A Simple Model for Epidemics

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These notes are based on lecture notes written by Professor Charles Peskin of the Courant Institute of Mathematical Sciences, NYU in April, 1981 for a presentation to a class called Mathink at NYU.

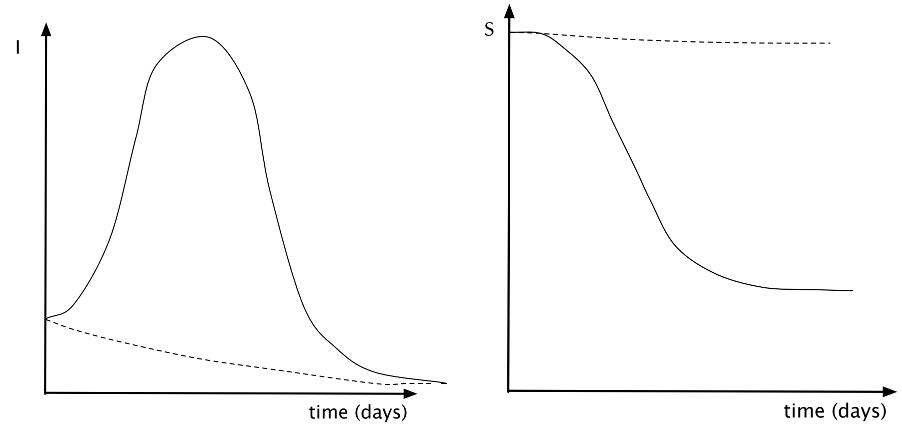
When a communicable disease breaks out in some community, public health officials naturally try to prevent it from becoming an epidemic. In order to think of effective intervention methods, one must understand as well as possible just what factors influence the spread of the disease. To this end, we make the simplest feasible model we can think of and see what we can learn from it.

We shall consider a community - perhaps an island or a remote village - that has a fixed population. We shall assume that the illness affects those who get it for a certain period, say one day, and that they are infectious only during that day. We suppose further that no individual can catch this disease more than once; this means people either become immune to it, or they die from it.

We know that the rate of infection rises with the number of sick people, and with the number of not-yet immune people; but we don't know precisely how these variables are related. We assume that we know roughly just how infectious the disease is, i.e. what the chances are that a person who has never had it will catch it.

Before we try constructing a simple model, we look at some qualitative pictures of possible courses the disease may take. We shall denote by I the number of people who are infected and can transmit the disease, also called infectives. Denote by S the number of people who have never had the disease and are therefore susceptible; and we sketch graphs of how these numbers may vary during the course of the disease in case the disease affects a relatively small part of the population (shown by - - - - - - - -), and in case it assumes epidemic proportions (shown by ————————) in Fig. 1. These graphs are based on our experience.

What determines which of the two phenomena, the mild or the epidemic, actually occurs? In the case of an epidemic, why does the number of sick first grow as fast as it does to some maximum and then decrease and die off, with the corresponding



(a) Number of infective people, as a function of time (b) Number of susceptible people, as a function of time

Figure 1: Possible sketches of the number of infectives and number of susceptibles as a function of time.

sharp drop in the healthy but susceptible population down to a steady number after the disease has run its course?

Our simple model is based on the assumptions outlined above, which we now summarize:

1. The population is considered fixed.
2. The illness lasts one day.
3. No individual can catch the disease a second time.

The fourth assumption, which we justify below, is that

4. The rate of infection is proportional to
   (a) the number of infective people
   (b) the number of people who have never had the disease.

With the notation

$$\begin{aligned} I &= \text{ number of infective people on a given day} \\ S &= \text{number of susceptible people on that day} \\ \hat{I} &= \text{number of infective people on the next day} \\ \hat{S} &= \text{number of susceptible people on the next day,} \end{aligned}$$

the last assumption above translates to

$$\hat{I} = AIS, \tag{1}$$

where the constant A measures the likelihood of catching the illness. We give two intuitive arguments below for the feasibility of (1), the symbolic expression of assumption 4. First, consider how the susceptible population changes. Since each sick person is no longer susceptible the next day (see assumptions 2, 3), the number of susceptible people decreases by the number of sick. So the number $\hat{S}$ of susceptible people the next day is

