Math.311 #23

4.6.(1, 2, 3, 4, 5, 6) (p.236)

4.6.1> Find the elements of surface area $d\vec{S}$ and dS, in terms of du and dv for the surface S given parametrically by $x = u^2$, $y = \sqrt{2}\, uv$, $z = v^2$.

$$\frac{\partial \vec{R}}{\partial u} = 2u\vec{i} + \sqrt{2}\, v\vec{j} \quad ; \quad \frac{\partial \vec{R}}{\partial v} = \sqrt{2}\, u\vec{j} + 2v\vec{k}$$

$$d\vec{S} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & \sqrt{2}\, v & 0 \\ 0 & \sqrt{2}\, u & 2v \end{vmatrix} du\, dv = \left(2\sqrt{2}\, v^2 \vec{i} - 4uv\vec{j} + 2\sqrt{2}\, u^2 \vec{k}\right) du\, dv$$

$$dS = du\, dv \sqrt{8v^4 + 16u^2v^2 + 8u^4} = 2\sqrt{2}\,(u^2+v^2)\, du\, dv$$

4.6.2> Find $d\vec{S}$ and dS in terms of $d\phi$ & $d\theta$, for the surface with parametric equation

$$x = (1+\cos\theta)\cos\phi, \quad y = (1+\cos\theta)\sin\phi, \quad z = \sin\theta$$

$$\frac{\partial \vec{R}}{\partial \phi} = -(1+\cos\theta)\sin\phi\, \vec{i} + (1+\cos\theta)\cos\phi\, \vec{j}$$

$$\frac{\partial \vec{R}}{\partial \theta} = -\sin\theta\cos\phi\, \vec{i} - \sin\theta\sin\phi\, \vec{j} + \cos\theta\, \vec{k}$$

$$\frac{\partial \vec{R}}{\partial \phi} \times \frac{\partial \vec{R}}{\partial \theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -(1+\cos\theta)\sin\phi & (1+\cos\theta)\cos\phi & 0 \\ -\sin\theta\cos\phi & -\sin\theta\sin\phi & \cos\theta \end{vmatrix} d\phi\, d\theta \Rightarrow$$

$$d\vec{S} = \left[\vec{i}\,(1+\cos\theta)\cos\theta\cos\phi + \vec{j}\,(1+\cos\theta)\cos\theta\sin\phi + \vec{k}\,(1+\cos\theta)\sin\theta\right] d\phi\, d\theta$$

$$dS = d\phi\, d\theta \left((1+\cos\theta)^2\left[\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta\right]\right)^{1/2}$$

$$= (1+\cos\theta)\, d\phi\, d\theta$$

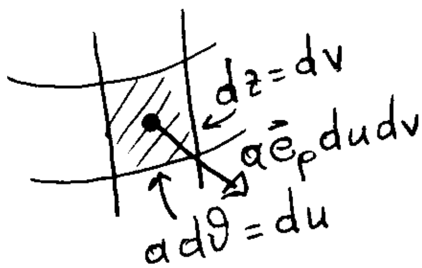
4.7.3) Determine the element of surface area dS for a right circular cylinder

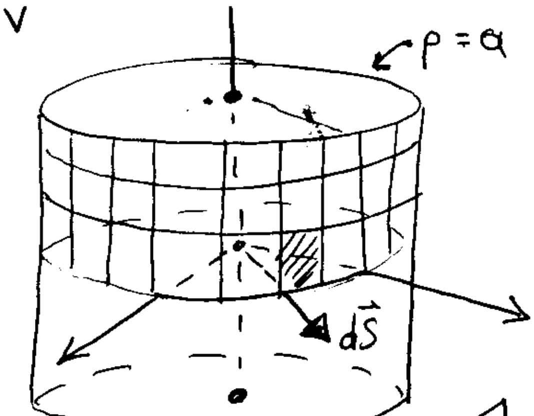
$$x = a\cos u,\quad y = a\sin u,\quad z = v$$

