ACCURACY AND STABILITY OF POLYNOMIAL INTERPOLATION SCHEMES FOR ADVECTION EQUATIONS*

HOON HONG† AND STANLY STEINBERG†

Abstract. We study explicit arbitrarily high-order polynomial interpolation schemes on uniform grids for the advection equations. The schemes have optimal accuracy and if the stencil for the scheme is supported in interval \([\alpha, \beta]\) then the stability region for the CFL number contains the set 

\[
[\mathfrak{L}_{\beta\alpha}, \mathfrak{R}_{\beta\alpha}] \cup \{\alpha, \ldots, \beta\}
\]

where

\[
\mathfrak{L}_{\beta\alpha} = \left[\frac{\beta + \alpha - 1}{2}\right], \quad \mathfrak{R}_{\beta\alpha} = \left[\frac{\beta + \alpha + 1}{2}\right]
\]

We conjecture that this set is exactly the stability region and provide a computer based proof of the conjecture for \(\beta - \alpha \leq 164\).

Key words. Stability, Numerical Schemes, Polynomial Interpolation, Advection equation.

AMS subject classification. 65M06, 65M12

1. Introduction. The advection or one-way wave equation is given by

\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0
\]

(1.1)

where \(c \in \mathbb{R}\). When devising numerical approximations for partial differential equations whose solutions display wave-like behavior, it is critical to understand the behavior of the numerical method on this simplest of wave equations, and conversely, high-quality algorithm for complex problems can be based on high-quality algorithms for this simple problem, as is made clear in the numerical partial differential equations texts [5, 11, 12].

In [1, 2, 3, 4] Goodrich and Dyson investigated some properties of a polynomial interpolation scheme for the advection equations (with a five point central stencil), via numerical sampling and experiments. The main goal of this paper is to provide mathematical proofs for those properties for schemes with arbitrary finite stencils:

\[
u(x, t + \Delta t) = \sum_{\nu=\alpha}^{\beta} C_{\alpha\beta\nu} u(x + \nu \Delta x, t)
\]

where \(\alpha < \beta\) are integers and \(C_{\alpha\beta\nu}\) is the stencil for the scheme. We will show that the stability region for the polynomial interpolation scheme contains the CFL number in the set

\[
[\mathfrak{L}_{\beta\alpha}, \mathfrak{R}_{\beta\alpha}] \cup \{\alpha, \ldots, \beta\}
\]

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The intervals of stability for the optimal schemes

<table>
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<th>$\alpha \setminus \beta$</th>
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We conjecture that this set is exactly the stability region. We have proved that the conjecture holds for $\beta - \alpha \leq 164$, using a program written in a computer algebra system (Maple). However, we have not yet succeeded in finding a general proof and we leave it as an open problem. Of course, $\beta - \alpha \leq 164$ is more than enough for practical purposes. However we still believe that finding a general proof is worthwhile because the found proof technique/ideas might shed new light into the subject matter.

In 1962, Strang [10] proved the symmetric case, $\alpha = -\beta$, of our result, that is, the stability region in this case contains $[-1, 1]$. In [6, 7, 8, 9], Iserles, Jeltsch, Nørsett, Renaut, Smit and Strang show that if the stability region contains $[0, 1]$, or in fact a single point from $(0, 1)$, then the maximum order of accuracy of the any scheme must be less than or equal to

$$\min \{ \beta - \alpha, -2 \alpha + 2, 2 \beta \}, \quad \text{if } \beta \geq 0, \quad \alpha \leq 0, \quad c > 0. \quad (1.2)$$

Because we do not require that $c > 0$, our results are more symmetric than (1.2). The interpolation schemes have accuracy $\beta - \alpha$, and by (1.2), the accuracy cannot be any larger, so they are optimal. The stability intervals (omitting isolated points) for the polynomial interpolation scheme are displayed in table 1.1. In the case $\alpha = -2$ and $\beta = 1$, the stability region is $[-1, 0]$, so (1.2) doesn’t apply. In the case that $\alpha = -1$ and $\beta = 2$, the stability region is $[0, 1]$, so (1.2) does apply and gives the maximum order as $\beta - \alpha = 3$, the order of the interpolation scheme. When $\alpha = 0$ and $\beta = 3$, $\beta - \alpha = 3$, the accuracy of the interpolation scheme, but (1.2) gives the maximum order of accuracy as 2, in agreement with the stability region of the interpolation scheme not containing a point from $(0, 1)$. In the case $\alpha = 0$ and $\beta = 2$, and the case $\alpha = -1$ and $\beta = 3$, our results show that the stability region contains $[0, 2]$, a result we found surprising. More generally, if $\alpha + \beta = 2m + 1$ is odd, the stability region contains $[m, m + 1]$, while if $\alpha + \beta = 2m$ is even, the stability region contains $[m - 1, m + 1]$. Moreover, the schemes corresponding to the skew diagonals in table 1.1 (where $\beta - \alpha$ is constant) have the same stability region.

This paper is organized as follows. In the next section we recall the polynomial interpolation schemes for advection equations. In the following section, we state the contributions of the paper (two theorems and one conjecture). In the subsequent three sections, we prove the two theorems and provide some supporting evidence for the conjecture.

