UNIVERSITY OF WASHINGTON COLLEGE of ENGINEERING Civil & Environmental Engineering

A Community of Innovators

Locking Phenomena in the Material Point Method

Peter Mackenzie-Helnwein,

Pedro Arduino

Civil and Environmental Engineering

University of Washington, Seattle

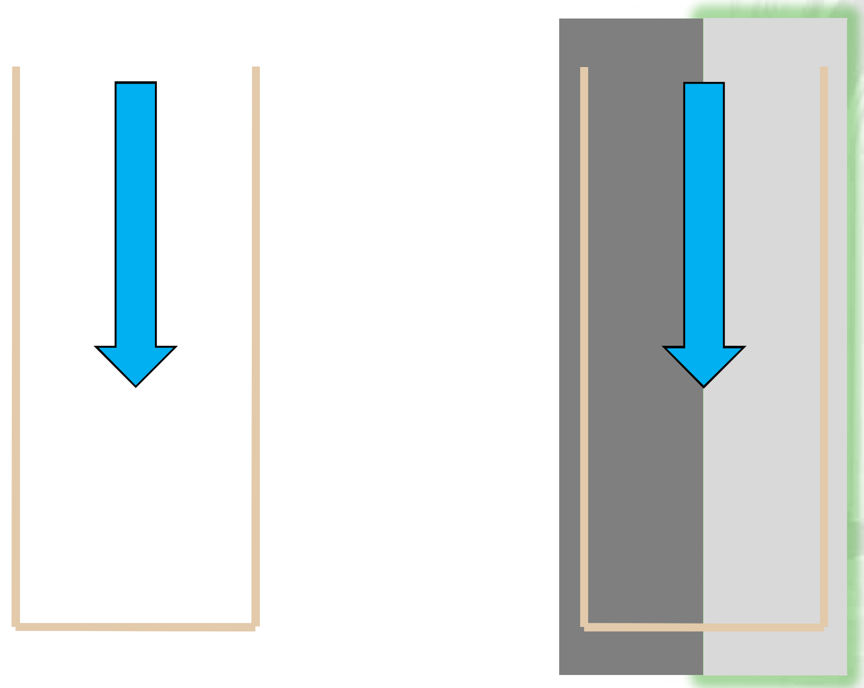
Overview

- **Introduction**
- **Sources of Locking**
  - Low order shape functions
  - Convection of a stressed body
- **Problem Resolution Approach**
  - Volumetric Locking
  - Shear Locking
- **Summary and Conclusions**

Introduction

- Last year's workshop:
  - Modeling fluid as nearly incompressible viscous material

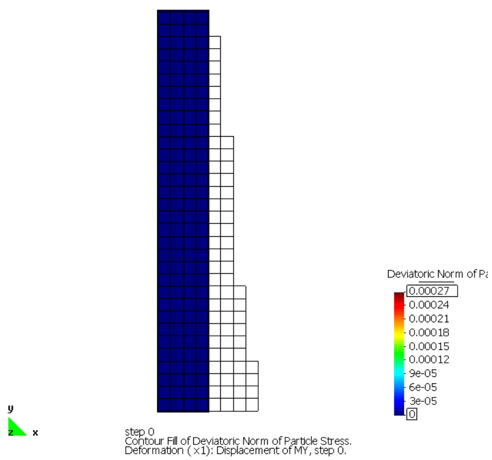
Burst of water into a glass



The Locking Problem

Volumetric locking intrinsic to the MPM

☹



- ***Proposed solution for locking***
  - *Kinematic constraints on the background grid relaxed by smoothened volume change*

$$\Rightarrow \quad \bar{\theta} = \frac{\sum_{p \in CV} \operatorname{tr} \boldsymbol{\varepsilon}_p \, m_p / \rho_p}{\sum_{p \in CV} m_p / \rho_p}$$

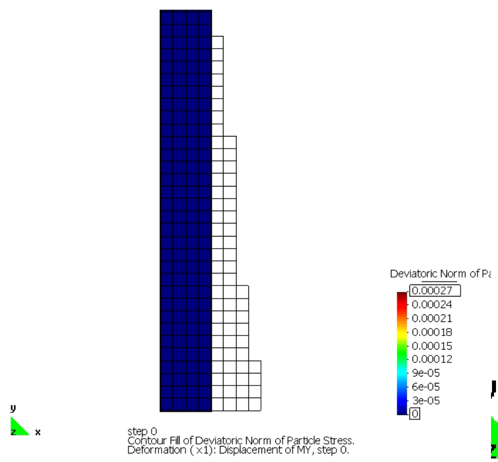
$$\Rightarrow \quad p := \bar{p}_{CV} = \rho \frac{\partial \bar{U}(\bar{\theta})}{\partial \bar{\theta}}$$

