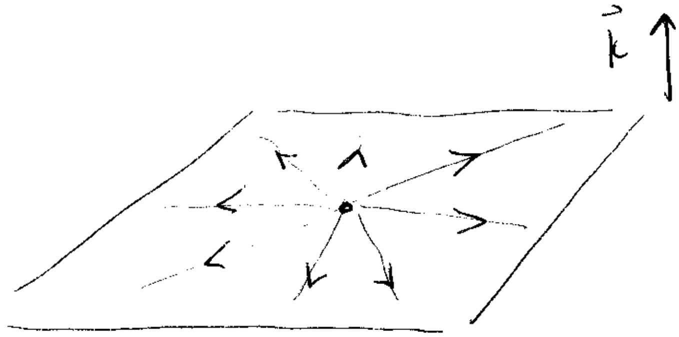
Math. 311
Set 18

p. 222 (4, 9(b,c))
Sec. 4.5.

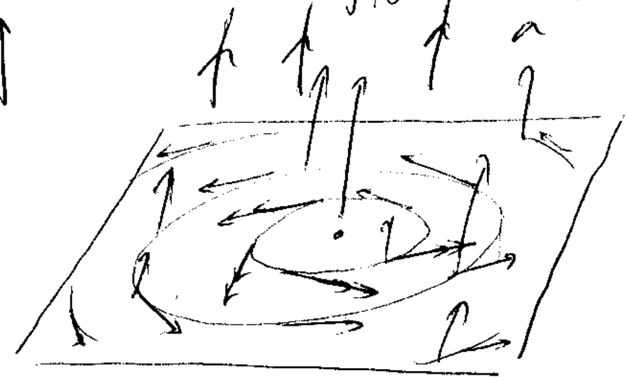
4.5.4 Verify directly that the field $\vec{G} = -[\log(x^2+y^2)\vec{k}]$ is a vector potential for $F = \frac{2}{x^2+y^2}\vec{k}\times\vec{R}$.

$$\nabla\times G = -\nabla\log(x^2+y^2)\times\vec{k} = -\frac{2x\vec{i}+2y\vec{j}}{x^2+y^2}\times\vec{k}$$

$$= k\times\frac{2\vec{R}}{R^2} = \vec{F}.$$



Flow lines for $\frac{2\vec{R}}{R^2}$



Flow lines for $\vec{F} = \vec{k}\times\left(\frac{2\vec{R}}{R^2}\right)$

Compute vector potentials

9(b) $F = xy\vec{i} - \frac{y^2}{2}\vec{j}$; $\nabla\cdot F = y - y = 0$

$$k\times F = \frac{y^2}{2}\vec{i} + xy\vec{j} = \nabla\left(\frac{xy^2}{2}\right)$$ (scalar potential, found by inspection)

$$\boxed{\vec{G} = \vec{k}\,\frac{xy^2}{2}}$$

9(c) $F = \frac{-y}{R^2}\vec{i} + \frac{x}{R^2}\vec{j}$; $\left(k\times F = \frac{x}{R^2}\vec{i} + \frac{y}{R^2}\vec{j}\right)$

this is $\frac{1}{2}$ the field in (4): $\vec{k}\times\vec{R} = -y\vec{i} + x\vec{j}$

i.e. $\vec{G} = -\frac{1}{2}\log R^2\,\vec{k}$

4.6.10 $\vec{F} = (x^2 - y^2)\vec{i} - 2xy\vec{j}$; Find scalar & vector potentials.

solenoidal

$\nabla \cdot F = 2x - 2y = 0$

$\nabla \times F = \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = \vec{k}(-2y + 2y) = 0$ irrotational

Scalar potential: $\phi_x = x^2 - y^2 \Rightarrow \phi = \frac{x^3}{3} - y^2 x + g(y)$

$\phi_y = -2xy \Rightarrow -2xy + g_y = -2xy$

$$\Rightarrow \boxed{\phi(x,y) = \frac{x^3}{3} - y^2 x + C} = \left(\frac{x^2}{2} - y^2\right) y + C$$

Vector potential: $\vec{k} \times \vec{F} = 2xy\,\vec{i} + (x^2 - y^2)\vec{j} = \nabla\psi$

$\psi_x = 2xy \Rightarrow \psi(x,y) = x^2 y + g(y)$

$\psi_y = x^2 - y^2 = x^2 + \frac{dg}{dy} \Rightarrow g = -\frac{y^3}{3} + C$

$$\text{i.e. } \boxed{\psi(x,y) = x^2 y - \frac{y^3}{3} + C} = \left(x^2 - \frac{y^2}{3}\right) y$$

$$\text{i.e. } \vec{F} = (x^2 - y^2)\vec{i} - 2xy\vec{j} = \vec{\nabla}\left(\frac{x^3}{3} - y^2 x + C\right)$$

$$\parallel$$

$$\nabla \times \left(\left(x^2 y - \frac{y^3}{3} + C\right)\vec{k}\right)$$

level curves for ϕ