2. Polynomial Interpolation Scheme. In this section, we will derive the polynomial interpolation scheme. We begin by recalling that every solution $u$ of the ad-
vection equation (1.1) satisfies the equation
\[ u(x, t + \Delta t) = u(x + c \Delta t, t) = u(x + \lambda \Delta x, t) \]
for arbitrary \( \Delta t \) and \( \Delta x \) and where \( \lambda = c \Delta t / \Delta x \) is the CFL number (\( \lambda \) will keep this meaning throughout this paper). Let \( T \) and \( S \) be the time and the space shift operators, that is,
\[ T u(x, t) = u(x, t + \Delta t) \]
\[ S u(x, t) = u(x + \Delta x, t) \]
Then we can rewrite the above equation into
\[ T u = S^\lambda u \]
Because \( \lambda \) is not necessarily an integer, we will approximate \( S^\lambda u \) using a polynomial interpolation over the points \( x + \nu \Delta x \), where \( \nu = \alpha, \ldots, \beta \). Using the well known Lagrange formula for polynomial interpolation, we get
\[ S^\lambda u \approx \sum_{\nu=\alpha}^{\beta} L_{\alpha\beta\nu}(\lambda) S^\nu u \]
where
\[ L_{\alpha\beta\nu}(\lambda) = \prod_{\substack{\alpha \leq \mu \leq \beta \\ \mu \neq \nu}} \frac{\lambda - m}{\nu - m} \]
From these expressions, we immediately obtain the following scheme.

**Definition 2.1 (Polynomial Interpolation Scheme).** For any \( \lambda \),
\[ T u \approx \sum_{\nu=\alpha}^{\beta} L_{\alpha\beta\nu}(\lambda) S^\nu u \]  \hspace{1cm} (2.1)

**3. Main Results.** In this section, we will list the main results of the paper (two theorems and a conjecture). In the subsequent sections, we will provide proofs of the theorems and supporting evidence for the conjecture.

**Theorem 3.1 (Accuracy).** For smooth initial data, the polynomial interpolation scheme has the optimal accuracy, that is, the accuracy of order \( \beta - \alpha \).

Let the stability region, \( R_{\alpha\beta} \), be the set of every \( \lambda \) value for which the polynomial interpolation scheme is stable. The following theorem provides a lower bound for \( R_{\alpha\beta} \).

**Theorem 3.2 (Lower Bound for the Stability Region).** We have
\[ R_{\alpha\beta} \supseteq [b_{\alpha\beta}, \bar{b}_{\alpha\beta}] \cup \{\alpha, \ldots, \beta\} \]
where
\[ b_{\alpha\beta} = \left\lfloor \frac{\beta + \alpha - 1}{2} \right\rfloor, \quad \bar{b}_{\alpha\beta} = \left\lceil \frac{\beta + \alpha + 1}{2} \right\rceil \]
**Remark 3.3.** This is the result that is important for computation. The usual domain-of-dependence argument will show that the stability region must be in the support of stencil of the scheme, that is \( R_{\alpha\beta} \subseteq [\alpha, \beta] \). We show this algebraically in proposition 6.1. In fact, the stability region is substantially smaller.

**Conjecture 3.4 (Exact Stability Region).** We conjecture that the above expression gives the exact stability region, that is,

\[
R_{\alpha\beta} = [2\alpha, 3\alpha, \ldots, 2\beta, 3\beta] \cup \{\alpha, \ldots, \beta\}
\]

We have proved that the conjecture holds for \( \beta - \alpha \leq 164 \), using a program written in a computer algebra system (Maple).

**Remark 3.5.** Numerical plots of the amplification factor support this conjecture.

4. **Proof of the Accuracy.** In this section, we will prove Theorem 3.1. The truncation error for an approximation to a first order in time differential equation is one less than the truncation error for the discrete solution operator (2.1) for the scheme, as one gets the solution operator by multiplying the difference approximation by \( \Delta t = \lambda \Delta x / c \). For smooth functions \( u \), the truncation error for the solution operator is given by

\[
w = w(\Delta x) = Tu - \sum_{\nu=\alpha}^{\beta} L_{\alpha\beta\nu}(\lambda) S^\nu u
\]

so we need to show that \( w(\Delta x) = O(\Delta x^{\beta-\alpha+1}) \). Recalling that \( Tu = S^\lambda u \), we have

\[
w = S^\lambda u - \sum_{\nu=\alpha}^{\beta} L_{\alpha\beta\nu}(\lambda) S^\nu u
\]

so we need to show that \( \frac{d^k w}{d\Delta x^k}(0) = 0 \) for \( k = 0, \ldots, \beta - \alpha \). Note that

\[
\frac{d^k w}{d\Delta x^k}(0) = = \frac{\partial^k u}{\partial x^k}(0) \lambda^k - \sum_{\nu=\alpha}^{\beta} L_{\alpha\beta\nu}(\lambda) \frac{\partial^k u}{\partial x^k}(0) \nu^k
\]

\[
= \frac{\partial^k u}{\partial x^k}(0) \left( \lambda^k - \sum_{\nu=\alpha}^{\beta} L_{\alpha\beta\nu}(\lambda) \nu^k \right)
\]

But the interpolating polynomial interpolates \( \lambda^k \) exactly for \( k \leq \beta - \alpha \), so we have \( \frac{d^k w}{d\Delta x^k}(0) = 0 \) for \( k \leq \beta - \alpha \). Hence

\[
w(\Delta x) = O(\Delta x^{\beta-\alpha+1})
\]

Thus the scheme has the accuracy of order \( \beta - \alpha \) and we have proved the theorem.

5. **Proof of the Lower Bound for Stability Region.** In this section, we will prove Theorem 3.2 using von Neumann Fourier stability analysis. For this, as usual, we replace the space shift operator \( S \) with \( e^{i\xi} \), obtaining the amplification factor:

\[
f_{\alpha\beta}(\lambda, \xi) = \sum_{\nu=\alpha}^{\beta} L_{\alpha\beta\nu}(\lambda) e^{i\nu\xi}
\]
So if
\[ g_{\alpha\beta}(\lambda, \xi) = 1 - |f_{\alpha\beta}(\lambda, \xi)|^2 \]
then the stability region is given by
\[ R_{\alpha\beta} = \{ \lambda \in \mathbb{R} : \forall \xi \in \mathbb{R}, g_{\alpha\beta}(\lambda, \xi) \geq 0 \} \]

In the following, we will find a “nice” expression for the function \( g_{\alpha\beta}(\lambda, \xi) \) from which we can easily “read-off” some information about the stability region \( R_{\alpha\beta} \). The derivation of such an expression is long and thus we divided it into several lemmas (Lemmas 5.1 through 5.4), ending with Lemma 5.4, which gives the nice expression. (We use the conventions that if \( m > n \), then \( \sum_{i=m}^{n} = 0 \) and \( \prod_{i=m}^{n} = 1 \).)

**Lemma 5.1.** We have
\[ g_{\alpha\beta}(\lambda, \xi) = h_{\alpha\beta}(\lambda, \xi) \prod_{\alpha \leq m \leq \beta} (\lambda - m) \]

where
\[ h_{\alpha\beta}(\lambda, \xi) = \sum_{\alpha \leq \nu_1, \nu_2 \leq \beta} \frac{1 - \cos [(\nu_1 - \nu_2) \xi]}{\prod_{m \neq \nu_1}^{\lambda \leq m \leq \beta} (\nu_1 - m)} \prod_{\alpha \leq m \leq \beta} (\lambda - m) \]

**Proof.** We repeatedly rewrite the defining expression of \( g_{\alpha\beta}(\lambda, \xi) \) as follows.
\[
g_{\alpha\beta}(\lambda, \xi) = 1 - |f_{\alpha\beta}(\lambda, \xi)|^2
\]
\[
= 1 - \left( \sum_{\alpha \leq \nu \leq \beta} L_{\alpha\nu}(\lambda) e^{i\nu\xi} \right) \left( \sum_{\alpha \leq \nu \leq \beta} L_{\alpha\nu}(\lambda) e^{-i\nu\xi} \right)
\]
\[
= 1 - \sum_{\alpha \leq \nu_1, \nu_2 \leq \beta} L_{\alpha\nu_1}(\lambda) L_{\alpha\nu_2}(\lambda) e^{i(\nu_1 - \nu_2)\xi}
\]
\[
= 1 - \sum_{\alpha \leq \nu_1, \nu_2 \leq \beta} L_{\alpha\nu_1}(\lambda) L_{\alpha\nu_2}(\lambda) \cos [(\nu_1 - \nu_2) \xi]
\]

Recalling the obvious fact
\[
\sum_{\alpha \leq \nu_1, \nu_2 \leq \beta} L_{\alpha\nu_1}(\lambda) L_{\alpha\nu_2}(\lambda) = \left( \sum_{\alpha \leq \nu \leq \beta} L_{\alpha\nu}(\lambda) \right) = 1
\]

we have
\[
g_{\alpha\beta}(\lambda, \xi) = \sum_{\alpha \leq \nu_1, \nu_2 \leq \beta} L_{\alpha\nu_1}(\lambda) L_{\alpha\nu_2}(\lambda)
\]
\[
- \sum_{\alpha \leq \nu_1, \nu_2 \leq \beta} L_{\alpha\nu_1}(\lambda) L_{\alpha\nu_2}(\lambda) \cos [(\nu_1 - \nu_2) \xi]
\]
\[
= \sum_{\alpha \leq \nu_1, \nu_2 \leq \beta} L_{\alpha\nu_1}(\lambda) L_{\alpha\nu_2}(\lambda) (1 - \cos [(\nu_1 - \nu_2) \xi])
\]
\[
= \sum_{\alpha \leq \nu_1, \nu_2 \leq \beta} \prod_{\alpha \leq m \leq \beta} (\lambda - m) \prod_{m \neq \nu_1} (\nu_1 - m) \prod_{m \neq \nu_2} (\nu_2 - m) (1 - \cos [(\nu_1 - \nu_2) \xi])
\]
\[
= h_{\alpha\beta}(\lambda, \xi) \prod_{\alpha \leq m \leq \beta} (\lambda - m)^2
\]
Next we recall two well known facts.

**Lemma 5.2.** We have

$$\cos(n\xi) - 1 = \sum_{k=1}^{n} \frac{2^k}{(2k)!} (\cos \xi - 1)^k$$

The next lemma deals with the binomial coefficient \(\binom{p}{q}\), which is defined only for the case \(0 \leq q \leq p\). In order to state the lemma, we need to extend this definition to arbitrary integers \(p\) and \(q\) as follows:

$$\binom{p}{q} = \begin{cases} \frac{p(p-1)\ldots(p-q+1)}{q(q-1)\ldots1} & q > 0 \\ 1 & q = 0 \\ 0 & q < 0 \end{cases}$$

**Lemma 5.3.** For \(s, p, t \in \mathbb{Z}\), we have

$$\binom{s}{p+t} = \sum_{m=0}^{p} (-1)^{p-m} \binom{p}{m} \binom{s+m}{t+m}$$

Using the above three lemmas, we prove the key lemma of this section. We shall see that the theorem follows immediately from the key lemma.

**Lemma 5.4 (Key Lemma).** We have

$$g_{\alpha\beta}(\lambda, \xi) = \sum_{r \leq \beta - \alpha, \alpha+r \geq \beta-r} c_{\alpha\beta r}(\xi) \ w_{\alpha\beta r}(\lambda)$$

where

$$c_{\alpha\beta r}(\xi) = \frac{2^{r+1}}{(2r)!} \binom{2r-1}{\beta - \alpha} (1 - \cos \xi)^r$$

$$w_{\alpha\beta r}(\lambda) = \binom{\lambda - \alpha - r}{\alpha - r} \prod_{\alpha \leq m \leq \beta - r} (\lambda - m) \prod_{\alpha+r \leq m \leq \beta} (m - \lambda)$$