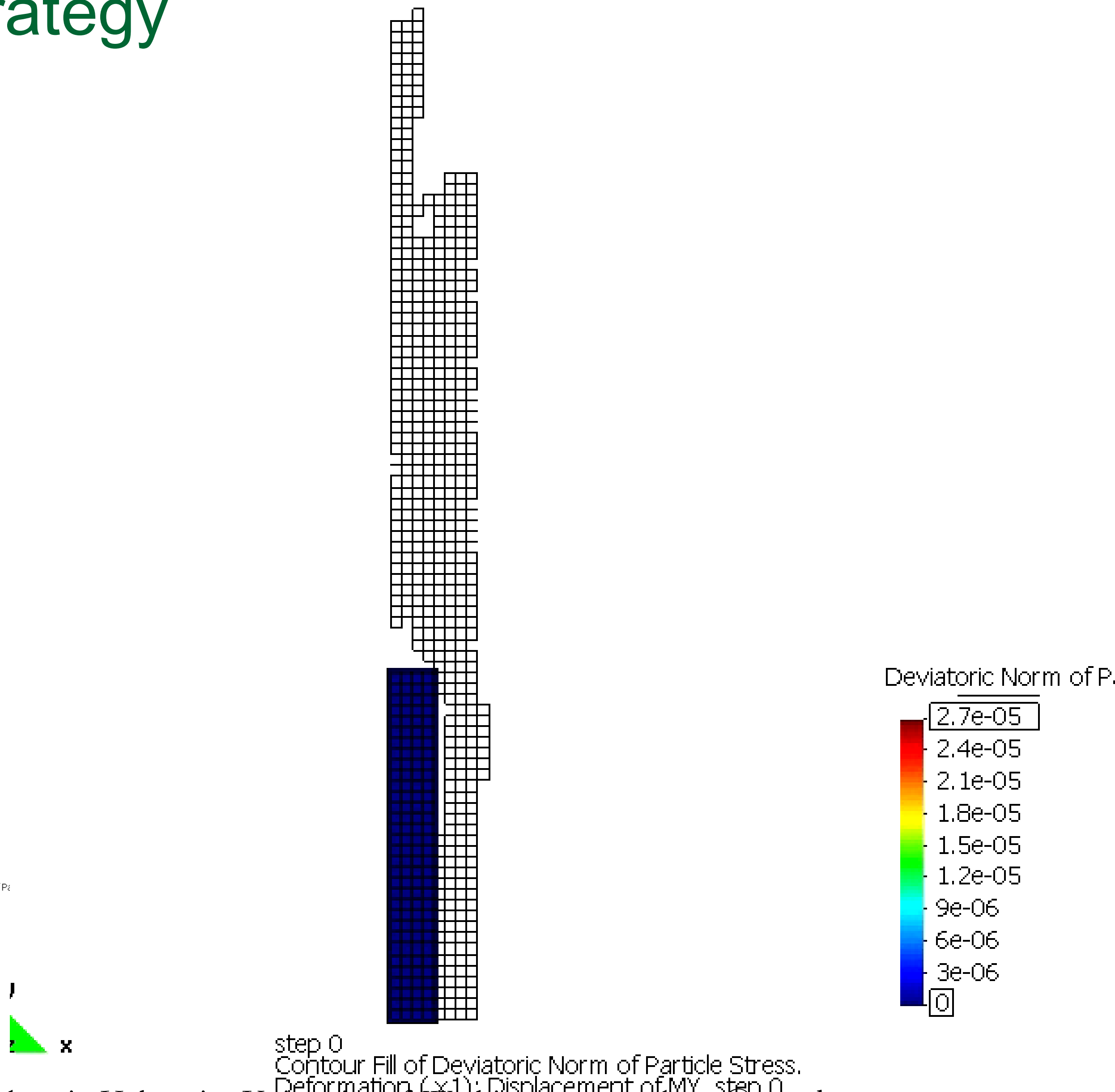
Anti-Locking Strategy

Modified MPM

locking resolved !

Standard MPM

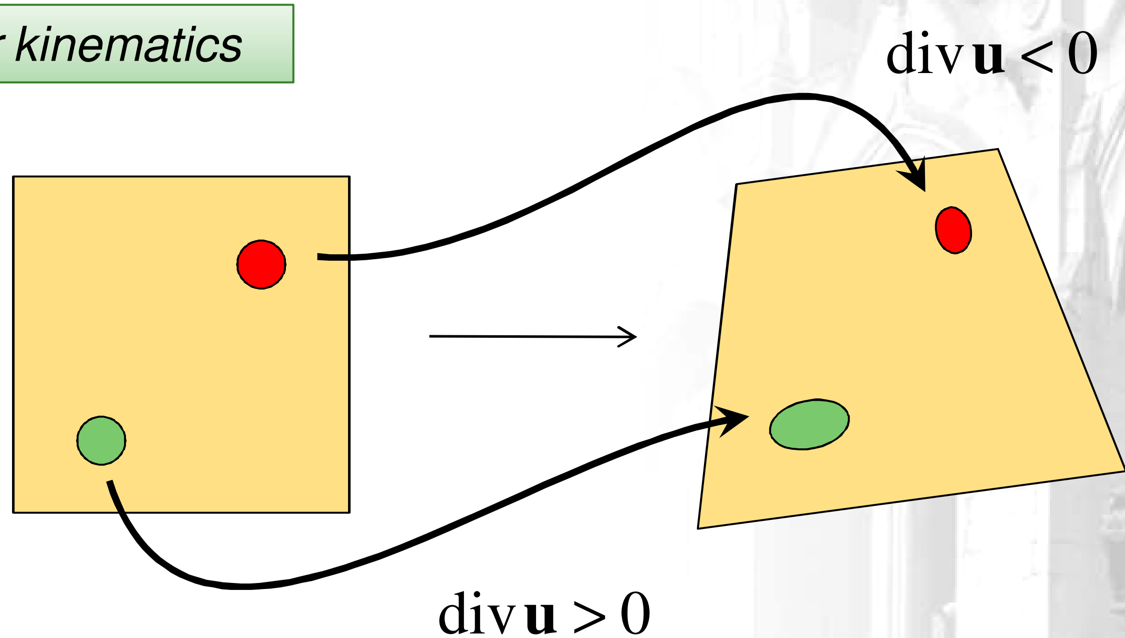




Sources of Locking

- Shape change >> "Volumetric Locking"

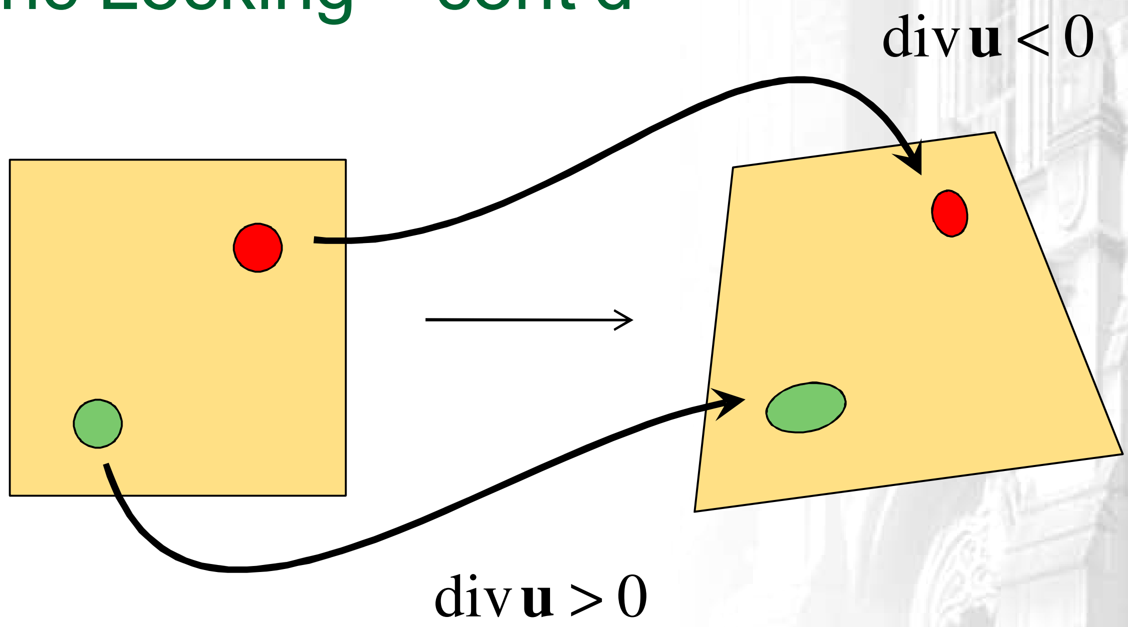
Low-order kinematics



Excess energy

$$E = \int_{\Omega} \frac{1}{2} k (\operatorname{div} \mathbf{u})^2 \, d\Omega > 0$$