$$\hat{S} = S - \hat{I}. \tag{2}$$

Now, to see why (1) seems reasonable, suppose the disease in question is a viral infection, and that it is caught by breathing in airborne viruses. Then the number V of viruses floating around is proportional to the number I of infective people:

$$V = A_1 I,$$

where the constant of proportionality A_1 is some measure of how many viruses a sick person exhales. Now the probability that a susceptible person succumbs to the disease is proportional to the number of viruses in the breathing space, so

$$\text{probability per person of becoming ill} = A_2 V = A_2(A_1 I), \tag{3}$$

where the constant of proportionality A_2 is some measure of the virulence of the virus. Finally, to find the number of sick people the next day, we multiply the chance per person (3) by the number of susceptible people, arriving at

$$\hat{I} = A_2 A_1 IS.$$

Denoting the product $A_1 A_2$ of the two constants by a single symbol A yields the consequence (1) in favor of our fourth assumption.

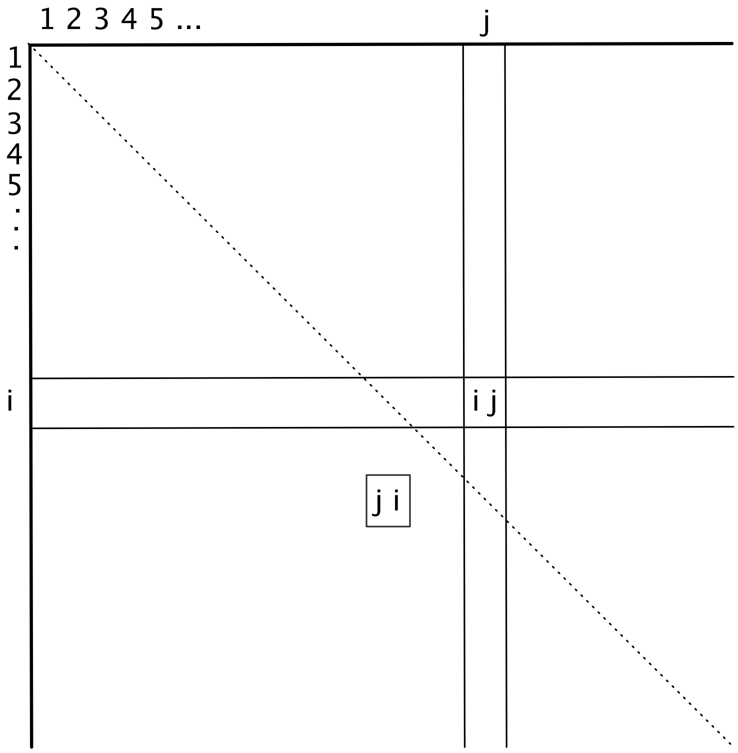


Figure 2:

[In mathematical language, we found that the per person chance (3) is a linear function A_2V of V, which, in turn, is a linear function $V = A_1I$ of I, yielding the composite function (3), linear in I.]

Alternatively, let us suppose that the disease is transmitted by actual physical contact, say handshakes, and that the members of the community in question actually encounter each other once a day. We can represent these encounters by numbering all the people, writing these numbers on the top row and the first column of a square array and interpreting the box in the ith row, jth column as a handshake between persons i and j. Since the box in the jth row, ith column represents a handshake between the same two people, we must not count any of the boxes below the diagonal drawn in the above array, and since handshakes of a person with himself do not count as encounters, we eliminate the boxes in the diagonal as well. Now suppose we assign the numbers $1, 2, \ldots, I$ to the infective people first, and $I+1, I+2, \ldots, I+S$ to the susceptible people. Clearly, encounters of a sick person with another sick person, or of an uninfected person with another uninfected person cannot lead to additional incidence of our disease. We are therefore interested only in the number of encounters between sick and susceptible people, and these are represented by the $I \times S$ boxes in the rectangular array above. Thus there are IS encounters between sick and susceptible people; the number of the sick the next day is therefore proportional to the number IS of encounters, and the constant A of proportionality is the probability of catching the disease, when a person is exposed to it: $\hat{I} = AIS$.