**Proof.** From Lemma 5.1 and Lemma 5.2, we have

$$h_{\alpha\beta}(\lambda, \xi) = 2 \sum_{\alpha \leq \nu_1, \nu_2 \leq \beta} \frac{1 - \cos[(\nu_1 - \nu_2)\xi]}{\prod_{\alpha \leq m \leq \beta} (\nu_1 - m) \prod_{\alpha \leq m \leq \beta} (\nu_2 - m)} \prod_{\alpha \leq m \leq \beta} (\lambda - m)$$

$$= 2 \sum_{\alpha \leq \nu_1, \nu_2 \leq \beta} (-1)^{\nu_1 - \nu_2} \frac{2^k}{(2k)!} (\cos \xi - 1)^k$$

$$\frac{(\nu_1 - \nu_2) \prod_{m=-k+1}^{k-1} (\nu_1 - \nu_2 + m) \prod_{m \neq \nu_1}^{k-1} (\nu_1 - m) \prod_{m \neq \nu_2}^{k-1} (\nu_2 - m)}{\prod_{\alpha \leq m \leq \beta} (\nu_1 - m) \prod_{\alpha \leq m \leq \beta} (\nu_2 - m) (\lambda - \nu_1) (\lambda - \nu_2)}$$

By re-ordering the summation symbols, we have

$$h_{\alpha\beta}(\lambda, \xi) = \sum_{r \leq \beta - \alpha} (-2)^{r+1} (2r)! (1 - \cos \xi)^r q_{\alpha\beta r}(\lambda)$$
where
\[
q_{\alpha \beta r}(\lambda) = \sum_{\substack{\alpha \leq \nu_1, \nu_2 \leq \beta \\
\nu_1 - \nu_2 \geq r}} (\nu_1 - \nu_2) \prod_{m=r+1}^{r-1} (\nu_1 - \nu_2 + m) \prod_{m \neq \mu}^{m=1} (\nu_1 - m) \prod_{m \neq \nu_2}^{m=1} (\nu_2 - m) \prod_{\alpha \leq m \leq \beta \atop m \neq \mu} (\lambda - m) \prod_{\alpha \leq m \leq \beta \atop m \neq \nu_2} (\lambda - \nu_1) (\lambda - \nu_2)
\]

For \( \mu \in \mathbb{Z}, \alpha \leq \mu \leq \beta \), we have
\[
q_{\alpha \beta r}(\mu) = q_{\alpha \beta r}(\mu) + \mathcal{I}_{\alpha \beta r}(\mu)
\]

where
\[
q_{\alpha \beta r}(\mu) = \sum_{\substack{\alpha \leq \nu_1, \nu_2 \leq \beta \\
\mu - \nu_2 \geq r}} (\mu - \nu_2) \prod_{m=r+1}^{r-1} (\mu - \nu_2 + m) \prod_{m \neq \mu}^{m=1} (\mu - m) \prod_{m \neq \nu_2}^{m=1} (\nu_2 - m) \prod_{\alpha \leq m \leq \beta \atop m \neq \mu} (\mu - m) \prod_{\alpha \leq m \leq \beta \atop m \neq \nu_2} (\mu - \nu_1) (\mu - \nu_2)
\]
\[
\mathcal{I}_{\alpha \beta r}(\mu) = \sum_{\substack{\alpha \leq \nu_1, \nu_2 \leq \beta \\
\nu_1 - \mu \geq r}} (\nu_1 - \mu) \prod_{m=r+1}^{r-1} (\nu_1 - \mu + m) \prod_{m \neq \nu_1}^{m=1} (\nu_1 - m) \prod_{m \neq \nu_2}^{m=1} (\nu_2 - m) \prod_{\alpha \leq m \leq \beta \atop m \neq \nu_1} (\nu_1 - m) \prod_{\alpha \leq m \leq \beta \atop m \neq \nu_2} (\mu - \nu_1) (\mu - \nu_2)
\]

We repeatedly rewrite \( q_{\alpha \beta r}(\mu) \) as follows.
\[
q_{\alpha \beta r}(\mu) = \sum_{\substack{\alpha \leq \nu_1 \leq \beta \\
\mu - \nu_2 \geq r}} \prod_{m=r+1}^{r-1} (\mu - \nu_2 + m) \prod_{\alpha \leq m \leq \beta \atop m \neq \mu} (\mu - m) (\nu_2 - m) (\nu_1 - m) (\nu_1 - \mu + m) (\beta - \mu) (\beta - \nu) (\beta - r)
\]
\[
= \sum_{\alpha \leq \nu \leq \mu - r} (-1)^{\beta - \nu} (\mu - \nu + r - 1)! (\mu - \nu - r)! (\nu - \alpha)! (\beta - \nu)!
\]
\[
= \frac{(2r - 1)!}{(\beta - \alpha)!} \sum_{\alpha \leq \nu \leq \mu - r} (-1)^{\beta - \nu} (\beta - \alpha) (\beta - \nu) (\mu - \nu + r - 1) (\mu - \nu - r)
\]

We repeatedly rewrite \( \mathcal{I}_{\alpha \beta r}(\mu) \) as follows.
\[
\mathcal{I}_{\alpha \beta r}(\mu) = - \sum_{\substack{\alpha \leq \nu_1 \leq \beta \\
\nu_1 - \mu \geq r}} \prod_{m=r+1}^{r-1} (\nu_1 - \mu + m) \prod_{\alpha \leq m \leq \beta \atop m \neq \nu_1} (\nu_1 - m) (\nu_1 - \mu + m) (\beta - \mu) (\beta - \nu) (\beta - r)
\]
\[
= - \sum_{\nu + r \leq \mu \leq \beta} (-1)^{\beta - \nu} (\nu - \mu + r - 1)! (\nu - \mu - r)! (\nu - \alpha)! (\beta - \nu)!
\]
\[
= \frac{(2r - 1)!}{(\beta - \alpha)!} \sum_{\nu + r \leq \mu \leq \beta} (-1)^{\beta - \nu} (\beta - \alpha) (\beta - \nu) (\nu - \mu + r - 1) (\nu - \mu - r)
\]
\[
= \frac{(2r - 1)!}{(\beta - \alpha)!} (\alpha - \mu + r - 1) (\beta - \mu - r)
\]

Put together we have
\[
q_{\alpha \beta r}(\mu) = \frac{(2r - 1)!}{(\beta - \alpha)!} \left[ (-1)^{\beta - \alpha} (\mu - \beta + r - 1) (\mu - r - \alpha) - (\beta - \mu - r) \right]
\]

We will simplify this expression for various cases.
• Case: $\alpha + r > \beta - r$
  - Subcase: $\mu \geq \alpha + r$
    \[
    q_{\alpha \beta r}(\mu) = \frac{(2r-1)!}{(\beta - \alpha)!} \left( (-1)^{\beta-\alpha} \left( \frac{\mu - \beta + r - 1}{\mu - \alpha} \right) - 0 \right) 
    = \frac{(2r-1)!}{(\beta - \alpha)!} \left( (-1)^{\beta-\alpha} \left( \frac{\mu - \beta + r - 1}{2r - 1 - \beta + \alpha} \right) \right) 
    = \frac{(2r-1)!}{(\beta - \alpha)!} \left( (-1)^{\beta-\alpha} \frac{\prod_{m=\beta-r+1}^{\alpha+r-1} (\mu - m)}{(2r - 1 - \beta + \alpha)!} \right) 
    = (-1)^{\beta-\alpha} \left( \frac{2r - 1}{\beta - \alpha} \right)^{\alpha+r-1} \prod_{m=\beta-r+1}^{\alpha+r-1} (\mu - m) 
    
    - Subcase: $\beta - r \geq \mu$.
    \[
    q_{\alpha \beta r}(\mu) = \frac{(2r-1)!}{(\beta - \alpha)!} \left( (-1)^{\beta-\alpha} \cdot 0 - \left( \frac{\alpha - \mu + r - 1}{\beta - \mu - r} \right) \right) 
    = \frac{(2r-1)!}{(\beta - \alpha)!} \left( (-1)^{\beta-\alpha} \left( \frac{\alpha - \mu + r - 1}{2r - 1 - \beta + \alpha} \right) \right) 
    = (-1)^{\beta-\alpha} \frac{(2r-1)!}{(\beta - \alpha)!} \left( \frac{\prod_{m=\beta-r+1}^{\alpha+r-1} (\mu - m)}{(2r - 1 - \beta + \alpha)!} \right) 
    = (-1)^{\beta-\alpha} \left( \frac{2r - 1}{\beta - \alpha} \right)^{\alpha+r-1} \prod_{m=\beta-r+1}^{\alpha+r-1} (\mu - m) 
    
    - Subcase: $\beta - r + 1 \leq \mu \leq \alpha + r - 1$.
    \[
    q_{\alpha \beta r}(\mu) = \frac{(2r-1)!}{(\beta - \alpha)!} \left( (-1)^{\beta-\alpha} \cdot 0 - 0 \right) 
    = 0 
    = (-1)^{\beta-\alpha} \left( \frac{2r - 1}{\beta - \alpha} \right)^{\alpha+r-1} \prod_{m=\beta-r+1}^{\alpha+r-1} (\mu - m) 
    
    - Putting these three subcases together we see that for $\alpha \leq \mu \leq \beta$, we have
    \[
    q_{\alpha \beta r}(\mu) = (-1)^{\beta-\alpha} \left( \frac{2r - 1}{\beta - \alpha} \right)^{\alpha+r-1} \prod_{m=\beta-r+1}^{\alpha+r-1} (\mu - m) 
    \]
    Since $q_{\alpha \beta r}(\lambda)$ is of degree at most $\beta - \alpha + 1 - 2$, we have
    \[
    q_{\alpha \beta r}(\lambda) = (-1)^{\beta-\alpha} \left( \frac{2r - 1}{\beta - \alpha} \right)^{\alpha+r-1} (\lambda - m) 
    \]

• Case: $\alpha + r \leq \beta - r$
  - Subcase: $\mu < \alpha + r$.
    \[
    q_{\alpha \beta r}(\mu) = \frac{(2r-1)!}{(\beta - \alpha)!} \left( (-1)^{\beta-\alpha} \cdot 0 - 0 \right) = 0
    
    \]
- Subcase: \( \mu > \beta - r \).

\[
q_{\alpha, \beta r}(\mu) = \frac{(2r - 1)!}{(\beta - \alpha)!} \left( -1 \right)^{\beta - \alpha} \cdot 0 - 0 = 0
\]

- Subcase: \( \alpha + r \leq \mu \leq \beta - r \). If

\[
q_{\alpha, \beta r}(\mu) = \frac{(2r - 1)!}{(\beta - \alpha)!} q_{\alpha, \beta r}(\mu)
\]

then

\[
q_{\alpha, \beta r}(\mu) = (-1)^{\beta - \alpha} \left( -1 \right)^{\mu - r - \alpha} \left( -\frac{\mu - \beta + r - 1 + \mu - r - \alpha - 1}{\mu - r - \alpha} \right)
\]

\[
-(-1)^{\beta - \mu - r} \left( -\frac{\alpha - \mu + r - 1 + \beta - \mu - r - 1}{\beta - \mu - r} \right)
\]

\[
= (-1)^{\beta - \mu - r} \left( \frac{\beta - 2r - \alpha}{\mu - r - \alpha} \right) - (-1)^{\beta - \mu - r} \left( \frac{\beta - 2r - \alpha}{\beta - \mu - r} \right)
\]

\[
= (-1)^{\beta - \mu - r} \left( \frac{\beta - 2r - \alpha}{\mu - r - \alpha} \right) - (-1)^{\beta - \mu - r} \left( \frac{\beta - 2r - \alpha}{\mu - r - \alpha} \right) = 0
\]

- Putting these three subcases together we see that for \( \alpha \leq \mu \leq \beta \), we have

\[
q_{\alpha, \beta r}(\mu) = 0
\]

Since \( q_{\alpha, \beta r}(\lambda) \) is of degree at most \( \beta - \alpha + 1 - 2 \), we have

\[
q_{\alpha, \beta r}(\lambda) = 0
\]

Putting the above two cases together we see that

\[
g_{\alpha, \beta}(\lambda, \xi) = \sum_{r \leq \beta - \alpha} (-1)^{\beta - \alpha + r + 1} \frac{2r + 1}{(2r)!} (1 - \cos \xi)^r \left( \frac{2r - 1}{\beta - \alpha} \right)
\]

\[
\prod_{m=\beta - r + 1}^{\alpha + r - 1} (\lambda - m) \prod_{m=\alpha}^{\beta} (\lambda - m)
\]

With the goal of eliminating \((-1)^{\beta - \alpha + r + 1}\), we repeatedly rewrite this expression as
follows

\[ g_{\alpha\beta}(\lambda, \xi) = \sum_{\substack{r \leq \beta - \alpha \leq \alpha + r \geq \beta - r \geq \alpha}} (-1)^{\beta - \alpha + r + 1} \frac{2^{r+1}}{(2r)!} (1 - \cos \xi)^r \left( \frac{2r - 1}{\beta - \alpha} \right) \]

\[ \prod_{m=\beta-r+1}^{\alpha+r-1} (\lambda - m) \prod_{m=\alpha}^{\beta} (\lambda - m) \]

\[ = \sum_{\substack{r \leq \beta - \alpha \leq \alpha + r \geq \beta - r \geq \alpha}} (-1)^{\beta - \alpha + r + 1} \frac{2^{r+1}}{(2r)!} (1 - \cos \xi)^r \left( \frac{2r - 1}{\beta - \alpha} \right) \]

\[ \left( \prod_{m=\beta-r+1}^{\alpha+r-1} (\lambda - m) \right)^2 \prod_{m=\alpha}^{\beta-r} (\lambda - m) \prod_{m=\alpha+r}^{\beta} (m - \lambda) \]

\[ = \sum_{\substack{r \leq \beta - \alpha \leq \alpha + r \geq \beta - r \geq \alpha}} c_{\alpha\beta r}(\xi) w_{\alpha\beta r}(\lambda) \]

Now we are ready to prove the theorem. From the key lemma (Lemma 5.4), it is immediate that \( w_{\alpha\beta}(\mu) = 0 \), hence \( g_{\alpha\beta}(\mu, \xi) = 0 \), for every \( \alpha \leq \mu \leq \beta \). Thus \( R_{\alpha\beta} \supseteq \{\alpha, \ldots, \beta\} \). It is obvious that \( c_{\alpha\beta r}(\xi) \geq 0 \) for all \( \xi \). It is also obvious that \( w_{\alpha\beta r}(\lambda) \geq 0 \), hence \( g_{\alpha\beta}(\lambda, \xi) \geq 0 \), for \( \lambda \in [\underline{a}_{\alpha\beta}, \overline{b}_{\alpha\beta}] \). Thus \( R_{\alpha\beta} \supseteq [\underline{a}_{\alpha\beta}, \overline{b}_{\alpha\beta}] \) and the theorem is proved.

6. Supporting Evidence for the Conjecture. In this section, we provide “supporting evidence” for Conjecture 3.4. The main idea is to narrow down the stability region by checking several \( \xi \) values. First using \( \xi = \pi \), we have obtained the following restriction.

**Proposition 6.1.** We have

\[ R_{\alpha\beta} \subseteq [\alpha, \beta] \]

**Proof.** Let \( q = \beta - \alpha \) and

\[ Q(\lambda) = f_{\alpha\beta}(\lambda, \pi) \]

where \( f_{\alpha\beta} \) is the amplification factor. It is obvious that \( Q \) is the unique polynomial of degree \( q \) such that

\[ Q(k) = e^{k\pi i} = (-1)^k, \quad \alpha \leq k \leq \beta \]

The Intermediate Value Theorem then implies that for \( \alpha \leq k \leq \beta - 1 \) there exist points \( \lambda_k \) such that \( k < \lambda_k < k + 1 \) and \( Q(\lambda_k) = 0 \). That is, \( Q \) is a polynomial of
degree \( q \) with \( q \) distinct zeros between \( \alpha \) and \( \beta \). Consequently, \( Q'(\lambda) \) is a polynomial of degree \( q - 1 \) with \( q - 1 \) distinct zero between \( \lambda_\alpha \) and \( \lambda_{\beta-1} \). So \( Q'(\lambda) \) must be of constant sign for \( \lambda < \lambda_\alpha \) and also for \( \lambda > \lambda_{\beta-1} \).

Now suppose that \( \alpha \) is even so that \( Q'(\alpha) = 1 \). Then the Mean Value Theorem implies that \( Q'(\lambda) < 0 \) for some \( \lambda \) with \( \alpha \leq \lambda \leq \lambda_\alpha \) and then the above argument gives \( Q'(\lambda) < 0 \) for \( \lambda < \lambda_\alpha \). So \( Q(\lambda) > 1 \) for \( \lambda < \alpha \). If one applies a similar argument to the case that \( \alpha \) is odd and the cases that \( \beta \) is even and odd, we find that

\[
|f_{\alpha\beta}(\lambda, \pi)| = |Q(\lambda)| > 1, \quad \text{for} \quad \lambda < \alpha \text{ or } \lambda > \beta
\]

So the scheme is unstable for \( \lambda \) outside the interval \( [\alpha, \beta] \) and we have proved the proposition. \( \square \)

Next using a sufficiently small \( \xi \), we have obtained the following restriction.

**Proposition 6.2.** We have

\[
R_{\alpha\beta} \subseteq T_{\alpha\beta}
\]

where

\[
T_{\alpha\beta} = \cdots \cup \left[ \frac{3}{2} \alpha - 4, \frac{3}{2} \alpha - 3 \right] \cup \left[ \frac{3}{2} \alpha - 2, \frac{3}{2} \alpha - 1 \right] \cup \left[ \frac{3}{2} \alpha, \frac{3}{2} \alpha + 1 \right] \cup \left[ \frac{3}{2} \alpha + 1, \frac{3}{2} \alpha + 2 \right] \cup \left[ \frac{3}{2} \alpha + 3, \frac{3}{2} \alpha + 4 \right] \cup \cdots
\]

**Proof.** From the key lemma (Lemma 5.4), we immediately see that

\[
g_{\alpha\beta}(\lambda, \xi) = l_{\alpha\beta}(\lambda) \left( 1 - \cos \xi \right)^r + O \left( (1 - \cos \xi)^{r+1} \right)
\]

where

\[
r = \left\lfloor \frac{\beta - \alpha + 1}{2} \right\rfloor
\]

and where

\[
l_{\alpha\beta}(\lambda) = \frac{2^{r+1}}{(2r)!} \left( \frac{2r - 1}{\beta - \alpha} \right) \left( \frac{\prod_{\beta - r + 1 \leq m \leq \alpha + r - 1} (\lambda - m)}{\prod_{\alpha \leq m \leq \beta - r} (\lambda - m)} \right)^2 \left( \frac{\prod_{\alpha + r \leq m \leq \beta} (m - \lambda)}{\prod_{\alpha \leq m \leq \beta - r} (\lambda - m)} \right)
\]

For a sufficient small \( \xi \), the sign of \( g_{\alpha\beta}(\lambda, \xi) \) is determined by that of \( l_{\alpha\beta}(\lambda) \). Thus if \( \lambda \in R_{\alpha\beta} \) then we should have \( l_{\alpha\beta}(\lambda) \geq 0 \), that is,

\[
\prod_{\alpha \leq m \leq \beta - r} (\lambda - m) \prod_{\alpha + r \leq m \leq \beta} (m - \lambda) \geq 0.
\]

The proposition merely states this fact explicitly. \( \square \)

Next we will show that the conjecture holds for many small values of \( \beta - \alpha \). We will do so by carrying out computations on computer algebra systems for each values of \( \alpha \) and \( \beta \). The following (naturally expected) lemma greatly reduces the number of cases (\( \alpha \) and \( \beta \)) to check.

**Lemma 6.3.** For any \( r \in \mathbb{Z} \) we have

\[
R_{\alpha + r, \beta + r} = R_{\alpha, \beta} + r
\]
that is

\[ R_{\alpha+r,\beta+r} = \{ \lambda + r : \lambda \in R_{\alpha,\beta} \} \]

\textbf{Proof.} Note

\[ L_{\alpha+r,\beta+r,\nu} (\lambda) = \prod_{\substack{\alpha+r \leq m \leq \beta+r \atop m \neq \nu}} \frac{\lambda - m}{\nu - m} \]

\[ = \prod_{\substack{\alpha+r \leq m \leq \beta+r \atop m \neq \nu}} \frac{(\lambda - r) - (m - r)}{(\nu - r) - (m - r)} \]

\[ = \prod_{\alpha \leq m \leq \beta \atop m \neq \nu} \frac{(\lambda - r) - m}{(\nu - r) - m} = L_{\alpha,\beta,\nu-r} (\lambda - r) \]

Thus

\[ f_{\alpha+r,\beta+r} (\lambda, \xi) = \sum_{\alpha+r \leq \nu \leq \beta+r} L_{\alpha+r,\beta+r,\nu} (\lambda) \ e^{i\nu \xi} = \sum_{\alpha+r \leq \nu \leq \beta+r} L_{\alpha,\beta,\nu-r} (\lambda - r) \ e^{i\nu \xi} \]

\[ = e^{i\xi} \sum_{\alpha \leq \nu \leq \beta} L_{\alpha,\beta,\nu-r} (\lambda - r) \ e^{i\nu \xi} = e^{i\xi} f_{\alpha,\beta} (\lambda - r, \xi) \]

Thus

\[ |f_{\alpha+r,\beta+r} (\lambda, \xi)|^2 = |e^{i\xi} f_{\alpha,\beta} (\lambda - r, \xi)|^2 = |f_{\alpha,\beta} (\lambda - r, \xi)|^2 \]

Hence we see immediately that

\[ R_{\alpha+r,\beta+r} = \left\{ \lambda : \forall \xi \ |f_{\alpha+r,\beta+r} (\lambda, \xi)|^2 \leq 1 \right\} \]

\[ = \left\{ \lambda : \forall \xi \ |f_{\alpha,\beta} (\lambda - r, \xi)|^2 \leq 1 \right\} \]

\[ = \left\{ \lambda : \lambda - r \in R_{\alpha,\beta} \right\} \]

\[ = \{ \lambda + r : \lambda \in R_{\alpha,\beta} \} \]

\[ = R_{\alpha,\beta} + r \]

\[ \square \]

Note that the conjectured expression for \( R_{\alpha,\beta} \) also satisfies this translation property. Thus we only need to study the two cases: \( \alpha = -\beta \) and \( \alpha = -\beta + 1 \). We will only show the case \( \alpha = -\beta \) here. The same idea can be applied to the case \( \alpha = -\beta + 1 \).

First we see immediately from Theorem 3.2 and the previous propositions (Propositions 6.1 and 6.2) that the conjecture holds for \( \beta = 1 \) and \( \beta = 2 \). Next we will attempt to prove the conjecture for \( \beta \geq 3 \) by using the following strategy. Let \( \lambda \notin [-1,1] \cup \{-\beta, \ldots, -\beta \} \). We need to show that \( \lambda \notin R_{\alpha,\beta} \). From Proposition 6.2, we already know that if \( \lambda \notin T_{\alpha,\beta} \) then \( \lambda \notin R_{\alpha,\beta} \). Thus we assume further that \( \lambda \in T_{\alpha,\beta} \). We need to show that \( \lambda \notin R_{\alpha,\beta} \). For this, we need to find “witness” for \( \xi \) such that \( g(\lambda, \xi) < 0 \). By rewriting/reindexing the key lemma (Lemma 5.4), we have

\[ g_{\alpha\beta} (\lambda, \xi) = p(\lambda) \ q(\lambda, \xi) \]
where
\[
p(\lambda) = \frac{2\beta+1}{(2\beta)!} \lambda^2 \prod_{m=1}^{\beta} (m^2 - \lambda^2)
\]
\[
q(\lambda, \xi) = \sum_{k=0}^{\beta-1} \frac{2^k}{(\beta + 1 + k) (2k+1)!} \prod_{m=1}^{k} (m^2 - \lambda^2) \left( \frac{8}{\lambda^2} \right)^{\beta+1+k}
\]

Since \( \lambda \not\in [-1,1] \cup \{-\beta, \ldots, -1\} \) and \( \lambda \in T_{\alpha, \beta} \), we have \( p(\lambda) > 0 \). Hence we only need to find a witness such that
\[
q(\lambda, \xi) < 0
\]

Through numerous experiments (using the computer algebra system Maple), we came up with the following “candidate” witness (given implicitly):
\[
\lambda^2 (1 - \cos \xi) = 8
\]

Since \( \lambda \not\in [-1,1] \) and \( \lambda \in T_{\alpha, \beta} \), we have \( |\lambda| \geq 2 \). Thus the above expression is well defined, that is, there exists a value for \( \xi \) for each value of \( \lambda \). By inserting the candidate witness into \( q \), we have
\[
q = \sum_{k=0}^{\beta-1} \frac{2^k}{(\beta + 1 + k) (2k+1)!} \prod_{m=1}^{k} (m^2 - \lambda^2) \left( \frac{8}{\lambda^2} \right)^{\beta+1+k}
\]

It is sufficient to show that \( q < 0 \). By pulling \( S^{\beta+1} \lambda^{-4\beta} \) out from \( q \), we have
\[
q = S^{\beta+1} \lambda^{-4\beta} r
\]

where
\[
r = \sum_{k=0}^{\beta-1} \frac{2^k}{(\beta + 1 + k) (2k+1)!} \prod_{m=1}^{k} (m^2 - \lambda^2) \ 8^k \lambda^{2(\beta-k-1)}
\]

Since \( S^{\beta+1} \lambda^{-4\beta} > 0 \), it is sufficient to show that \( r < 0 \). Using the computer algebra system Maple, we explicitly computed the polynomial \( r \) for several small values of \( \beta \).

For \( \beta = 3 \), we get
\[
r = \frac{13}{180} \lambda^4 - \frac{56}{45} \lambda^2 + \frac{64}{45} = \frac{-13}{180} (16 - \lambda^2)(\lambda^2 - \frac{16}{13})
\]

We need to show that \( r < 0 \) for \( 3 > |\lambda| > 2 \). This is obviously true. Thus the conjecture holds for \( \beta = 3 \).

For \( \beta = 4 \), we get
\[
r = \frac{13}{315} \lambda^6 + \frac{12}{35} \lambda^4 - \frac{1184}{315} \lambda^2 + \frac{128}{35}
\]

We need to show that \( r < 0 \) for \( 3 > |\lambda| > 2 \) and for \( |\lambda| > 4 \). Let \( b = \lambda^2 - 2^2 \). Then we have
\[
r = \frac{13}{315} b^3 - \frac{16}{105} b^2 - \frac{944}{315} b - \frac{128}{15}
\]
We need to show that \( r < 0 \) for \( 5 > b > 0 \) and for \( b > 12 \). This is obviously true because all the coefficients are negative. In fact, \( r < 0 \) for \( b > 0 \). Thus the conjecture holds for \( \beta = 4 \).

For \( \beta = 5 \), by proceeding in the same way as for \( \beta = 4 \), we get

\[
    r = -\frac{563}{28350} b^4 - \frac{2588}{4725} b^3 - \frac{43856}{14175} b^2 - \frac{14272}{945} b - \frac{640}{21}
\]

Note again that all the coefficients are negative. Hence \( r < 0 \) for \( b > 0 \). Thus the conjecture holds for \( \beta = 5 \).

We proceeded in the same manner up to \( \beta = 82 \), verifying that all the coefficients of \( r \) (as a polynomial in \( b \)) are negative. (We could not try larger values of \( \beta \) due to the lack of computer memory). In the same way, we also verified the conjecture for the case \( \alpha = -\beta + 1 \) up to \( \beta = 82 \). Thus we have proved:

**Theorem 6.4.** The conjecture is true for \( \beta - \alpha \leq 164 \).

In order to extend this result to even larger values of \( \beta - \alpha \), one could use a computer with larger memory. But then, this approach will never give a proof that the conjecture holds for arbitrary \( \beta - \alpha \). Thus, it will be better to come up with an analytic proof for arbitrary \( \beta - \alpha \). For this, one might obtain explicit expressions for the coefficients of the polynomial \( r \) (given above) and show that they are negative. We tried this approach without success, mainly because the expressions for the coefficients are very complicated and we do not yet know how to simplify them enough to show that they are negative. We leave this as an open problem.

**REFERENCES**