Volumetric Locking – cont'd



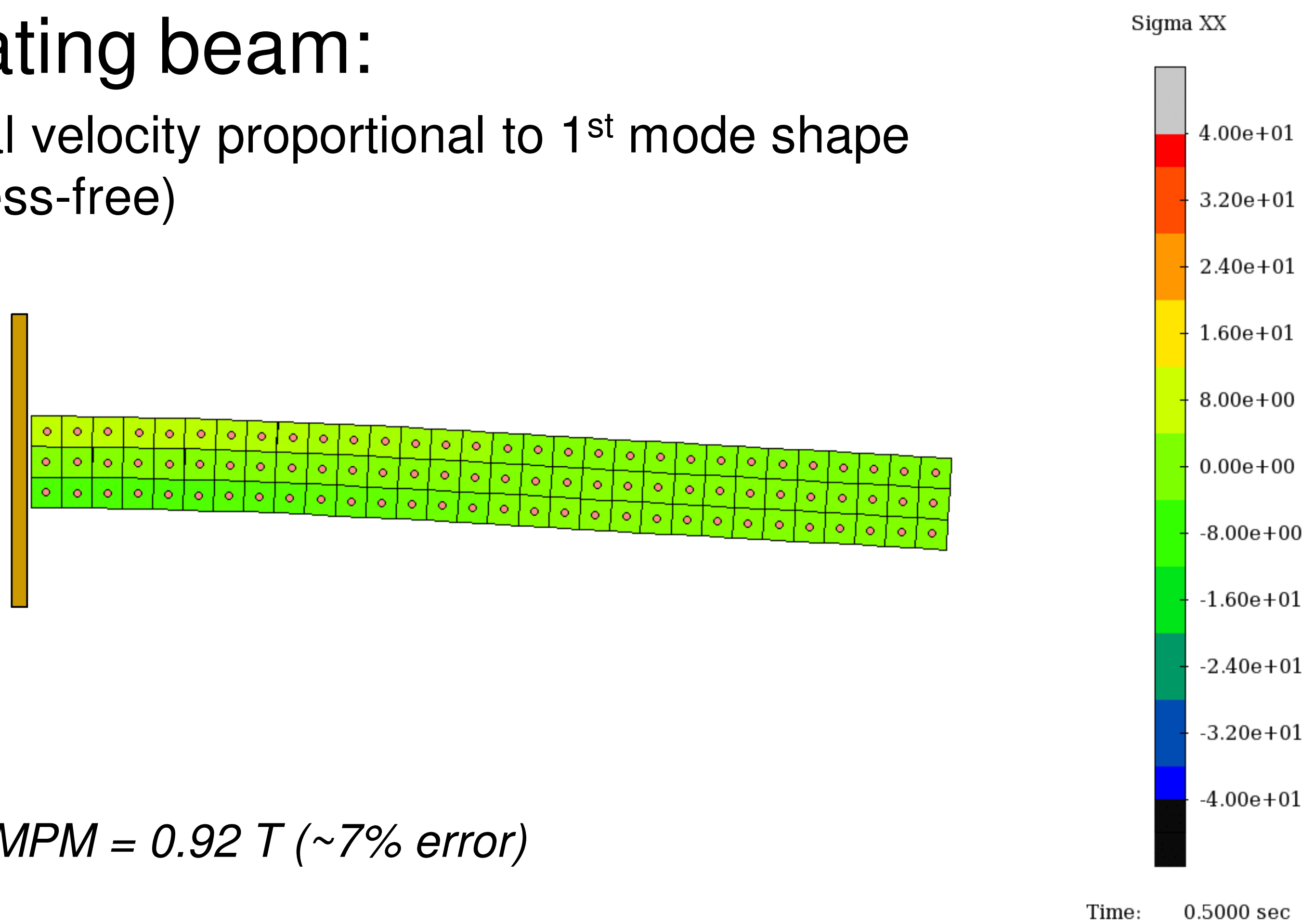
$$\int_{CV} (\bar{\theta} - \operatorname{div}\mathbf{u}) w \, dm = 0 \qquad \Rightarrow \bar{\theta} = \int_{CV} \operatorname{div}\mathbf{u} \, dm \Big/ \int_{CV} dm$$

$$\Rightarrow p = \rho \bar{k} \, \bar{\theta}$$

$$E = \int_{CV} \frac{1}{2} \bar{p} \operatorname{div}\mathbf{u} \, dm = \frac{1}{2} \bar{k} \bar{\theta}^2 m_\Omega \to 0$$

Sources of Locking

- Vibrating beam:
  - initial velocity proportional to 1st mode shape (stress-free)

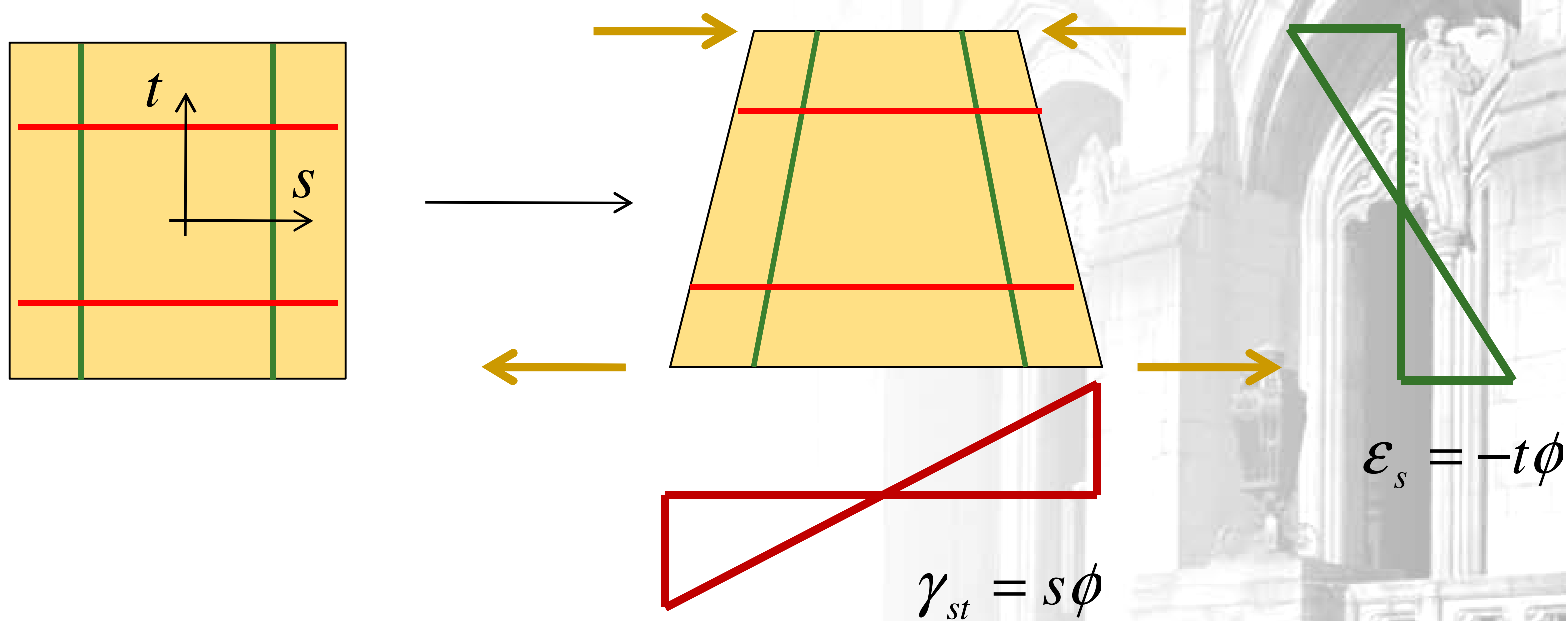


Period MPM = 0.92 T (~7% error)

