_h^k f)(x) = f^{(k)}(y)/k!.$$

2. *(Boas–Widder, Duke Math. J., 1940.) Suppose $k \geq 2$ is an integer, and $f : I \to \mathbb{R}$ is continuous and has all forward differences of order k non-negative on I:*

$$(\Delta_h^k f)(x) \geq 0 \qquad \textit{whenever } h > 0 \textit{ and } x, x+kh \in I.$$

Then on all of I, the function $f^{(k-2)}$ exists, is continuous and convex, and has non-decreasing left and right hand derivatives.

We can now conclude the proof of Theorem 1.6. Recall that $f_\delta, f_\delta', \dots, f_\delta^{(n-1)}$, where f_δ is the mollified version of f. By Theorem 1.9(1), the forward differences if f_δ of order $k = 0, 1, \dots, n-1$ are non-negative on $(0,\infty)$. The same holds for f since $f_\delta(x) \to f(x)$ for all $x \in (0,\infty)$. By Theorem 1.9(2), it now follows that f is C^{n-3} on $(0,\infty)$ and that $f^{(n-3)}$ is convex and non-decreasing on $(0,\infty)$. This conclude the proof of the theorem. $\square$

We now return to Proposition 1.7. The following lemma will be useful in the proof.

Lemma 1.10. *Let $I \subseteq \mathbb{R}$ be an interval and let $A : I \to \mathbb{R}^{n\times n}$ be a matrix valued function with columns $A_1(t), A_2(t), \dots, A_n(t)$, i.e.,*

$$A(t) = \begin{pmatrix} | & | & & | \\ A_1(t) & A_2(t) & \dots & A_n(t) \\ | & | & & | \end{pmatrix}.$$

and let $\Delta(t) := \det A(t)$. Then

$$\Delta'(t) = \sum_{j=1}^{n} \det \hat{A}_j(t),$$

where

$$\hat{A}_j := \begin{pmatrix} | & & | & | & | & & | \\ A_1(t) & \dots & A_{j-1}(t) & A_j'(t) & A_{j+1}(t) & \dots & A_n(t) \\ | & & | & | & | & & | \end{pmatrix},$$

and where $A_j'(t)$ denotes the entrywise derivative with respect to t of the column A_j.

Proof. By the Laplace expansion formula, for a matrix $A = (a_{ij})$,

$$\det A = \sum_{i=1}^{n} a_{ij} A_{ij},$$

where A_{ij} denotes the (i,j)-th cofactor of A. Hence,

$$\frac{\partial A}{\partial a_{ij}} = A_{ij}.$$

Now, in our case, each entry a_{ij} is a function of $f_{ij}(t)$. Using the chain rule, we obtain

$$\Delta'(t) = \sum_{i,j=1}^{n} \frac{\partial A(t)}{\partial a_{ij}} f_{ij}'(t) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} f_{ij}'(t) A_{ij}(t) \right) = \sum_{j=1}^{n} \hat{A}_j(t),$$

where the last equality follows by another application of Laplace's expansion formula. □

We now prove Proposition 1.7.

Proof of Proposition 1.7. Let $\mathbf{w}_j$ denote the jth column of $a\mathbf{1}_{n\times n} + t\mathbf{u}\mathbf{v}^T$; thus $\mathbf{w}_j$ has ith entry $a + tu_iv_j$. To differentiate $\Delta(t)$, we apply Lemma 1.10 repeatedly to

$$A(t) = \begin{pmatrix} | & | & & | \\ f[\mathbf{w}_1] & f[\mathbf{w}_2] & \dots & f[\mathbf{w}_n] \\ | & | & & | \end{pmatrix}.$$

We obtain:

$$\Delta^{(k)}(0) = \sum_{\substack{m_1,m_2,\dots,m_n \geq 0 \\ m_1+m_2+\dots+m_n=k}} \frac{k!}{m_1!m_2!\dots m_n!} \det \hat{A}_{m_1,m_2,\dots,m_n}(0), \tag{1.4}$$

where $\hat{A}_{m_1,m_2,\dots,m_n}(t)$ denotes the matrix whose j-th column is equal to

$$\frac{d^{m_j}}{dt^{m_j}} f[\mathbf{w}_j] = v_j^{m_j} \mathbf{u}^{\circ m_j} \circ f^{(m_j)}[\mathbf{w}_j].$$

In particular, at $t = 0$,

$$\frac{d^{m_j}}{dt^{m_j}} f[\mathbf{w}_j](0) = v_j^{m_j} \mathbf{u}^{\circ m_j} \circ f^{(m_j)}[a\mathbf{1}_{n\times 1}].$$

Thus,

$$\hat{A}_{m_1,m_2,\dots,m_n}(0) = \begin{pmatrix} v_1^{m_1} f^{(m_1)}(a)\mathbf{u}^{\circ m_1} & v_2^{m_2} f^{(m_2)}(a)\mathbf{u}^{\circ m_2} & \dots & v_n^{m_n} f^{(m_n)}(a)\mathbf{u}^{\circ m_n} \end{pmatrix}. \tag{1.5}$$

Notice that if any $m_j = m_k$ for $j \neq k$ then $\hat{A}_{m_1,m_2,\dots,m_n}(0)$ vanishes. Thus, the lowest degree derivative $\Delta^{(m)}(0)$ whose expansion contains a non-vanishing determinant is when $m = 0 + 1 + \dots + (n-1) = N = \binom{n}{2}$. This proves the first part of the result.

Now, consider $\Delta^{(N)}(0)$. Using the above observation, we obtain

$$\Delta^{(N)}(0) = \binom{N}{0, 1, \ldots, n-1} \sum_{\sigma \in S_n} \det \hat{A}_{\sigma_1 - 1, \sigma_2 - 1, \ldots, \sigma_n - 1}(0).$$

By Equation (1.5), we have

$$\det \hat{A}_{\sigma_1 - 1, \sigma_2 - 1, \ldots, \sigma_n - 1}(0) = \operatorname{sgn}(\sigma) \det(\mathbf{u}^{\circ 0} | \mathbf{u}^{\circ 1} | \ldots | \mathbf{u}^{\circ (n-1)}) \prod_{j=0}^{n-1} f^{(j)}(a) v_j^{\sigma_j - 1}.$$

Hence,

$$\begin{aligned}
\Delta^{(N)}(0) &= \binom{N}{0, 1, \ldots, n-1} V(\mathbf{u}) \prod_{j=0}^{n-1} f^{(j)}(a) \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{j=0}^{n-1} v_j^{\sigma_j - 1} \\
&= \binom{N}{0, 1, \ldots, n-1} V(\mathbf{u}) V(\mathbf{v}) \prod_{j=0}^{n-1} f^{(j)}(a).
\end{aligned}$$

$\square$

2. Polynomials preserving positivity

The Horn–Loewner theorem (Theorem 1.6) provides a necessary condition for preserving positivity. Unfortunately, a useful characterization of the full class of functions that preserve positivity on $\mathcal{P}_N$ for a fixed value of N remains out of reach as of today. A natural subclass of functions to examine is polynomials. In this lecture, we will provide a characterization of polynomials of degree at most N that preserve positivity on $\mathcal{P}_N$.

Let $p(z) = c_0 + c_1 z + \cdots + c_N z^N$ and assume $f[A] \in \mathcal{P}_N$ for all $A \in \mathcal{P}((0,\infty))$. Observe that:

1. By the Loewner–Horn theorem 1.6, the first $N-1$ coefficients $c_0, c_1, \dots, c_{N-1}$ of p have to be non-negative. In particular polynomials of degree $< N$ preserve positivity if and only if all their coefficients are non-negative.