◁ $\dfrac{\partial \vec{R}}{\partial u} = -a\sin u\,\vec{i} + a\cos u\,\vec{j}$; $\dfrac{\partial \vec{R}}{\partial v} = \vec{k}$

$$d\vec{S} = \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\,du\,dv = \left(a\cos u\,\vec{i} + a\sin u\,\vec{j}\right) du\,dv$$
$$= a\,\vec{e}_\rho\,du\,dv$$

$$dS = a\,du\,dv$$





▷

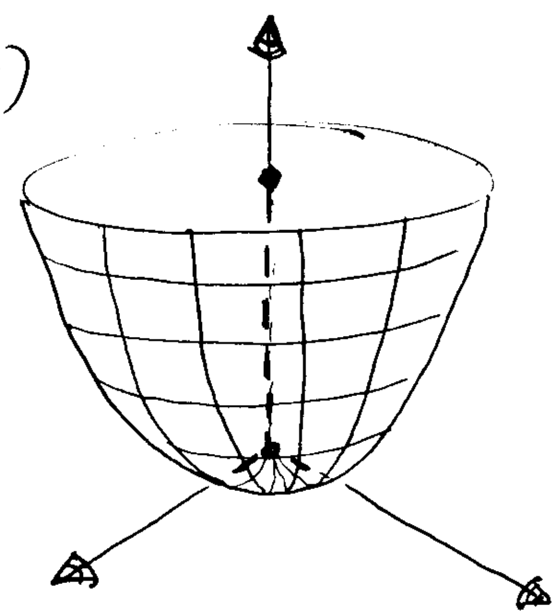
4.7.4) Determine the element of surface area dS in the special case of the paraboloid of revolution
$z = x^2 + y^2$

◁ Parametrize using polars: (ρ, θ)

$$\begin{cases} x = \rho\cos\theta, & y = \rho\sin\theta : \\ z = \rho^2 \end{cases}$$

$$\frac{\partial \vec{R}}{\partial \rho} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} + 2\rho\,\vec{k}$$

$$\frac{\partial \vec{R}}{\partial \theta} = -\rho\sin\theta\,\vec{i} + \rho\cos\theta\,\vec{j}$$



$$d\vec{S} = \frac{\partial \vec{R}}{\partial \rho} \times \frac{\partial \vec{R}}{\partial \vartheta}\, d\rho\, d\vartheta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\vartheta & \sin\vartheta & 2\rho \\ -\rho\sin\vartheta & \rho\cos\vartheta & 0 \end{vmatrix} d\rho\, d\vartheta$$

$$= \left(-\vec{i}\cdot 2\rho^2\cos\vartheta - \vec{j}\cdot 2\rho^2\sin\vartheta + \vec{k}\,\rho(\cos^2\vartheta + \sin^2\vartheta)\right) d\rho\, d\vartheta$$

$$= \left[-2\rho^2\left(\cos\vartheta\,\vec{i} + \sin\vartheta\,\vec{j}\right) + \vec{k}\rho\right] d\rho\, d\vartheta$$

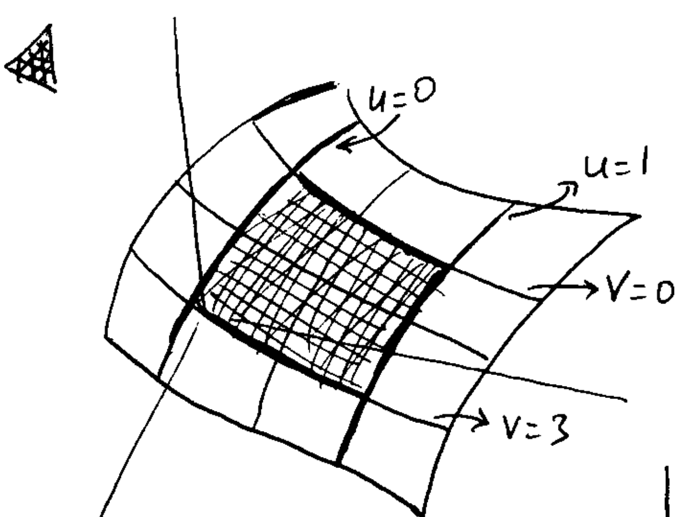
$$= \left(-2\rho^2\,\vec{e}_\rho + \vec{k}\rho\right) d\rho\, d\vartheta$$

$$dS = \left(4\rho^4 + \rho^2\right)^{1/2} d\rho\, d\vartheta = \underline{\rho\left(1+4\rho^2\right)^{1/2} d\rho\, d\vartheta}$$

$\boxed{4.7.5}$ Find the area of the section of the surface

$$x = u^2,\quad y = uv,\quad z = \frac{1}{2}v^2$$

bounded by the curves $u=0$, $u=1$, $v=0$ and $v=3$



$$\frac{\partial \vec{R}}{\partial u} = 2u\vec{i} + v\vec{j}, \quad \frac{\partial \vec{R}}{\partial v} = u\vec{j} + v\vec{k}$$

$$\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix} \cancel{du\,dv}$$

$$= \vec{i}\,v^2 - \vec{j}\,2uv + \vec{k}\cdot 2u^2$$

$$\left|\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}\right| = \left(v^4 + 4u^2v^2 + 4u^4\right)^{1/2}$$