Sources of Locking

- Bending deformation >> "Shear Locking"

Low-order kinematics

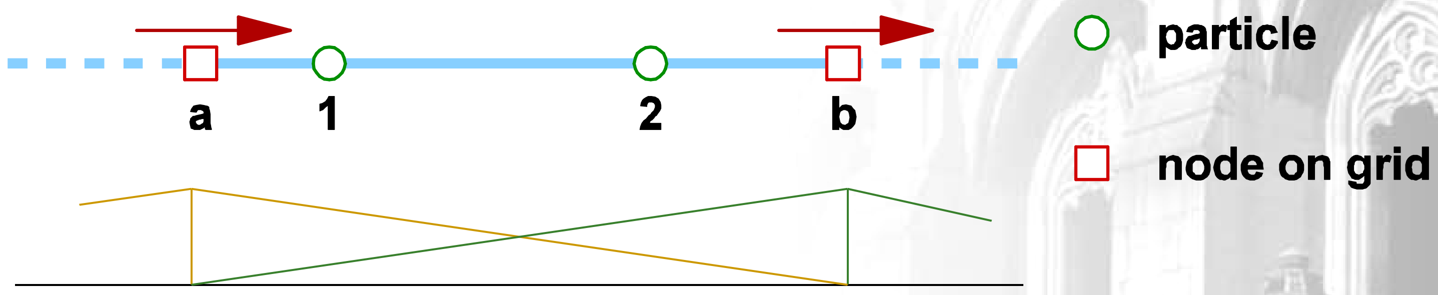


Excess energy

$$E^h = \frac{1.5+\nu}{1+\nu} E_{bending}$$

Sources of Locking

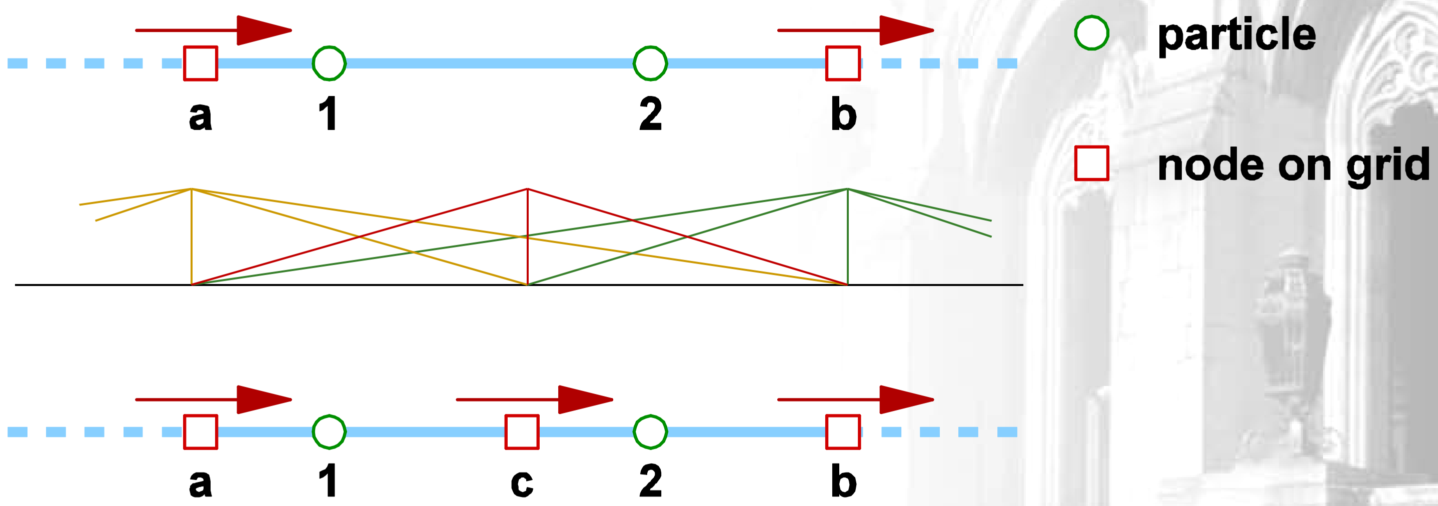
- Convection of a stressed body in quasi-static equilibrium
  - Known as “cell crossing error”



$$f_a = \frac{1}{h}(\sigma_1 m_1 + \sigma_2 m_2) \qquad f_b = -\frac{1}{h}(\sigma_1 m_1 + \sigma_2 m_2)$$

Sources of Locking

- Convection of a stressed body in quasi-static equilibrium
  - Known as "cell crossing error"

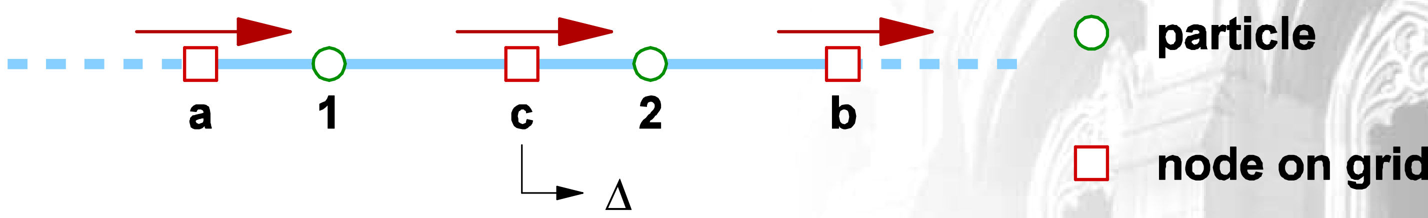


The black sheep

$$f_c = \frac{2}{h}(-\bar{\sigma}_1 m_1 + \bar{\sigma}_2 m_2) \neq 0$$

Sources of Locking

- Convection of a stressed body in quasi-static equilibrium

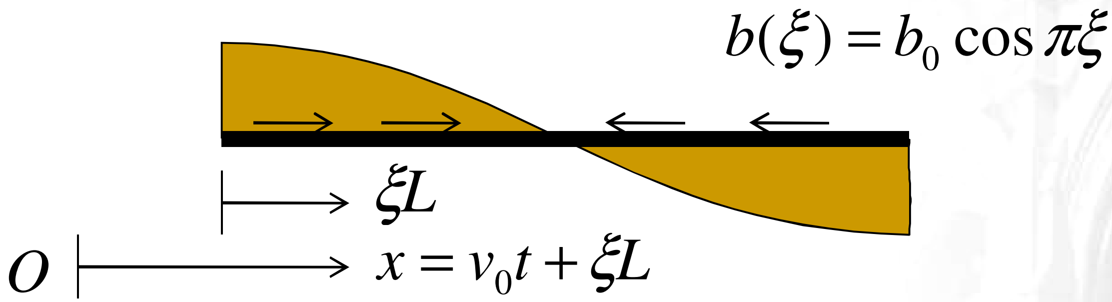


$$f_c = \frac{2}{h}(-\bar{\sigma}_1 m_1 + \bar{\sigma}_2 m_2) \neq 0 \qquad \Rightarrow \text{find } \Delta \text{ such that } f_c = 0$$

$$\bar{\sigma}_1 \to \frac{2m_1 + m_2}{m_1 + m_2}\bar{\sigma}_1 - \frac{m_2}{m_1 + m_2}\bar{\sigma}_2 \qquad \bar{\sigma}_2 \to -\frac{m_1}{m_1 + m_2}\bar{\sigma}_1 + \frac{m_1 + 2m_2}{m_1 + m_2}\bar{\sigma}_2$$

Problem Resolution Approach

- Convection of a stressed body in quasi-static equilibrium



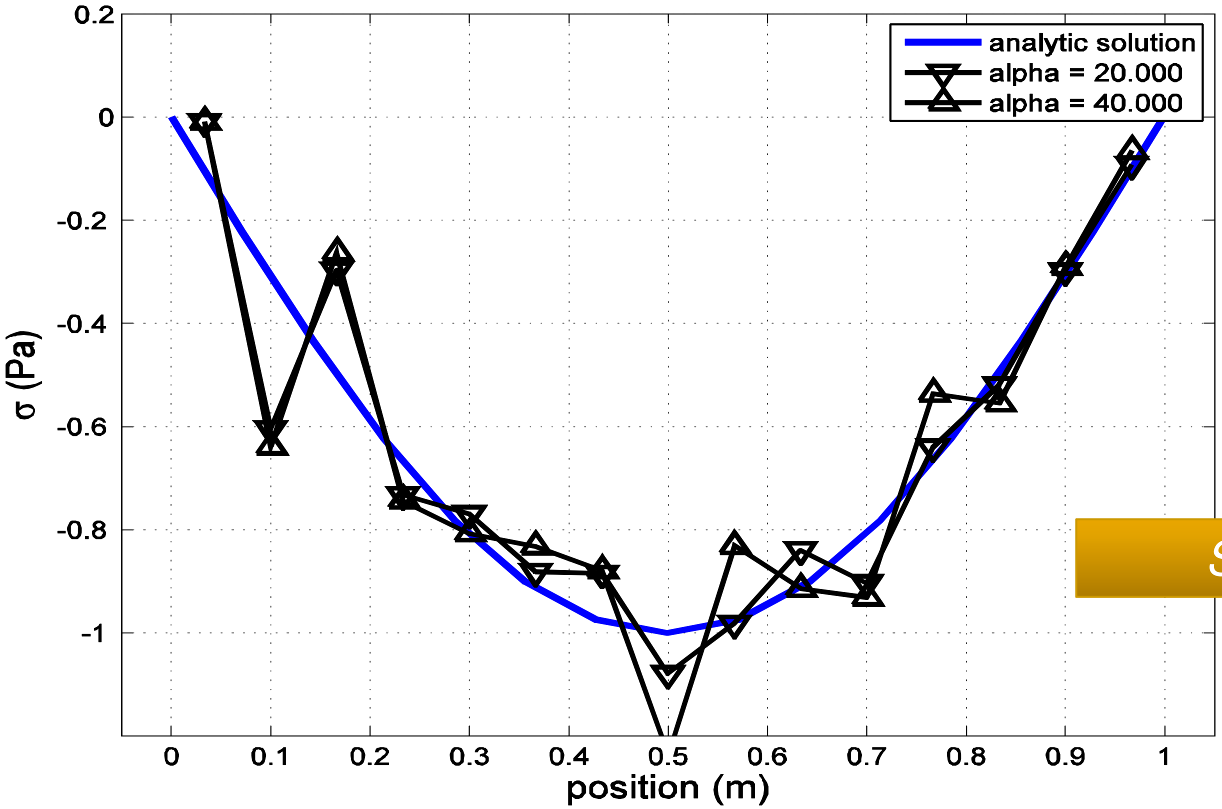
Analytic solution :

$$u(\xi) = v_0 t + \frac{b_0 L^2}{\pi^2 EA}\left(\cos \pi\xi - 1\right)$$

$$\sigma(\xi) = -\frac{b_0 L}{\pi A} \sin \pi\xi$$

Problem Resolution Approach

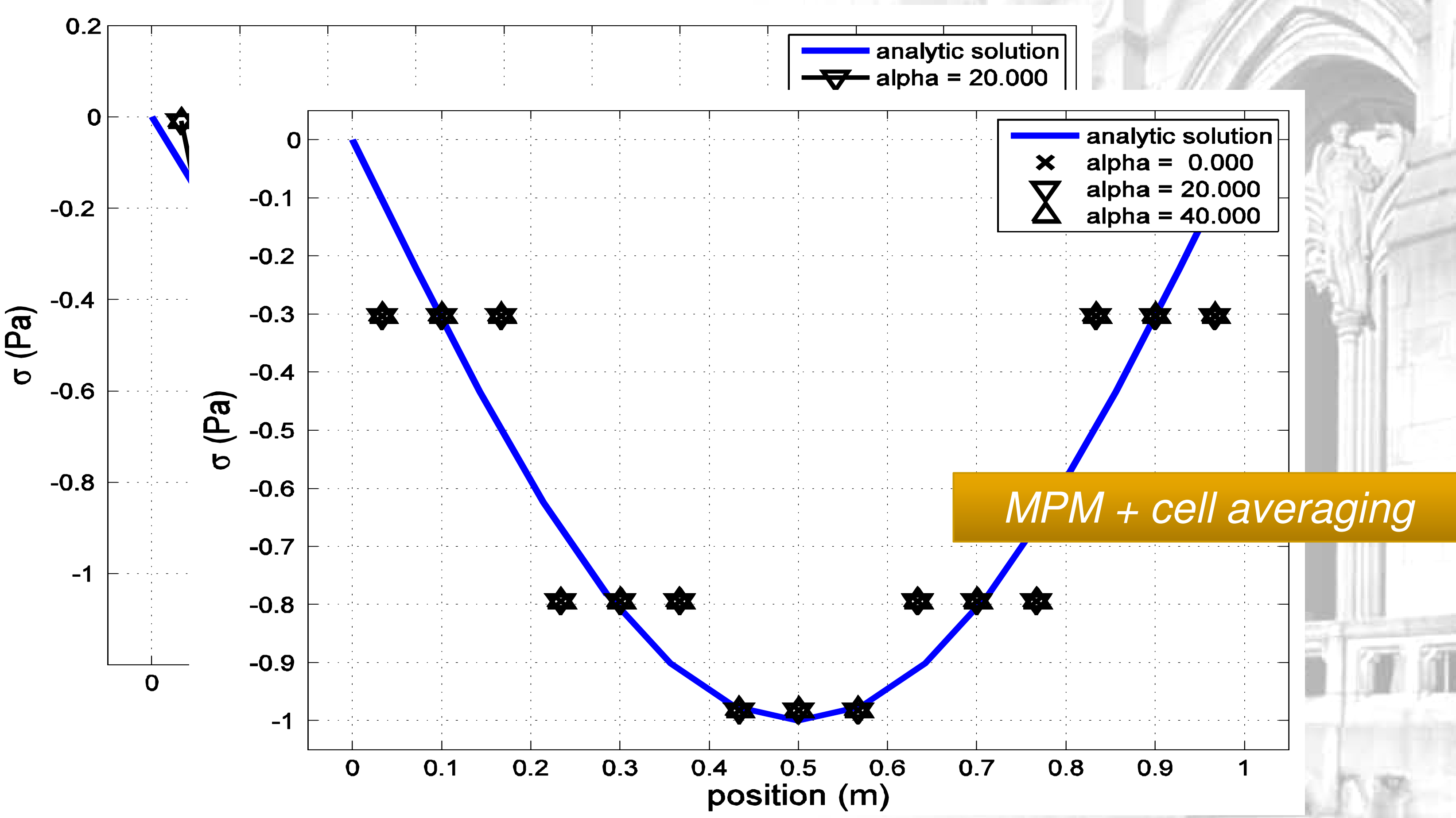
- Convection of αL in 1000 steps / *L (5 cells/L)*



Standard MPM

Problem Resolution Approach

- Convection of αL in 1000 steps / L (5 cells/L)



Problem Resolution Approach

- 2D/3D >> Filtering of strain/stress fields

$$\int_m \psi(\boldsymbol{\varepsilon}^h)\,dm - \int_m \bar{\mathbf{b}}\cdot\boldsymbol{\chi}\,dm - \int_{\partial V_\sigma} \bar{\mathbf{t}}\cdot\boldsymbol{\chi}\,dS + \int_m \bar{\boldsymbol{\sigma}}^h : (\boldsymbol{\varepsilon}(\mathbf{u}^h) - \boldsymbol{\varepsilon}^h)\,dm \to stationary$$

$$\{\bar{\boldsymbol{\sigma}}^h\} := \begin{Bmatrix} \bar{\sigma}_{ss} \\ \bar{\sigma}_{tt} \\ \bar{\sigma}_{st} \end{Bmatrix} \to \underbrace{\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & t & 0 \\ 0 & 1 & 0 & 0 & s \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]}_{=:\mathbf{S}} \underbrace{\begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{Bmatrix}}_{=:\boldsymbol{\beta}}$$

$$\{\boldsymbol{\varepsilon}^h\} := \begin{Bmatrix} \varepsilon_{ss} \\ \varepsilon_{tt} \\ 2\varepsilon_{st} \end{Bmatrix} \to [\mathbf{S}]\{\boldsymbol{\alpha}\}$$

Problem Resolution Approach

- 2D/3D >> Filtering of strain/stress fields

For each cell c

$$[\mathbf{H}_c] := \sum_{p\in c} [\mathbf{S}_p]^t [\mathbf{S}_p] m_p$$

$$\{\mathbf{R}_c\} := \sum_{p\in c} [\mathbf{S}_p]^t \{\overline{\boldsymbol{\sigma}}_p\} m_p$$

$$\Rightarrow \{\boldsymbol{\beta}_c\} = [\mathbf{H}_c]^{-1}\{\mathbf{R}_c\}$$

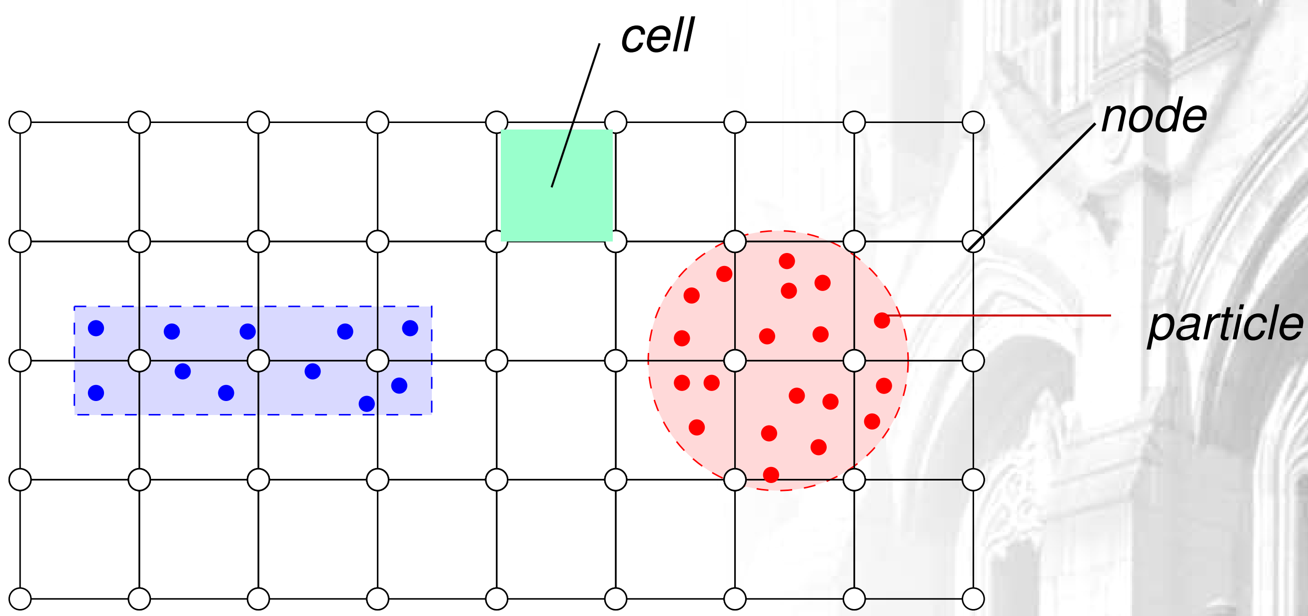
For each particle p in cell c

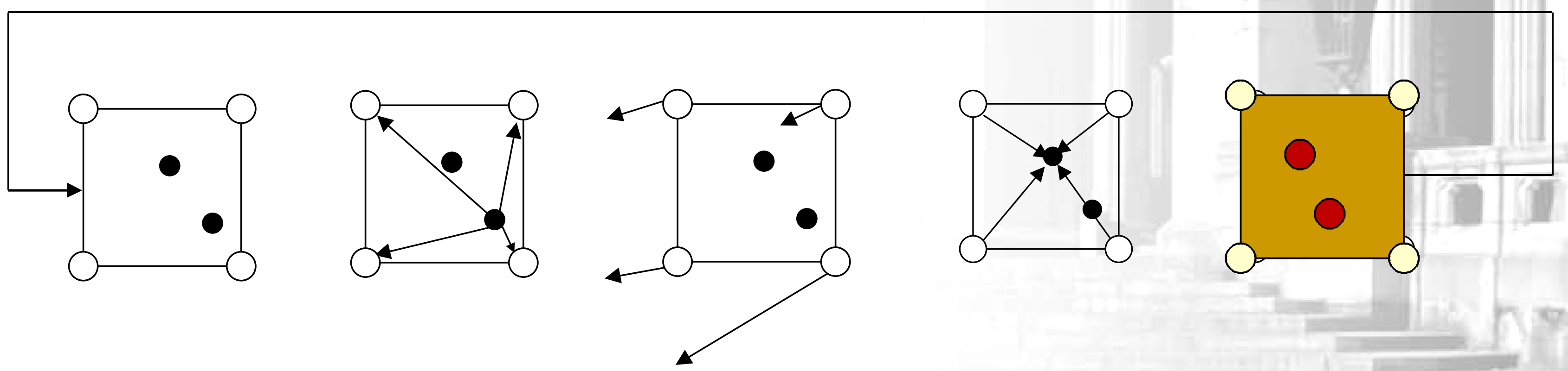
$$\{\overline{\boldsymbol{\sigma}}_p\} \rightarrow [\mathbf{S}_p]\{\boldsymbol{\beta}_c\} \quad \Rightarrow$$

elastic $\Rightarrow \{\boldsymbol{\varepsilon}_p\} = [\mathbf{C}]^{-1}\{\overline{\boldsymbol{\sigma}}_p\}$

plastic $\{\boldsymbol{\varepsilon}_p\} \rightarrow [\mathbf{S}_p]\{\boldsymbol{\alpha}_c\}$

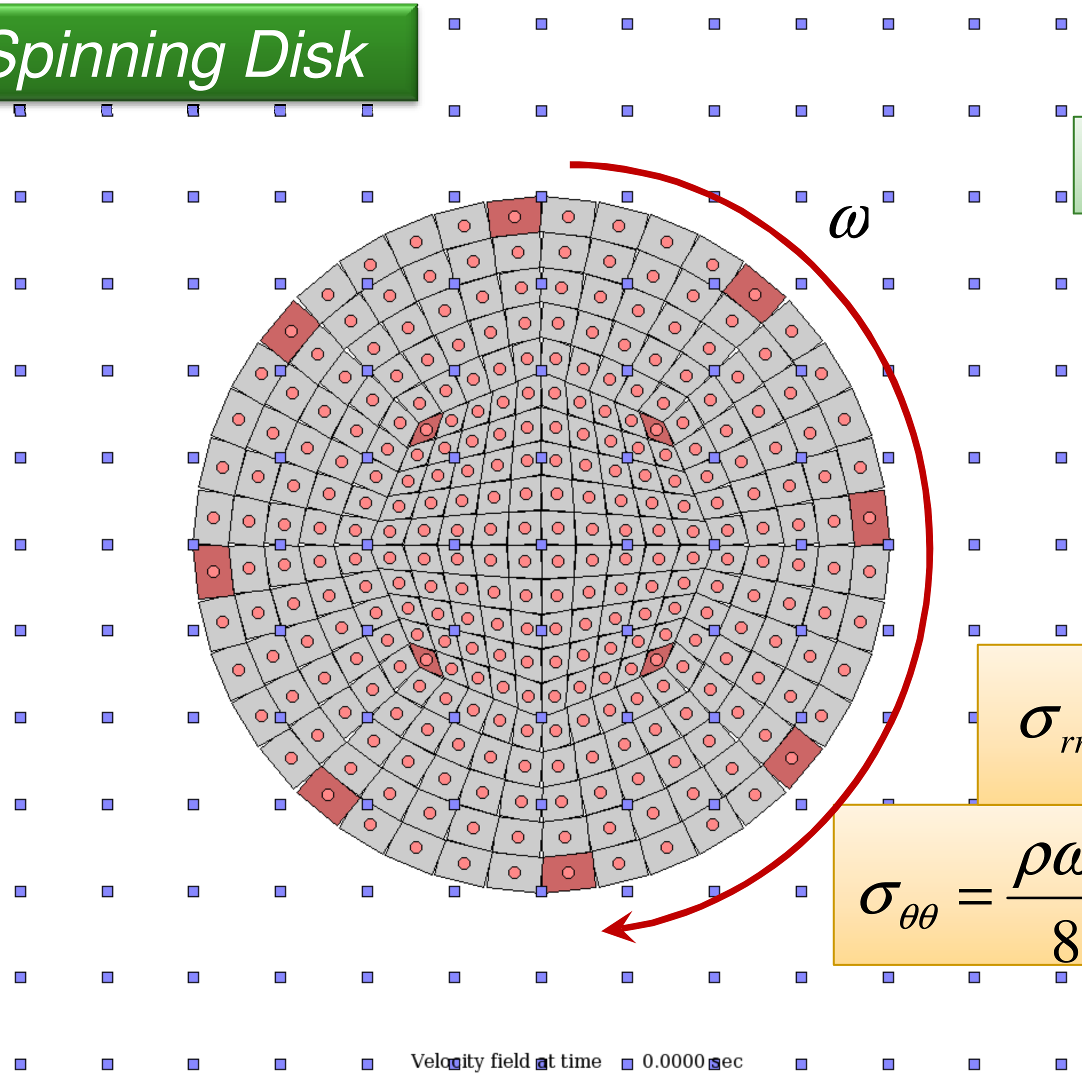
Adding Smoothing to the Basic Algorithm





The Ultimate Challenge

Spinning Disk



Initial conditions

$$u_r = 0$$

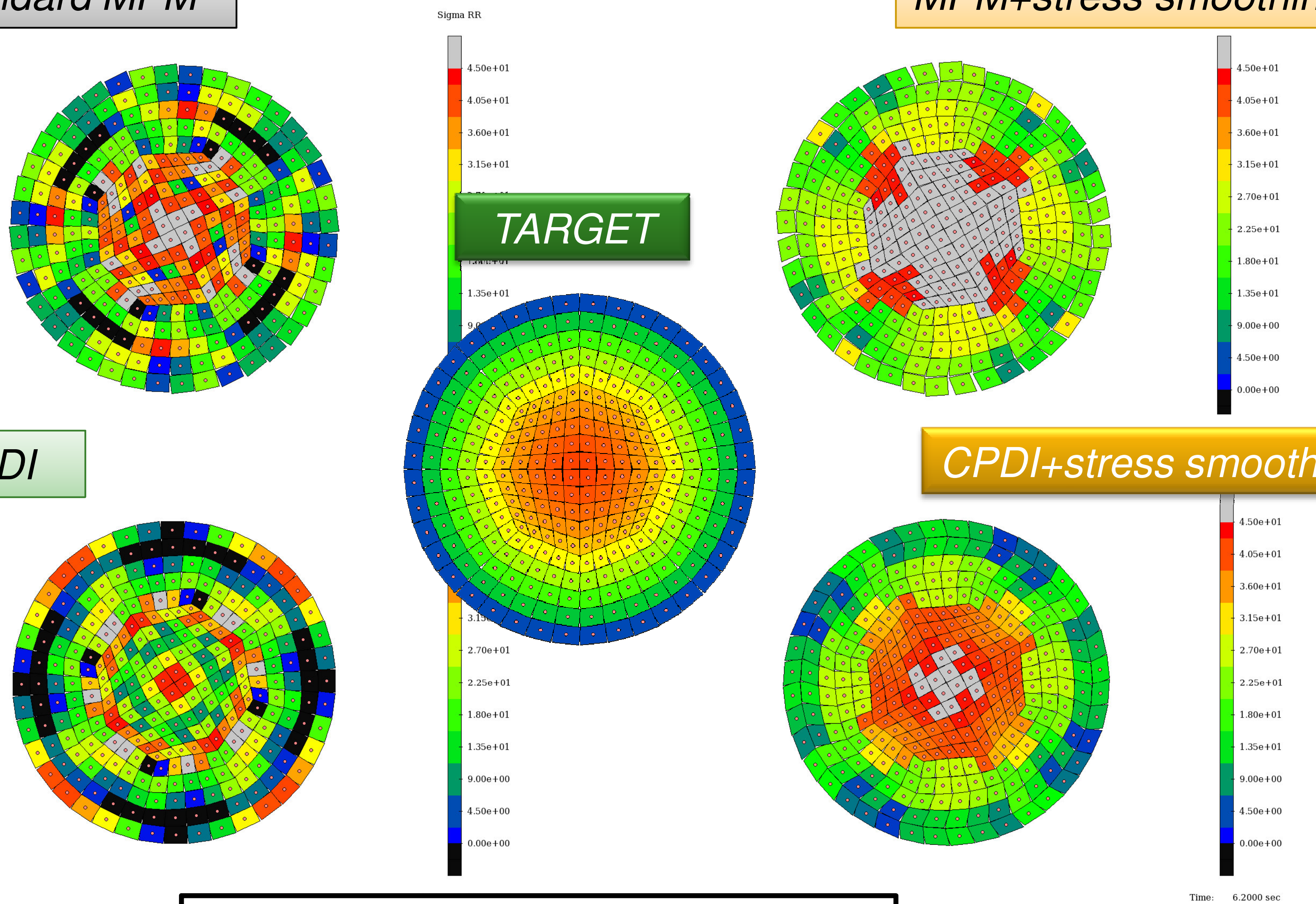
$$u_\theta = r\omega$$

$$\sigma_{rr} = \frac{\rho\omega^2}{8}(3+\nu)(a^2 - r^2)$$

$$\sigma_{\theta\theta} = \frac{\rho\omega^2}{8}\left((3+\nu)a^2 - (1+3\nu)r^2\right)$$

$$\sigma_{r\theta} = 0$$

The Ultimate Challenge: Spinning Disk $\sigma_{rr} = const.?$



After almost one full revolution

The Ultimate Challenge: Spinning Disk $\sigma_{r\theta} = 0$?

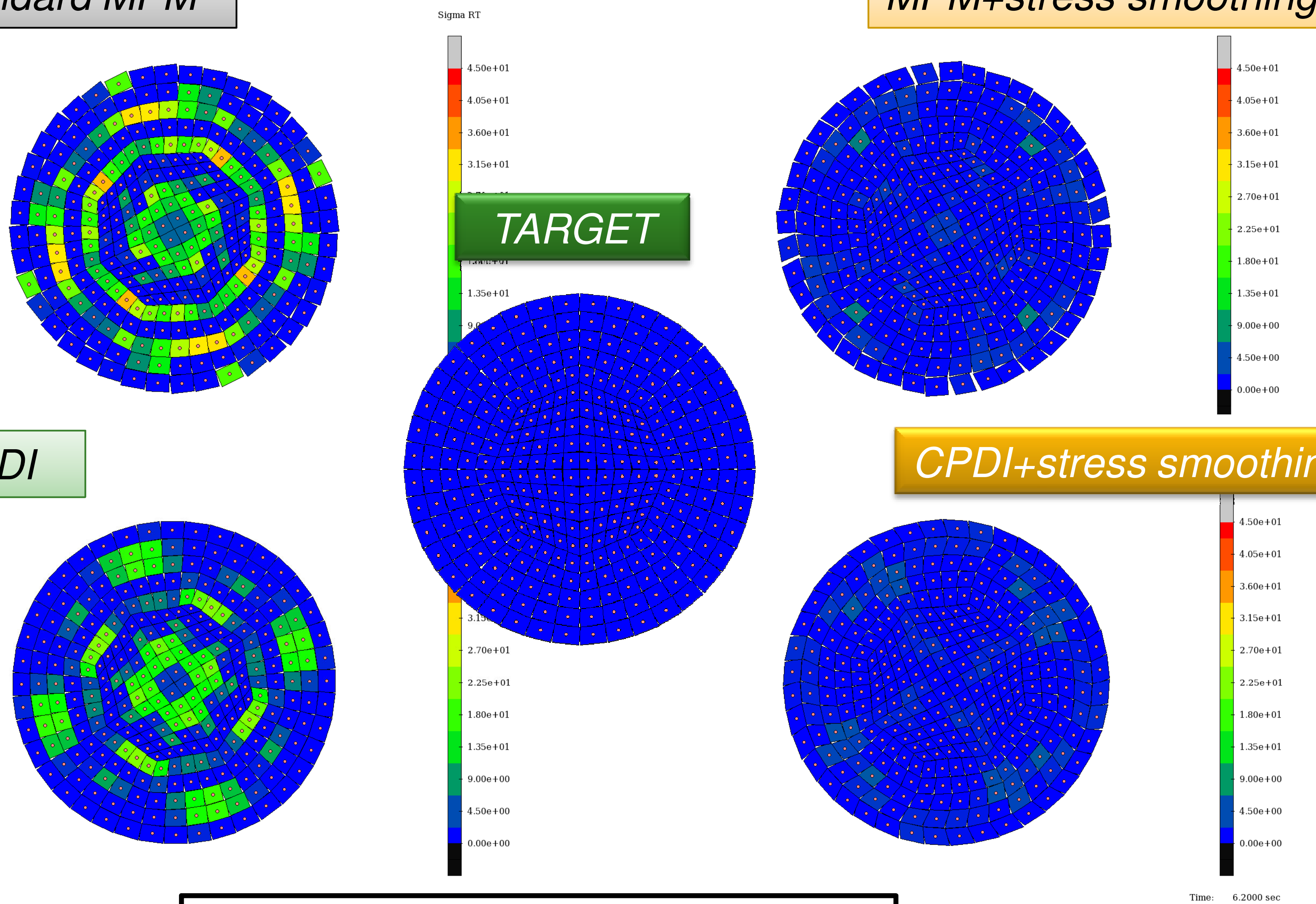
Standard MPM

MPM+stress smoothing

TARGET

CPDI

CPDI+stress smoothing



After almost one full revolution

pmackenz@uw.edu

Summary & Conclusions

- We identified that, thanks to using identical shape functions, MPM inherited locking from the FEM.
- We interpret the "cell crossing error" as "convection locking".
- We propose a cell-based smoothing strategy that filters/removes locking stresses/strains from the numerical solution.
- The proposed technique improves solutions for both standard MPM and CPDI but does not completely eliminate the problem (yet) .

Questions ?