2. It suffices to determine how negative can c_N be for given values of $c_0, c_1, \dots, c_{N-1}$. Note that, a priori, it is not even clear if c_N can be negative...

The following theorem provides the sharp bound on how negative can a_N be.

Theorem 2.1 (Belton-Guillot-Khare-Putinar [1]). *Fix $\rho > 0$ and integers $N \geq 1$, $M \geq 0$ and let $f(z) = \sum_{j=0}^{N-1} c_j z^j + c' z^M$ be a polynomial with real coefficients. Also denote by $\overline{D}(0,\rho)$ the closed disc in $\mathbb{C}$ with radius $\rho > 0$ and center the origin. For any vector $\mathbf{d} := (d_0, \dots, d_{N-1})$ with non-zero entries, let*

$$\mathcal{C}(\mathbf{d}) = \mathcal{C}(\mathbf{d}; z^M; N, \rho) := \sum_{j=0}^{N-1} \binom{M}{j}^2 \binom{M-j-1}{N-j-1}^2 \frac{\rho^{M-j}}{d_j}, \tag{2.1}$$

and let $\mathbf{c} := (c_0, \dots, c_{N-1})$. The following are equivalent.

1. *$f[-]$ preserves positivity on $\mathcal{P}_N(\overline{D}(0,\rho))$.*

2. *The coefficients c_j satisfy either $c_0, \dots, c_{N-1}, c' \geq 0$, or $c_0, \dots, c_{N-1} > 0$ and $c' \geq -\mathcal{C}(\mathbf{c})^{-1}$.*

3. *$f[-]$ preserves positivity on $\mathcal{P}_N^1((0,\rho))$, the set of matrices in $\mathcal{P}_N((0,\rho))$ having rank at most 1.*

In particular, observe that there exists polynomials preserving positivity on $\mathcal{P}_N$ and with $c_N < 0$. The theorem also provides examples of polynomials that preserve positivity on $\mathcal{P}_N$ but not on $\mathcal{P}_{N+1}$. In the special case $M = N$, Theorem 2.1 provides an exact description of the coefficients of polynomials of degree N that preserve positivity on $\mathcal{P}_N$. Note the surprising fact that preserving positivity on rank 1 matrices with entries in $(0,\rho)$ immediately implies preserving positivity on all of $\mathcal{P}_N(\overline{D}(0,\rho))$.

2.1. Schur polynomials

The proof of Theorem 2.1 reveals new exciting connections between positivity preservers and the theory of symmetric functions. To elaborate, we need a few notions from algebra.

Definition 2.2. A **partition** λ of an integer n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ such that

1. The terms are weakly decreasing, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$;

2. $\lambda_1 + \lambda_2 + \cdots + \lambda_N = n$.

For example, there are seven possible partitions of 5:

$$5,\ 41,\ 32,\ 311,\ 221,\ 2111,\ 11111.$$

Associated to every partition $\lambda = (\lambda_1, \dots \lambda_N)$ is a $N \times N$ determinant:

$$a_\lambda := \det(x_i^{\lambda_j}).$$

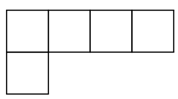
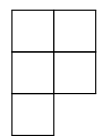
In particular, if $\delta := (N-1, N-2, \dots, 1, 0)$, then a_δ is the Vandermonde determinant $V(\mathbf{x}) = \prod_{1\leq j<k\leq N}(x_k - x_j)$.

Definition 2.3. The Schur polynomial $s_\lambda : \mathbb{R}^N \to \mathbb{R}$ is defined as (the unique polynomial extension to $\mathbb{R}^N$ of)

$$s_\lambda := \frac{a_{\lambda+\delta}}{a_\delta}.$$

Schur polynomials also have a nice combinatorial definition. Partitions are represented using **Young diagrams**: a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths in non-increasing order.

Example:

 corresponds to $4+1$;  corresponds to $2+2+1$.

A *Young tableau* is obtained by filling the boxes of a Young diagram with symbols taken from some alphabet.

Definition 2.4. A *semi-standard* (or column strict) Young tableau with shape $\lambda = (\lambda_1, \dots, \lambda_N)$ and cell entries $1, 2, \dots, m$ is obtained by filling the Young diagram corresponding to λ using only integers in $\{1, \dots, m\}$ in such a way that (1) the numbers in each row are non-decreasing, and (2) the numbers in each column are strictly increasing.

One can show that

$$s_\lambda(x_1, \dots, x_N) = s_\lambda(\mathbf{x}) := \sum_T \mathbf{x}^T = \sum_T x_1^{t_1} \dots x_N^{t_N}, \tag{2.2}$$

where the summation is over all semistandard Young tableaux T of shape λ with entries in $\{1, \dots, N\}$. Each Schur polynomial s_λ is a homogeneous symmetric polynomial with integer coefficients. Schur polynomials form a basis of the space of homogeneous symmetric polynomials, and may be interpreted as characters of irreducible polynomials representations of the Lie group $GL_n(\mathbb{C})$ [9]. At the heart of Theorem 2.1 is a novel identity for symmetric functions over any field $\mathbb{F}$.

Theorem 2.5 (Belton–Guillot–Khare–Putinar, *Adv. Math.* [1])**.** *Fix integers $k \geq 0$ and $M \geq N \geq 1$. Given a variable t and scalars $c_0, \dots, c_{N-1} \in \mathbb{F}^\times$, let the polynomial*

$$p_t(x) := t(c_0 x^k + \dots + c_{N-1}x^{k+N-1}) - x^{k+M}.$$

Also define the hook partition $\mu(M, N, j) := (M-N+1, 1, \dots, 1, 0, \dots, 0)$, where $N-j-1$ entries after the first are 1 and the remaining j entries are 0. Then for all $\mathbf{u} = (u_1, \dots, u_N)^T, \mathbf{v} := (v_1, \dots, v_N)^T \in \mathbb{F}^N$:

$$\det p_t[\mathbf{u}\mathbf{v}^T] = t^{N-1} V(\mathbf{u}) V(\mathbf{v}) \prod_{j=1}^N c_{j-1} u_j^k v_j^k \cdot \Big(t - \sum_{j=0}^{N-1} \frac{s_{\mu(M,N,j)}(\mathbf{u}) s_{\mu(M,N,j)}(\mathbf{v})}{c_j}\Big). \tag{2.3}$$

Let us see how the constant $\mathcal{C}(\mathbf{d}; z^M; N, \rho)$ arises from Proposition 2.5.

Sketch of proof of the (3) $\implies$ (2) *implication in Theorem 2.1.* To see how the constant $\mathcal{C}(\mathbf{c}; z^M, N, \rho)$ in Theorem 2.1 arises from Theorem 2.5, we examine the case where $A = \mathbf{u}\mathbf{u}^T$, i.e., the (3) $\implies$ (2) implication in Theorem 2.1. Assume $c_M < 0 < c_0, \ldots, c_{N-1}$. Define $p_t(z)$ as in Theorem 2.5, and set $t := |c_M|^{-1}$. Suppose $p_t[A] \in \mathcal{P}_N$ for all $A = \mathbf{u}\mathbf{u}^T \in \mathcal{P}_N^1((0,\rho))$. By Equation (2.3),

$$0 \leq \det p_t[\mathbf{u}\mathbf{u}^T] = t^{N-1} V(\mathbf{u})^2 c_0 \cdots c_{N-1} \Big(t - \sum_{j=0}^{N-1} \frac{s_{\mu(M,N,j)}(\mathbf{u})^2}{c_j} \Big). \tag{2.4}$$

Set $u_k := \sqrt{\rho}(1 - t'\epsilon_k)$, with pairwise distinct $\epsilon_k \in (0,1)$, and $t' \in (0,1)$. Thus, $V(\mathbf{u}) \neq 0$. Taking the limit as $t' \to 0^+$, since the final term in (2.4) must be non-negative, it follows by the monomial positivity of Schur polynomials (2.2) that

$$t = |c_M|^{-1} \geq \sum_{j=0}^{N-1} \frac{s_{\mu(M,N,j)}(\sqrt{\rho}, \ldots, \sqrt{\rho})^2}{c_j} = \sum_{j=0}^{N-1} s_{\mu(M,N,j)}(1, \ldots, 1)^2 \frac{\rho^{M-j}}{c_j} = \mathcal{C}(\mathbf{c}; z^M; N, \rho). \tag{2.5}$$

That the condition is also sufficient to preserve positivity on all of $\mathcal{P}_N(\overline{D}(0,\rho))$ is highly non-trivial. The reader is referred to [1] for the details. $\square$

2.2. Determinant identities for entrywise powers

The proof of of Proposition 2.5 depends on the following application of the Cauchy–Binet formula.

Proposition 2.6. *Let $A := \mathbf{u}\mathbf{v}^T$, where $\mathbf{u} = (u_1, \ldots, u_N)^T$ and $\mathbf{v} := (v_1, \ldots, v_N)^T \in \mathbb{F}^N$ for $N \geq 1$. Given m-tuples of non-negative integers $\mathbf{n} = (n_m > n_{m-1} > \cdots > n_1)$ and scalars $(c_{n_1}, \ldots, c_{n_m}) \in \mathbb{F}^m$, the following determinantal identity holds:*

$$\det \sum_{j=1}^m c_{n_j} A^{\circ n_j} = V(\mathbf{u})V(\mathbf{v}) \sum_{\mathbf{n}' \subset \mathbf{n},\ |\mathbf{n}'| = N} s_{\lambda(\mathbf{n}')}(\mathbf{u}) s_{\lambda(\mathbf{n}')}(\mathbf{v}) \prod_{k=1}^N c_{n'_k}. \tag{2.6}$$

Here, $\lambda(\mathbf{n}') := (n'_N - N + 1 \geq n'_{N-1} - N + 2 \geq \cdots \geq n'_1)$ is obtained by subtracting the staircase partition $(N-1, \ldots, 0)$ from $\mathbf{n}' := (n'_N > \cdots > n'_1)$, and the sum is over all subsets $\mathbf{n}'$ of cardinality N. In particular, if $m < N$ then the determinant is zero.

Proof. If there are $m < N$ summands then the matrix in question has rank at most $m < N$, so it is singular; henceforth we suppose $m \geq N$. Note first that if $\mathbf{c} := (c_{n_1}, \ldots, c_{n_m})$ and

$$X(\mathbf{u}, \mathbf{n}, \mathbf{c}) := (\sqrt{c_{n_k}} u_j^{n_k})_{1 \leq j \leq N, 1 \leq k \leq m}$$

where we work over an algebraic closure of $\mathbb{F}$, then

$$\sum_{j=1}^m c_{n_j} A^{\circ n_j} = X(\mathbf{u}, \mathbf{n}, \mathbf{c}) X(\mathbf{v}, \mathbf{n}, \mathbf{c})^T. \tag{2.7}$$

Next, let $\mathbf{c}|_{\mathbf{n}'} := (c_{n'_1}, \ldots, c_{n'_N})$ and note that, by the Cauchy–Binet formula applied to (2.7),

$$\begin{aligned} \det \sum_{j=1}^m c_{n_j} A^{\circ n_j} &= \sum_{\mathbf{n}' \subset \mathbf{n},\ |\mathbf{n}'| = N} \det\big(X(\mathbf{u}, \mathbf{n}', \mathbf{c}|_{\mathbf{n}'}) X(\mathbf{v}, \mathbf{n}', \mathbf{c}|_{\mathbf{n}'})^T \big) \\ &= \sum_{\mathbf{n}' \subset \mathbf{n},\ |\mathbf{n}'| = N} \det X(\mathbf{u}, \mathbf{n}', \mathbf{c}|_{\mathbf{n}'}) \det X(\mathbf{v}, \mathbf{n}', \mathbf{c}|_{\mathbf{n}'}) \\ &= \sum_{\mathbf{n}' \subset \mathbf{n},\ |\mathbf{n}'| = N} \det(u_j^{n'_k}) \det(v_j^{n'_k}) \prod_{k=1}^N c_{n'_k}. \end{aligned}$$

Each of the last two determinants is precisely the product of the appropriate Vandermonde determinant times the Schur polynomial corresponding to $\lambda(\mathbf{n}')$. This observation completes the proof. $\square$

We can now prove Proposition 2.6.

Proof of Proposition 2.6. Proof. Let $A = \mathbf{u}\mathbf{v}^T$ and note first that $\det(A^{\circ R} \circ B) = \prod_{j=1}^N u_j^R v_j^R \cdot \det B$ for any $N \times N$ matrix B, so it suffices to prove the result when $R = 0$, which we assume from now on.

Recall the Laplace formula: if B and C are $N \times N$ matrices, then

$$\det(B + C) = \sum_{\mathbf{n} \subset \{1,\dots,N\}} \det M_{\mathbf{n}}(B; C), \tag{2.8}$$

where $M_{\mathbf{n}}(B; C)$ is the matrix formed by replacing the rows of B labelled by elements of $\mathbf{n}$ with the corresponding rows of C. In particular, if $B = \sum_{j=0}^{N-1} c_j A^{\circ j}$ then

$$\det p_t[A] = \det(tB - A^{\circ M}) = t^N \det B - t^{N-1} \sum_{j=1}^{N} \det M_{\{j\}}(B; A^{\circ M}), \tag{2.9}$$

since the determinants in the remaining terms contain two rows of the rank-one matrix $A^{\circ M}$. By Proposition 2.6 applied with $n_j = j - 1$, we obtain

$$\det B = V(\mathbf{u})V(\mathbf{v})c_0 \cdots c_{N-1}.$$

To compute the coefficient of t^{N-1}, note that taking $t = 1$ in Equation (2.9) gives that

$$\sum_{j=1}^{N} \det M_{\{j\}}(B; A^{\circ M}) = \det B - \det p_1[A].$$

Moreover, $\det p_1[A]$ can be computed using Proposition 2.6 with $m = N + 1$ and $c_{n_{N+1}} = -1$:

$$\det p_1[A] = \det B - V(\mathbf{u})V(\mathbf{v})c_0 \cdots c_{N-1} \sum_{j=0}^{N-1} \frac{s_{\mu(M,N,j)}(\mathbf{u}) s_{\mu(M,N,j)}(\mathbf{v})}{c_j},$$

since $\mu(M, N, j) = \lambda\big((M, N-1, N-2, \dots, j+1, \widehat{j}, j-1, \dots, 0)\big)$ for $0 \le j \le N-1$. The identity (2.3) now follows. $\square$

$\square$

3. Sparsity constraints

The problem of characterizing positive entrywise maps can naturally be put in a much more general context, where matrices have additional structures of zeros.

3.1. The cones $\mathcal{P}_G$

Given $I \subset \mathbb{R}$ and a graph $G = (V, E)$ on the finite vertex set $V = \{1, \dots, N\}$, we define the cone of positive-semidefinite matrices with *zeros according to* G:

$$\mathcal{P}_G(I) := \{A = (a_{jk}) \in \mathcal{P}_N(I) : a_{jk} = 0 \text{ if } (j,k) \notin E \text{ and } i \neq j\}. \tag{3.1}$$

Note that if $(j, k) \in E$, then the entry a_{jk} is unconstrained; in particular, it is allowed to be 0. Consequently, the cone $\mathcal{P}_G := \mathcal{P}_G(\mathbb{R})$ is a closed subset of $\mathcal{P}_N$. Of course, when $G = K_N$, the complete graph on N vertices, the cone $\mathcal{P}_G$ reduces to $\mathcal{P}_N$.

Very little is known about functions preserving positivity on $\mathcal{P}_G$ for a given graph G.

Definition 3.1. Given a function $f : \mathbb{R} \to \mathbb{R}$ and $A \in \mathcal{S}_{|G|}(\mathbb{R})$, denote by $f_G[A]$ the matrix such that

$$f_G[A]_{jk} := \begin{cases} f(a_{jk}) & \text{if } (j,k) \in E \text{ or } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

The next proposition shows that, in fact, the notation f_G is not really necessary since if $f_G[-]$ preserves positivity on a non-complete connected graph with a least 3 vertices, then $f(0) = 0$.

Proposition 3.2 (see Guillot–Khare–Rajaratnam [5, Proposition 3.5])**.** *Let $0 \in I \subseteq \mathbb{R}$ be any interval. Let G be any non-complete connected graph on at least 3 vertices and let $f : I \to \mathbb{R}$. Suppose $f_G[A] \in \mathcal{P}_{|G|}$ for any $A \in \mathcal{P}_G(I)$. Then $f(0) = 0$.*

Proof. Without loss of generality, assume the vertices of $G = (V, E)$ are labeled by $V = \{1, 2, \dots, n\}$, that $(1, 2), (2, 3) \in E$, and that $(1, 3) \notin E$. Let $A = \mathbf{0}_{|G|\times|G|} \in \mathcal{P}_G(I)$. Then $f_G[A]$ contains the following matrix as a principal submatrix

$$B = f(0) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Since B is a principal matrix of $f_G[A]$, it has to be in $\mathcal{P}_{A_3}$. A simple calculation shows that this is the case if and only if $f(0) = 0$. □

The first main result in [5] is an explicit characterization of the entrywise positive preservers of $\mathcal{P}_G$ for any collection of trees (other than copies of K_2). Following Vasudeva's classification for $\mathcal{P}_{K_2}$ in Theorem 1.2, trees are the only other graphs for which such a classification is currently known.

Theorem 3.3 (Guillot–Khare–Rajaratnam [5, Theorem A])**.** *Suppose $I = [0, R)$ for some $0 < R \leq \infty$, and $f : I \to \mathbb{R}_+$. Let G be a tree with at least 3 vertices, and let A_3 denote the path graph on 3 vertices. The following are equivalent.*

1. *$f_G[A] \in \mathcal{P}_G$ for every $A \in \mathcal{P}_G(I)$;*
2. *$f_T[A] \in \mathcal{P}_T$ for all trees T and all matrices $A \in \mathcal{P}_T(I)$;*

3. $f_{A_3}[A] \in \mathcal{P}_{A_3}$ *for every* $A \in \mathcal{P}_{A_3}(I)$*;*

4. The function f *satisfies*

$$f\big(\sqrt{xy}\big)^2 \leq f(x)f(y) \qquad \text{for all } x, y \in I \tag{3.2}$$

and is super-additive on I*, that is,*

$$f(x+y) \geq f(x) + f(y) \qquad \text{whenever } x, y, x+y \in I. \tag{3.3}$$

In order to prove Theorem 3.3, we first examine the case of star graphs. A key tool the will be needed to analyze the positivity of matrices later is the notion of the *Schur complement* of a matrix.

Definition 3.4. Let M be a matrix written in block form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.4}$$

where $A = A_{m\times m}$, $B = B_{m\times n}$, $C = C_{n\times m}$, and $D = D_{n\times n}$. Assuming D is invertible, we define the *Schur complement* of D in M to be

$$M/D := A - BD^{-1}C.$$

The Schur complement of a matrix has important properties.

Proposition 3.5. *Let* M *be a block matrix as in Equation* (3.4)*, with* D *invertible. Then*

1. $\det M = \det D \cdot \det(M/D)$.

2. $M \in \mathcal{P}_{n+m}$ *if and only if* $D \in \mathcal{P}_n$ *and* $M/D \in \mathcal{P}_m$.

Proof. Both results follow immediately from the factorization:

$$M = \begin{pmatrix} I_m & BD^{-1} \\ 0 & I_n \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_m & 0 \\ D^{-1}C & I_n \end{pmatrix}.$$

□

3.2. Preserving positivity on star graphs

Recall that a star graph has $d+1$ vertices for some $d \geq 0$, d edges, and a unique vertex of degree d. The following result characterizes positive semidefinite matrices with zeros according to a star. Note that every nonempty graph contains a star subgraph, so the result yields useful information about $\mathcal{P}_G$ for all nonempty G, and will be crucial in proving Theorem 3.3.

Proposition 3.6. *Suppose* $d \geq 0$ *and*

$$A = \left(\begin{array}{c|ccc} p_1 & \alpha_2 & \cdots & \alpha_{d+1} \\ \hline \alpha_2 & p_2 & & 0 \\ \vdots & & \ddots & \\ \alpha_{d+1} & 0 & & p_{d+1} \end{array}\right) \tag{3.5}$$

is a real-valued symmetric matrix with zeros according to a star graph. Then A *is positive semidefinite if and only if the following three conditions hold:*

1. $p_i \geq 0$ *for all* $1 \leq i \leq d+1$;
2. *for all* $2 \leq i \leq d+1$, $p_i = 0 \implies \alpha_i = 0$;
3. $p_1 \geq \displaystyle\sum_{\{i>1\,:\,p_i \neq 0\}} \alpha_i^2/p_i$.

Proof. The result follows easily by examining the Schur complement of the lower right block in A, and applying Proposition 3.5(2). $\square$

Proof of Theorem 3.3. Clearly (2) $\Rightarrow$ (1) $\Rightarrow$ (3). We now prove that (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2).

(3) $\Rightarrow$ (4). If $f \equiv 0$ on I then the result is obvious. Now assume $f_{A_3}[A] \in \mathcal{P}_{A_3}$ for every $A \in \mathcal{P}_{A_3}(I)$. In particular, $f[A] \in \mathcal{P}_{K_2}$ for every $A \in \mathcal{P}_{K_2}(I)$. Therefore, by Theorem 1.2, f satisfies (3.2) on I. Now consider the matrix A in Equation (3.5) for $d = 2$. By Proposition 3.6, for $0 < p_i, \alpha_i \in I$, we have $A \in \mathcal{P}_{A_3}(I)$ if and only if $p_1 \geq \alpha_2^2/p_2 + \alpha_3^2/p_3$. Now suppose $0 < \alpha_2, \alpha_3, \alpha_2 + \alpha_3 \in I$; then $f(\alpha_2), f(\alpha_3) > 0$ by Theorem 1.2. Now, by Proposition 3.6, we have

$$B = \begin{pmatrix} \alpha_2 + \alpha_3 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_2 & 0 \\ \alpha_3 & 0 & \alpha_3 \end{pmatrix} \in \mathcal{P}_{A_3}.$$

Hence, $f_{A_3}[B] \in \mathcal{P}_{A_3}$ and it follows by Proposition 3.6 that

$$f(p_1) = f(\alpha_2 + \alpha_3) \geq \frac{f(\alpha_2)^2}{f(p_2)} + \frac{f(\alpha_3)^2}{f(p_3)} = f(\alpha_2) + f(\alpha_3).$$

This proves f is superadditive on $(0, R)$. The case when α_2 or α_3 is zero follows from Proposition 3.2.

(4) $\Rightarrow$ (2).

Once again, if $f \equiv 0$ on I then the result is immediate. Now suppose f is superadditive, not identically zero on I, and satisfies (3.2) on I. Let $0 \leq y < x \in I$. Then $x - y \in (0, x] \subset I$, so by the superadditivity of f,

$$f(x) = f(y + x - y) \geq f(y) + f(x - y) \geq f(y).$$

Moreover, if $0 \in I$, then $0 \leq f(0) \geq f(0) + f(0)$ by super-additivity, so $f(0) = 0$. This shows that f is nonnegative and nondecreasing on I. Hence by Theorem 1.2, $f[A] \in \mathcal{P}_{K_2}$ for every $A \in \mathcal{P}_{K_2}((0, \infty))$.

Now since $f \not\equiv 0$ on I, hence $f(p) > 0$ for all $0 < p \in I$ by Theorem 1.2. Moreover, Equation (3.2) trivially holds if x or y is zero (and $0 \in I$). Now assume that $x, y > 0$; then (3.2) can be restated as:

$$p, \frac{\alpha^2}{p} \in I,\ p > 0 \quad \implies \quad f\left(\frac{\alpha^2}{p}\right) \geq \frac{f(\alpha)^2}{f(p)}. \tag{3.6}$$

We now prove that (2) holds for any tree T by induction on $|T| \geq 3$. Suppose first that T is a tree with 3 vertices, i.e., $T = A_3$. Then, by Proposition 3.6, $f_{A_3}[A] \in \mathcal{P}_{A_3}$ for every $A \in \mathcal{P}_{A_3}$ if and only if

$$f\left(\frac{\alpha_2^2}{p_2} + \frac{\alpha_3^2}{p_3}\right) \geq \frac{f(\alpha_2)^2}{f(p_2)} + \frac{f(\alpha_3)^2}{f(p_3)}, \tag{3.7}$$

(or if one of p_2, p_3 is zero, in which case the assertion is easy to verify). Now suppose $0 < p_2, p_3 \in I$. If $A \in \mathcal{P}_{A_3}(I)$, then $p_1 \in I$, so $\frac{\alpha_2^2}{p_2} + \frac{\alpha_3^2}{p_3} \in [0, p_1]$ is also in I. Hence (3.7) follows immediately by the superadditivity of f and by (3.6).

Therefore (4) $\Rightarrow$ (2) holds for a tree with $n = 3$ vertices. Now assume that $A \in \mathcal{P}_{T'}(I)$ implies $f_{T'}[A] \in \mathcal{P}_{T'}$ for any tree T' with n vertices, and consider a tree T with $n+1$ vertices. Let $\widetilde{T}$ be a sub-tree obtained by removing a vertex connected to only one other node. Without loss of generality, assume the vertex that is removed is labeled $n+1$ and its neighbor is labeled n. Let $A \in \mathcal{P}_T(I)$; then A has the form

$$A = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\widetilde{A}_{n\times n}} & \begin{array}{c}\mathbf{0}_{(n-1)\times 1} \\ a\end{array} \\ \hline \mathbf{0}_{1\times(n-1)} & a & \alpha \end{array} \right).$$

If $\alpha = 0$ then $a = 0$ since A is positive semidefinite, and thus $f_T[A] \in \mathcal{P}_G$ since $f(0) = 0$. When $\alpha \neq 0$, the Schur complement S_A of α in A is $S_A = \widetilde{A} - (a^2/\alpha)E_{n,n}$. Here, $E_{i,j}$ denotes the $n \times n$ elementary matrix with the (i,j) entry equal to 1, and every other entry equal to 0. Since $A \in \mathcal{P}_T(I)$, hence $\widetilde{A} \in \mathcal{P}_{\widetilde{T}}(I)$, and $S_A \in \mathcal{P}_{\widetilde{T}}(I)$ from the above analysis (since $(S_A)_{nn} = \widetilde{a}_{nn} - a^2/\alpha \in [0, \widetilde{a}_{nn}) \subset I$). Therefore, by the induction hypothesis, $f_{\widetilde{T}}[\widetilde{A}], f_{\widetilde{T}}[S_A] \in \mathcal{P}_{\widetilde{T}}$. Consider now the matrix $f_T[A]$. Using Schur complements, $f_T[A] \in \mathcal{P}_T$ if and only if $f_{\widetilde{T}}[\widetilde{A}] \in \mathcal{P}_{\widetilde{T}}$ and the Schur complement $S_{f_T[A]}$ of $f(\alpha) > 0$ in $f_T[A]$, given by

$$S_{f_T[A]} = f_{\widetilde{T}}[\widetilde{A}] - \frac{f(a)^2}{f(\alpha)} E_{n,n},$$

belongs to $\mathcal{P}_{\widetilde{T}}$. Now, notice that $f_{\widetilde{T}}[S_A] = f_{\widetilde{T}}[\widetilde{A}] + [f(b) - f(\widetilde{a}_{nn})]\, E_{n,n}$, where $b := (S_A)_{nn} = \widetilde{a}_{nn} - \frac{a^2}{\alpha} \in I$ from the above analysis. Since $f_{\widetilde{T}}[S_A] \in \mathcal{P}_{\widetilde{T}}$ from above, to conclude the proof, it suffices to show that

$$-\frac{f(a)^2}{f(\alpha)} \geq f(b) - f(\widetilde{a}_{nn}). \tag{3.8}$$

Indeed, by using the superadditivity of f and (3.6), we compute:

$$f(\widetilde{a}_{n,n}) = f\left(\frac{a^2}{\alpha} + b\right) \geq f\left(\frac{a^2}{\alpha}\right) + f(b) \geq \frac{f(a)^2}{f(\alpha)} + f(b),$$

which proves (3.8). Therefore (4) $\Rightarrow$ (2) holds for a tree with $n+1$ vertices. This completes the induction and the proof of the theorem. $\square$

4. Power functions and critical exponents

A natural approach to tackle the problem of characterizing entrywise positive maps in fixed dimension is to examine if some natural simple functions preserve positivity. One such family is the collection of power functions, $f(x) = x^\alpha$ for $\alpha > 0$. Characterizing which fractional powers preserve positivity entrywise has recently received much attention in the literature.

4.1. FitzGerald and Horn's result

One of the first results in this area reads as follows.

Theorem 4.1 (FitzGerald and Horn [3, Theorem 2.2])**.** *Let $N \geq 2$ and let $A = (a_{jk}) \in \mathcal{P}_N\big(\mathbb{R}_+\big)$. For any real number $\alpha \geq N-2$, the matrix $A^{\circ\alpha} := (a_{jk}^\alpha)$ is positive semidefinite. If $0 < \alpha < N-2$ and α is not an integer, then there exists a matrix $A \in \mathcal{P}_N\big((0,\infty)\big)$ such that $A^{\circ\alpha}$ is not positive semidefinite.*

Theorem 4.1 shows that every real power $\alpha \geq N-2$ entrywise preserves positivity, while no non-integers in $(0, N-2)$ do so. This surprising "phase transition" phenomenon at the integer $N-2$ is referred to as the "critical exponent" for preserving positivity. Studying which powers entrywise preserve positivity is a very natural and interesting problem. It also often provides insights to determine which general functions preserve positivity. For example, Theorem 4.1 suggests that functions that entrywise preserve positivity on $\mathcal{P}_N$ should have a certain number of non-negative derivatives, which is indeed the case by Theorem 1.6.

Proof of Theorem 4.1. The first part of Theorem 4.1 relies on an ingenious idea integral trick. We proceed by induction over the dimension N of the matrix. The result is obvious for $N = 2$. Let us assume it holds for some $N-1 \geq 2$, let $A \in \mathcal{P}_N(\mathbb{R}_+)$, and let $\alpha \geq N-2$. Write A in block form,

$$A = \begin{bmatrix} B & \xi \\ \xi^T & a_{NN} \end{bmatrix},$$

where B has dimension $(N-1) \times (N-1)$ and $\xi \in \mathbb{R}^{N-1}$. Assume without loss of generality that $a_{NN} \neq 0$ (as the case where $a_{NN} = 0$ follows from the induction hypothesis) and let $\zeta := (\xi^T, a_{NN})^T/\sqrt{a_{NN}}$. Then $A - \zeta\zeta^T = (B - \xi\xi^T)/a_{NN} \oplus 0$, where $(B - \xi\xi^T)/a_{NN}$ is the Schur complement of a_{NN} in A. Hence $A - \zeta\zeta^T$ is positive semidefinite. By the fundamental theorem of calculus, for any x, $y \in \mathbb{R}$,

$$x^\alpha = y^\alpha + \alpha \int_0^1 (x-y)(\lambda x + (1-\lambda)y)^{\alpha-1}\,\mathrm{d}\lambda.$$

Using the above expression entrywise, we obtain

$$A^{\circ\alpha} = \zeta^{\circ\alpha}(\zeta^{\circ\alpha})^T + \int_0^1 (A - \zeta\zeta^T) \circ (\lambda A + (1-\lambda)\zeta\zeta^T)^{\circ(\alpha-1)}\,\mathrm{d}\lambda.$$

Observe that the entries of the last row and column of the matrix $A - \zeta\zeta^T$ are all zero. Using the induction hypothesis and the Schur product theorem, it follows that the integrand is positive semidefinite, and therefore so is $A^{\circ\alpha}$.

Finally, that non-integer powers do not preserve positivity in general follows from Theorem 1.6. $\square$

Note that Theorem 1.6 guarantees that for every non-integer $\alpha \in (0, N-2)$, there exists a matrix $A_\alpha \in \mathcal{P}(\mathbb{R}_+)$ for which $A^{\circ\alpha} \notin \mathcal{P}$. Tanvi Jain was recently able to greatly generalize this result.

Theorem 4.2 (Jain [8])**.** *Let*

$$A := (1 + u_j u_k)_{j,k=1}^N = \mathbf{1}_{N\times N} + \mathbf{u}\mathbf{u}^T,$$

where $N \geq 2$ *and* $\mathbf{u} := (u_1, \ldots, u_N)^T \in (0,\infty)^N$ *has distinct entries. Then* $A^{\circ\alpha}$ *is positive semidefinite for* $\alpha \in \mathbb{R}$ *if and only if* $\alpha \in \mathbb{Z}_+ \cup [N-2,\infty)$.

4.2. Critical exponents of graphs

A natural refinement of Theorem 4.1 involves studying powers that entrywise preserve positivity on $\mathcal{P}_G$. In that case, the flavor of the problem changes significantly, with the discrete structure of the graph playing a prominent role.

Definition 4.3 (Guillot–Khare–Rajaratnam [4])**.** Given a simple graph $G = (V, E)$, let

$$\mathcal{H}_G := \{\alpha \in \mathbb{R} : A^{\circ\alpha} \in \mathcal{P}_G \text{ for all } A \in \mathcal{P}_G(\mathbb{R}_+)\}. \tag{4.1}$$

Define the *Hadamard critical exponent of* G to be

$$CE(G) := \min\{\alpha \in \mathbb{R} : [\alpha,\infty) \subset \mathcal{H}_G\}. \tag{4.2}$$

Using this notions, Theorem 4.1 is equivalent to $\mathcal{H}_{K_N} = N-2$, where K_N denotes the complete graph on N vertices. Notice that Theorem 4.1 also guarantees that the critical exponent $CE(G)$ exists for every graph $G = (V, E)$, and lies in $[\omega(G)-2, |V|-2]$, where $\omega(G)$ is the size of the largest complete subgraph of G, that is, the clique number. To compute such critical exponents is natural and highly non-trivial.

Theorem 1.2 and Theorem 3.3 immediately provides the powers that preserve positivity on trees.

Theorem 4.4. *Let* G *be any tree. Then* $\mathcal{H}_G = [1,\infty)$.

A natural family of graphs that encompasses both complete graphs and trees is that of chordal graphs. Observe that trees are graphs with no cycles of length $n \geq 3$.

Definition 4.5. A graph is *chordal* if it does not contain an induced cycle of length $n \geq 4$.

Figure 1: Examines of chordal (left) and non-chordal (right) graphs.

In [4], the authors were able to compute the full set of powers preserving positivity for chordal graphs. Remarkably, the critical exponent can be fully described combinatorially.

Theorem 4.6 (Guillot–Khare–Rajaratnam, *J. Combin. Theory Ser. A* [4])**.** *Let* $K_r^{(1)}$ *denote the complete graph with one edge removed, and let* G *be a finite simple connected chordal graph. Then the critical exponent for preserving positivity of* G *is* $r-2$, *where* r *is the largest integer such that* K_r *or* $K_r^{(1)}$ *is a subgraph of* G. *More strongly, the set of entrywise powers preserving* $\mathcal{P}_G$ *is* $\mathcal{H}_G = \mathbb{N} \cup [r-2,\infty)$, *with* r *as above.*

The critical exponents of cycles and bipartite graphs are also known.

Theorem 4.7 (Guillot–Khare–Rajaratnam, *J. Combin. Theory Ser. A* [4])**.** *The critical exponent of cycles and bipartite graphs is* 1.

Surprisingly, the critical exponent does not depend on the size of the graph for cycles and bipartite graphs. In particular, it is striking that any power greater than 1 can preserve positivity for families of dense graphs such as bipartite graphs.

We will prove the result for complete bipartite graphs.

Proof (complete bipartite graphs). We will prove that the complete bipartite graph $K_{n,n}$ satisfies: $\mathcal{H}_{K_{n,n}} = [1, \infty)$ for all $n \geq 2$. Indeed, $P_3 \subset K_{n,n}$ since $n \geq 2$, so we conclude via Theorem 4.4 that $\mathcal{H}_{K_{n,n}} \subset \mathcal{H}_{P_3} = [1, \infty)$. To show the reverse inclusion, let $\alpha > 0$, $m, n \in \mathbb{N}$, and let

$$A = \begin{pmatrix} D_{m\times m} & X_{m\times n} \\ X^T & D'_{n\times n} \end{pmatrix} \in \mathcal{P}_{K_{m,n}}([0,\infty)),$$

with $\max(m, n) > 1$, and where D, D' are diagonal matrices. Given $\epsilon > 0$, define the matrix

$$X_{D,D'}(\epsilon, \alpha) := (D + \epsilon \operatorname{Id}_m)^{\circ(-\alpha/2)} \cdot X^{\circ\alpha} \cdot (D' + \epsilon \operatorname{Id}_n)^{\circ(-\alpha/2)}.$$

Also observe that for all block diagonal matrices A of the above form and all $\epsilon, \alpha > 0$,

$$(A + \epsilon \operatorname{Id}_{m+n})^{\circ\alpha} = \mathbf{D}_\epsilon \begin{pmatrix} \operatorname{Id}_m & X_{D,D'}(\epsilon,\alpha) \\ X_{D,D'}(\epsilon,\alpha)^T & \operatorname{Id}_n \end{pmatrix} \mathbf{D}_\epsilon,$$

where

$$\mathbf{D}_\epsilon := \begin{pmatrix} (D + \epsilon \operatorname{Id}_m)^{\circ\alpha/2} & \mathbf{0} \\ \mathbf{0} & (D' + \epsilon \operatorname{Id}_n)^{\circ\alpha/2} \end{pmatrix}.$$

We now compute for $\alpha, \epsilon > 0$:

$$\begin{aligned} & (A + \epsilon \operatorname{Id}_{m+n})^{\circ\alpha} \in \mathcal{P}_{K_{m,n}}([0,\infty)) \\ \iff & \begin{pmatrix} \operatorname{Id}_m & X_{D,D'}(\epsilon,\alpha) \\ X_{D,D'}(\epsilon,\alpha)^T & \operatorname{Id}_n \end{pmatrix} \in \mathcal{P}_{K_{m,n}}([0,\infty)) \\ \iff & \operatorname{Id}_m - X_{D,D'}(\epsilon,\alpha) X_{D,D'}(\epsilon,\alpha)^T \in \mathcal{P}_m(\mathbb{R}) \\ \iff & \|u\| \geq \|X_{D,D'}(\epsilon,\alpha)^T u\|, \ \forall u \in \mathbb{R}^n \\ \iff & \sigma_{\max}(X_{D,D'}(\epsilon,\alpha)) \leq 1, \end{aligned}$$

where $\sigma_{\max}$ denotes the largest singular value. Now note that if $m = n$, then the above calculation shows that $(A + \epsilon \operatorname{Id}_{2n})^{\circ\alpha} \in \mathcal{P}_{K_{n,n}}([0,\infty))$ if and only if $\rho(X_{D,D'}(\epsilon,\alpha)) \leq 1$, where ρ denotes the spectral radius.

To finish this first step of the proof, now suppose $\alpha \geq 1$ and $A \in \mathcal{P}_{K_{n,n}}([0,\infty))$. Then $A + \epsilon \operatorname{Id} \in \mathcal{P}_{K_{n,n}}([0,\infty))$ for all $0 < \epsilon \ll 1$, so by the above analysis with $\alpha = 1$, $\rho(X_{D,D'}(\epsilon, 1)) \leq 1$ for all $0 < \epsilon \ll 1$. By the Perron–Frobenius theorem, if a matrix has positive entries, then the eigenvector corresponding to the largest eigenvalue of the matrix has positive entries. From this, it follows that

$$\rho(X_{D,D'}(\epsilon,\alpha)) \leq \rho(X_{D,D'}(\epsilon,1))^\alpha \leq 1.$$

(See [7, Lemma 5.7.8] for more details.) It follows from the above analysis and the continuity of entrywise powers that $A^{\circ\alpha} \in \mathcal{P}_{K_{n,n}}([0,\infty))$. Thus $[1,\infty) \subset \mathcal{H}_{K_{n,n}}$. □

5. Moment transforms

We conclude by examining a different, but closely connected, setting. First, recall the notion of the *moments* of a measure.

Definition 5.1. Let μ be a measure defined on a sigma algebra of subsets of a set Ω. The k-th moment of μ is defined by

$$s_k(\mu) := \int_\Omega x^k \, d\mu(x) \qquad (k \geq 0)$$

if the above integral exists.

Moments play an important role in probability theorem. For example, if X is a random variable with a density f with respect to the Lebesgue measure dx on $\mathbb{R}$, then the moments of X are defined to be the moments of the associated measure $d\mu = f \, dx$:

$$E(X^k) = \int_{\mathbb{R}} x^k f(x) \, dx.$$

In particular, the first moment of X is its *expected value* $E(X)$. Its second moment $E(X^2)$ is closely related to the *variance* of X: $\mathrm{Var}(X) = E(X^2) - E(X)^2$.

An important problem in analysis and probability theory is to determine if a given sequence $(s_k)_{k\geq 0}$ of real numbers is the moment sequence of a measure with a given support. Another important problem is to determine if such a measure is unique when it exists.

5.1. The hamburger moment problem

The moment problem on $\mathbb{R}$ is known as the *Hamburger moment problem* (in honor of Hans Ludwig Hamburger, 1889–1956).

Hamburger moment problem. Given a sequence $(m_k)_{k\geq 0}$, does there exist a positive Borel measure such that

$$m_k = \int_{-\infty}^{\infty} x^k \, d\mu(x) \qquad \forall k = 0, 1, \dots,$$

i.e., a measure such that $m_k = s_k(\mu)$ for all $k \geq 0$.

This problem has a very interesting connection to matrix positivity.

Definition 5.2. A $n \times n$ matrix $A = (a_{ij})_{i,j=1}^n$ is said to be a *Hankel matrix* if $a_{ij} = x_{i+j}$ for some sequence $x_2, x_3, \dots, x_{2n}$.

In other words, a Hankel matrix is a matrix whose entries are constant on its anti-diagonals.

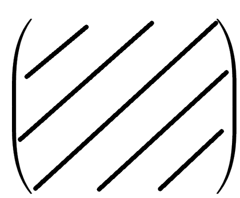


Figure 2: Illustration of a Hankel matrix. Entries are constant on anti-diagonals.

Theorem 5.3 (Hamburger). *A sequence $(m_k)_{k\geq 0}$ satisfies $x_k = s_k(\mu)$ for some positive Borel measure μ if and only if the associated Hankel matrices*

$$A = (a_{ij})_{i,j=1}^n = (m_{i+j})_{i,j=0}^n$$

are positive semidefinite for all $n \geq 1$.

Conditions are also known to guarantee the uniqueness of the measure μ.

Theorem 5.4 (Carleman's condition)**.** *Let μ be a positive Borel measure on $\mathbb{R}$ with finite moments. Suppose*

$$\sum_{j=1}^{\infty} s_{2j}(\mu)^{-\frac{1}{2j}} = \infty.$$

Then μ is the only measure on $\mathbb{R}$ with moment sequence $m_k = s_k(\mu)$.

5.2. Transforming moments

In probability theory, it is common to transform the moments of a given measure. It is then natural to ask if the resulting sequence is the moment sequence of a new measure. Here, we focus on transformations of the form $F(s_k(\mu))$ where $F : \mathbb{R} \to \mathbb{R}$. Hence, we would like to characterize the functions $F : \mathbb{R} \to \mathbb{R}$ with the property that for every positive Borel measure μ,

$$F(s_k(\mu)) = s_k(\nu) \qquad k = 0, 1, \ldots$$

for some positive Borel measure ν (that depends on μ). Several characterization were recently obtained in [2], on various domains. The following illustrates the type of results one can prove.

Theorem 5.5 (Belton–Guillot–Khare–Putinar [2])**.** *Let $F : \mathbb{R} \to \mathbb{R}$. The following are equivalent:*

1. *For every positive borel measure with* $\operatorname{supp} \mu \subset [-1, 1]$, *there exists a measure positive Borel measure ν with* $\operatorname{supp} \nu \subset [-1, 1]$ *and such that*

$$F(s_k(\mu)) = s_k(\nu) \qquad \forall k \geq 0.$$

2. *$F[A] \in \mathcal{P}_n$ for all $A \in \mathcal{P}_n \cap$ Hankel and all $n \geq 1$.*

3. *$F[A] \in \mathcal{P}_n$ for all $A \in \mathcal{P}_n$ and all $n \geq 1$.*

4. *$F(z) = \sum_{j=0}^{\infty} c_j z^z$ with $c_j \geq 0$.*

To illustrate some of the ideas used in the proof, we will only prove the following.

Lemma 5.6. *Let $F : \mathbb{R} \to \mathbb{R}$. Suppose that for every positive borel measure with* $\operatorname{supp} \mu \subset [-1, 1]$, *there exists a measure positive Borel measure ν such that*

$$F(s_k(\mu)) = s_k(\nu) \qquad \forall k \geq 0.$$

Then F is continuous on $\mathbb{R}$.

Proof. We prove the result in different steps.
Step 1. F is continuous on $(0, \infty)$.

(a) F is monotone on $(0, \infty)$. Let $x \leq y$. Then the matrix

$$A = \begin{pmatrix} y & x \\ x & y \end{pmatrix}$$

is a positive semidefinite Hankel matrix. Hence $f[A] \in \mathcal{P}_2$ and it follows that $F(x) \le F(y)$.

(b) Since F is monotone on $(0,\infty)$, it is measurable on $(0,\infty)$.

(c) F is multiplicatively mid-convex on $(0,\infty)$, i.e., $F(\sqrt{xy}) \le F(x)F(y)$ for all $x, y > 0$. This follows from considering the positive semidefinite Hankel matrix

$$\begin{pmatrix} x & \sqrt{xy} \\ \sqrt{xy} & y \end{pmatrix}.$$

(d) Now, $g(x) := \log F(e^x)$ is midpoint convex and measurable. One can show that this implies g is convex, which in turn, implies g is continuous. We conclude that F is continuous on $(0,\infty)$.

Step 2. F is continuous on $(-\infty, 0]$. We will use the following *key idea*: if $p(t) = a_0 + a_1 t + \cdots + a_d t^d \ge 0$ on $[-1,1]$, then:

$$\boxed{0 \le \int_{-1}^{1} p(t)\ d\nu = \sum_{j=0}^{d} a_j s_j(\nu) = \sum_{j=0}^{d} a_j F(s_j(\mu)).} \tag{5.1}$$

We will apply Equation (5.1) to well-chosen μ and p. Let

$$p_{\pm}(t) = (1 \pm t)(1 - t^2).$$

Note that $p_{\pm} \ge 0$ on $[-1,1]$. Fix $v_0 \in (0,1)$, let $\beta \ge 0$ and define:

$$a := \beta + b v_0, \qquad \mu := a\delta_{-1} + b\delta_{v_0},$$

where δ_x denotes the Delta measure supported at x. The following table provides the first moments of μ:

k	$s_k(\mu)$
0	$a + b$
1	$-a + b v_0$
2	$a + b v_0^2$
3	$-a + b v_0^3$

Using Equation (5.1), we obtain

$$F(a+b) - F(a + b v_0^2) \ge \pm \left(F(-a + b v_0) - F(-a + b v_0^3) \right).$$

Equivalently, we have

$$F(\beta + b + b v_0) - F(\beta + b v_0 + b v_0^2) \ge \left| F(-\beta) - F(-\beta + b(v_0^3 - v_0)) \right|$$

Letting $b \to 0^+$ and using the fact that F is continuous on $(0,\infty)$, we conclude that F is left-continuous at $-\beta$ for any $\beta \ge 0$. A similar argument can be used to obtain the right-continuity. $\square$

A. Basic properties of symmetric and positive semidefinite matrices

We record in this section some basic results from linear algebra.

Recall that a $n \times n$ matrix is *symmetric* if $A = A^T$.

Proposition A.1. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then*

1. *All the eigenvalues of A are real.*

2. *Eigenvectors of A associated to distinct eigenvalues are orthogonal.*

3. *The matrix A is diagonalizable. More precisely, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A and $u_1, \ldots, u_n$ are associated eigenvectors, then*

$$A = UDU^T = \sum_{i=1}^{n} \lambda_i u_i u_i^T$$

where $U = (u_1, \ldots, u_n) \in \mathbb{R}^{n\times n}$ is the matrix containing the eigenvectors of A as columns, and $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix containing the eigenvalues of A on its diagonal.

Definition A.2. A symmetric matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is said to be *positive semidefinite* if $x^T Ax = \sum_{i,j=1}^{n} a_{ij}x_i x_j \geq 0$ for all $x \in \mathbb{R}^n$. It is said to be *positive definite* if $x^T Ax > 0$ for all $x \in \mathbb{R}^n$.

We denote the set of $n \times n$ symmetric positive semidefinite matrices with real entries by $\mathcal{P}_n$.

The following result shows that $\mathcal{P}_n$ is closed under addition, multiplication by a nonnegative scalar, and congruences.

Theorem A.3. *Let $A, B \in \mathcal{P}_n$ and let $C \in \mathbb{R}^{n\times m}$. Then*

1. *$\lambda A \in \mathcal{P}_n$ for all $\lambda \geq 0$.*

2. *$A + B \in \mathcal{P}_n$.*

3. *$C^T AC \in \mathcal{P}_m$.*

The following theorem summarizes several important characterizations of positive semidefinite matrices. Let $\alpha, \beta \subseteq \{1, \ldots, n\}$ and let $A \in \mathbb{R}^{n\times n}$. We denote by $A[\alpha, \beta]$ the *submatrix* of A with rows indices in α and column indices in β, i.e.,

$$A[\alpha, \beta] := (a_{ij})_{i\in\alpha, j\in\beta}.$$

To simplify the notation, when $\alpha = \beta$, we let $A[\alpha] := A[\alpha, \alpha]$. A *principal minor* of A is a determinant $\det A[\alpha]$ for some $\alpha \subseteq \{1, \ldots, n\}$.

Theorem A.4. *Let A be a $n \times n$ symmetric matrix. Then the following are equivalent:*

1. *The matrix A is positive semidefinite.*

2. *All eigenvalues of A are nonnegative.*

3. *All the principal minors of A are nonnegative.*

4. *All the leading principal minors of A are nonnegative.*

5. *There exists a matrix $B \in \mathbb{R}^{m\times n}$ such that $A = B^T B$. Equivalently, A is a Gram matrix, i.e., there exist vectors $v_1, \ldots, v_n \in \mathbb{R}^m$ such that $a_{ij} = v_i^T v_j$.*

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