[In mathematical language: we construct a square $(I+S)$ by $(I+S)$ matrix. It is symmetric, the number of entries in the upper triangular array count the total number $\frac{1}{2}(I+S)(I+S-1)$ of encounters, and the I by S rectangular sub-array counts the number IS of encounters that potentially lead to more infection.] So, either way, we arrive at our equation $\hat{I} = AIS$. Notice that we assumed that all possible encounters between individuals occurs. The term used in the literature is that the system is well mixed.

We are now ready to use our pair of equations

$$\hat{I} = AIS$$
$$\hat{S} = S - \hat{I}$$

to compute, from given data A (depending only on the disease and not changing during its course), I (the number of sick on a given day and S (the number of susceptible on that day), the numbers $\hat{I}$ and $\hat{S}$ of sick and susceptible the next day. We use those results to determine the corresponding numbers for the day after that, then for the day after that, and so on, for as long as the disease seems to last. How

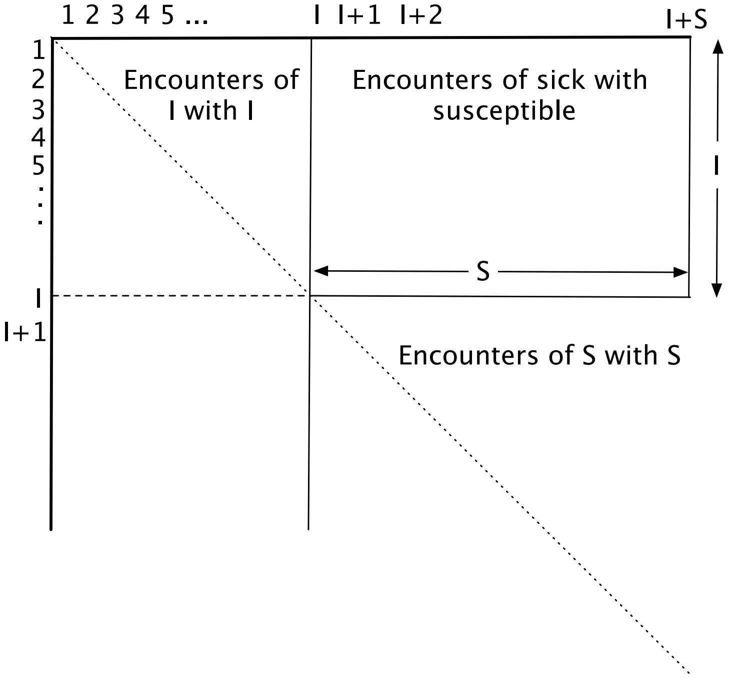

1 2 3 4 5 ...
I I+1 I+2
I+S
1
2
3
4
5
I
I+1
Encounters of I with I
Encounters of sick with susceptible
S
I
Encounters of S with S

Figure 3:

Day	I	S
1	20	980
2	39	941
3	74	867
4	128	739
5	189	550
6	208	342
7	142	200
8	57	143
9	16	127
10	4	123
11	1	122
12	0.24	121
13	0.06	121

Table 1: Example 1: $A = 0.002$, $I = 10$, $S = 1000$ to start.

do we tell how long that is? By noticing that practically no new people get sick after a certain number of days.

Let us actually carry out some calculations, record our results, then interpret their meaning for the disease in question. Let's start by assuming the number of susceptibles is $S = 1000$ and the number of infectives is $I = 10$. In the first example, let's set $A = 0.002$ and in the second, set $A = 0.0005$. The tables and the accompanying graphs show that for $A = 0.002$, an epidemic breaks out while for $A = 0.0005$, the number of sick decreases from the very first day.

The constant A in Example 2 is 1/4 the value of A in Example 1. This means that the disease of Example 2 was only 1/4 as transmittable as that previously considered, and this had the dramatic effect of turning the epidemic into an illness that affected only a small portion of the population: each day the number of new sick decreases by about 50%.