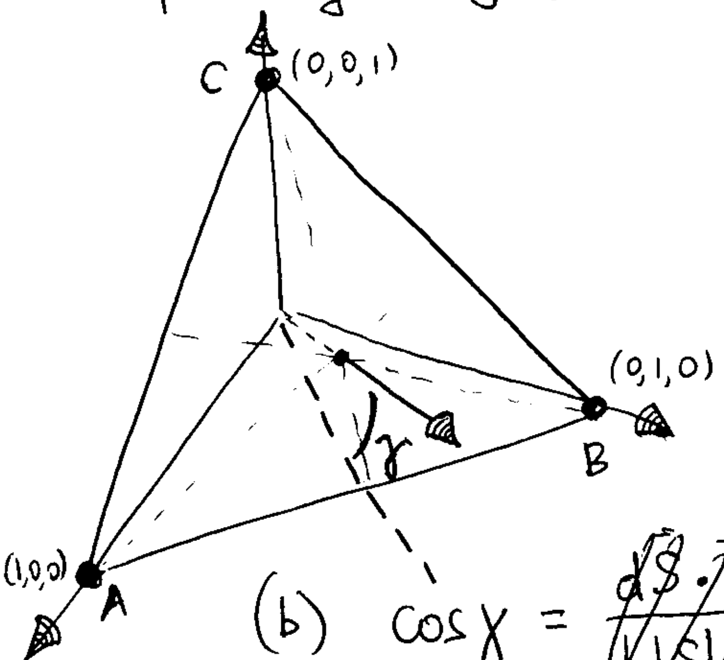
$$= \left(2u^2 + v^2\right)$$

$$S = \int_{u=0}^{1}\int_{v=0}^{3} \left(2u^2 + v^2\right) dv\, du = \int_{u=0}^{1} \left(2u^2 v + \frac{v^3}{3}\right)\Big|_{v=0}^{3} du \longrightarrow$$

$= \int (6u^2 + 9)\,du = 2u^3 + 9u \Big|^1 = 11$

$(0,1,0)$, and $(0,0,1)$.

(a) Find a unit vector $\vec{u}$ normal to this triangle, pointing away from the origin.



$\vec{AB} = \vec{j} - \vec{i}$, $\vec{AC} = \vec{k} - \vec{i}$

$\vec{AB} \times \vec{AC} = (\vec{j} - \vec{i}) \times (\vec{k} - \vec{i})$

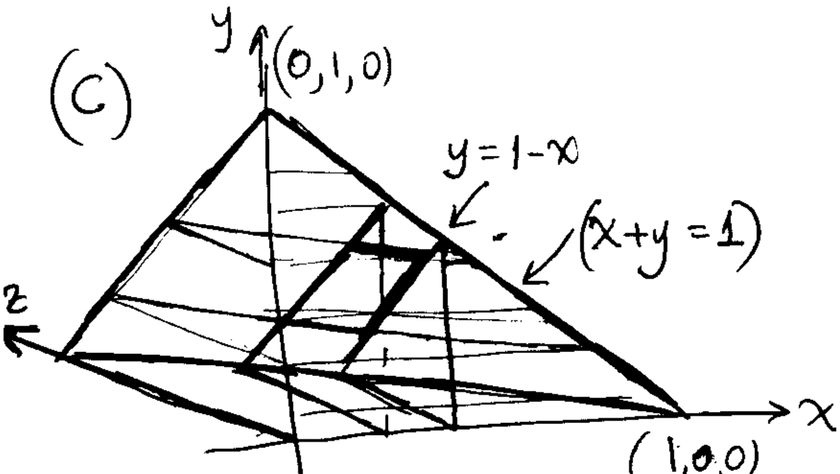
$= (\vec{i} + \vec{k} + \vec{j})$

$|\vec{AB} \times \vec{AC}| = \sqrt{3}$

$\vec{n} = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})$

(b) $\cos\gamma = \cancel{\frac{d\vec{S}\cdot\vec{k}}{|dS|} = \frac{1}{\sqrt{3}}}$ $\quad \vec{n}\cdot\vec{k} = \frac{1}{\sqrt{3}}$

(c)



The base of the triangle has the description

$$\left.\begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{array}\right\}$$

(d) $S = \int_{x=0}^{1}\int_{y=0}^{1-x} \frac{dx\,dy}{1/\sqrt{3}} = \sqrt{3}\int_0^1 \left(y\Big|_{y=0}^{1-x} \right) dx = \sqrt{3}\int_0^1 (1-x)\,dx$

$= \sqrt{3}\left(x - \frac{x^2}{2}\right)\Big|_0^1 = \sqrt{3}/2$