Suppose we reconsider Example 1, but this time we immunize three quarters of the susceptible population, thus reducing S to $\frac{1}{4}S$. Then our equation $\hat{I} = AIS$ becomes $\hat{I} = AI\frac{S}{4}$, and for Example 2 it becomes $\hat{I} = \frac{A}{4}IS$ with $A = 0.002$. In both cases, the number of people who become ill the next day shrunk to 1/4 of what it was in the original example.

We postpone a fuller discussion of how changes in the parameters affect the course of the disease until we have run more examples.

Day	I	S
1	5	995
2	2.5	993
3	1.2	991
4	0.6	991
5	0.3	990
6	0.2	990
7	0.1	990
8	0	990

Table 2: Example 2: $A = 0.0005$, $I = 10$, $S = 1000$ to start.

The difference between Examples 1 and 2 is that in Example 1, the number of new sick each day, for six days, is larger than that on the previous day, while in Example 2, the number of sick each day decreases, right from the first day on. We saw that when A or S were "too large" we have an epidemic, and if they are "small" the disease dies down immediately. What is "too large"? In our model, having fewer new sick people one day than the day before means $\hat{I} < I$ or $\hat{I} - I < 0$ or

$$\hat{I} - I = AIS - I = I(AS - 1) < 0.$$

Since I is positive, this will happen if $AS < 1$. If $AS > 1$, then $\hat{I} > I$, i.e. more people are getting sick today than yesterday. Indeed, in Example 1, $AS = (0.003)10^3 = 2 > 1$, and there was an epidemic; while in Example 2, $AS = (0.0005)10^3 = .5 < 1$, and the disease took its mild course.

Even in an epidemic, the number of new sick people, after its peak, decreases. Since A doesn't change, this tells us that S must decrease, and when it gets so small that $AS < 1$, then the disease dies down. This is quite feasible, since the number of susceptible people decreases precisely by the number of sick the previous day.

Let us take another value of A leading to a product of AS just a little larger than 1 and see if we get a very mild epidemic. In Example 3, use $A = 0.0015$. In fact, we do get a milder epidemic than Example 1.

What is the total population? How many people actually get sick?

Day	I
1	15
2	22
3	32
4	45
5	59
6	73
7	83
8	84
9	73
10	57
11	39
12	24
13	14
14	8
15	5
16	2
17	1
18	1
19	0
20	0

Table 3: Example 3: $A = 0.0015$, $I = 10$, $S = 1000$ to start. ($AS = 1.5$)

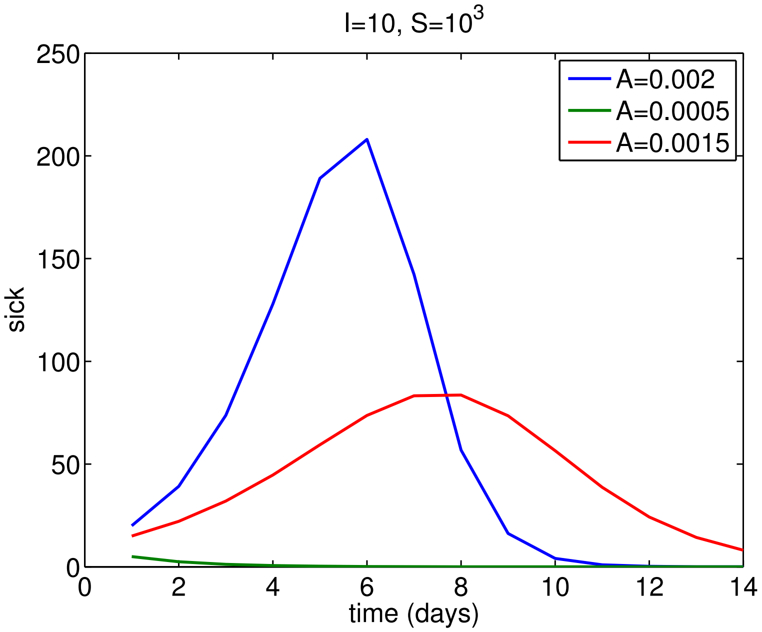

I=10, S=10^3
A=0.002
A=0.0005
A=0.0015
sick
time (days)
0
50
100
150
200
250
0
2
4
6
8
10
12
14

Figure 4: