Math. 311
Spring 99

Set III

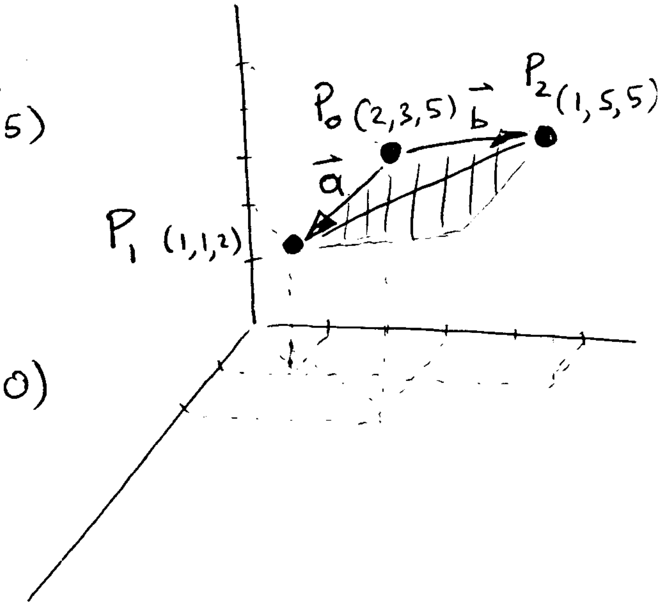
p.51, 1.12(3, 11)
p.57, 1.13 (3, 6)
p.60, 1.14 (11)

**p.51, 1.12.3** Find the area of the triangle with vertices (1,1,2), (2,3,5) and (1,5,5).

Let

$$\vec{a} = \overrightarrow{P_0P_1} = \overrightarrow{(1,1,2) - (2,3,5)} = \overrightarrow{(-1,-2,-3)}$$

$$\vec{a} = -\vec{i} - 2\vec{j} - 3\vec{k}$$




$$\vec{b} = (\overrightarrow{P_0P_2}) = \overrightarrow{(1,5,5) - (2,3,5)} = \overrightarrow{(-1,2,0)}$$

$$\vec{b} = -\vec{i} + 2\vec{k}$$

Then the area of the triangle $P_0P_1P_2$ is ½ the area of the parallelogram defined by $\vec{a}$ and $\vec{b}$, i.e.

$$\text{Area}(P_0P_1P_2) = \frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}\left\| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & -3 \\ -1 & 2 & 0 \end{matrix} \right\|$$

Now $\vec{a}\times\vec{b} = \vec{i}\begin{vmatrix} -2 & -3 \\ 2 & 0 \end{vmatrix} + \vec{j}\begin{vmatrix} -3 & -1 \\ 0 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} -1 & -2 \\ -1 & 2 \end{vmatrix} = 6\vec{i} + 3\vec{j} - 4\vec{k}$

$$|\vec{a}\times\vec{b}| = \sqrt{6^2+3^2+4^2} = \sqrt{61} \Rightarrow \boxed{\text{Area} = \frac{1}{2}\sqrt{61}}$$

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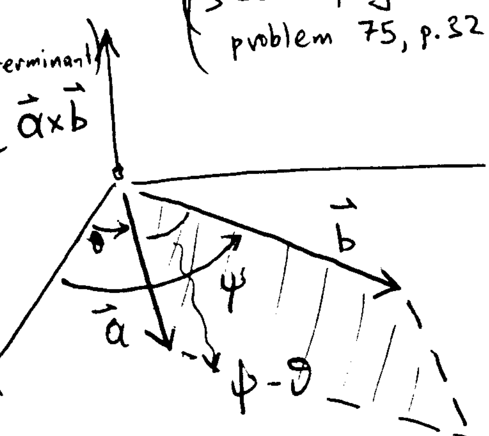
**p.52, 1.12.11** By taking the cross product of $(\cos\vartheta)\vec{i} + (\sin\vartheta)\vec{j}$ and $(\cos\varphi)\vec{i} + (\sin\varphi)\vec{j}$ and interpreting geometrically, derive a well-known trig. identity.

(see Spiegel, problem 75, p.32)

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\vartheta & \sin\vartheta & 0 \\ \cos\varphi & \sin\varphi & 0 \end{vmatrix} = \vec{k}(\cos\vartheta\sin\varphi - \sin\vartheta\cos\varphi) \quad \text{(by determinant)}$$

$$= \underset{1}{|\vec{a}|}\,\underset{1}{|\vec{b}|}\,\sin(\varphi-\vartheta)\,\vec{k} = \vec{a}\times\vec{b} \quad \text{(directly)}$$

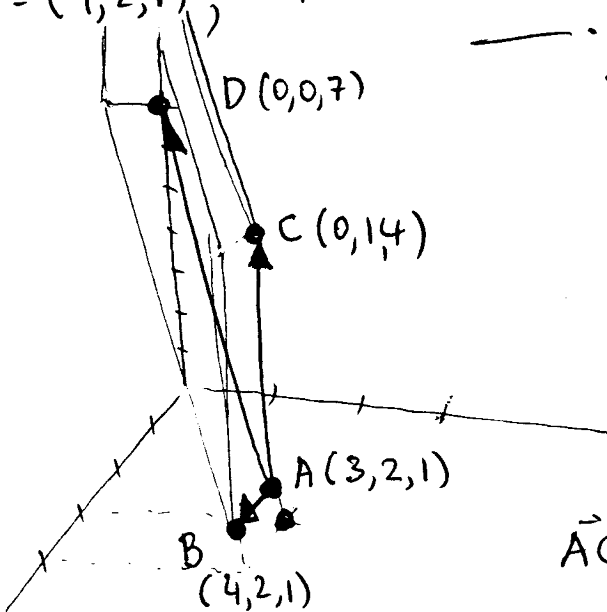



$\vec{a} = \cos\vartheta\,\vec{i} + \sin\vartheta\,\vec{j}$

$|\vec{a}| = \sqrt{\cos^2\vartheta + \sin^2\vartheta} = 1$ ; similarly $|\vec{b}| = 1$, $\vec{b} = \cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}$

p.57, 1.13(3) Find the volume of the parallelepiped with coterminal edges AB, AC and AD, where $A=(3,2,1)$, $B=(4,2,1)$, ~~and~~ $C=(0,1,4)$ and $D=(0,0,7)$.

$\frac{3}{2/4}$




The volume is given by the triple scalar product:

$$V = \vec{AB}\cdot(\vec{AC}\times\vec{AD})$$

where

$\vec{AB} = (4,2,1)-(3,2,1) = (1,0,0) = \vec{i}$

$\vec{AC} = (0,1,4)-(3,2,1) = (-3,-1,3) = -3\vec{i}-\vec{j}+3\vec{k}$

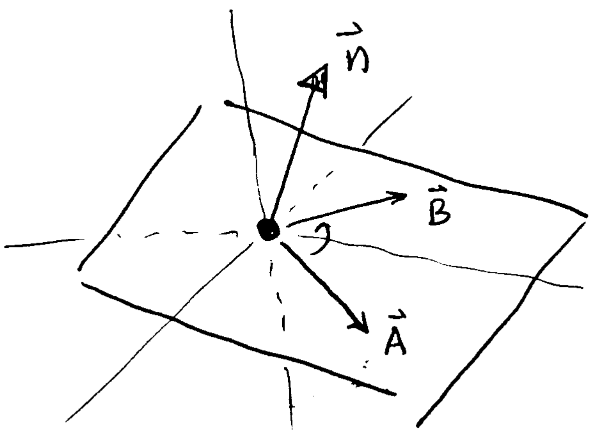
$\vec{AD} = (0,0,7)-(3,2,1) = (-3,-2,6) = -3\vec{i}-2\vec{j}+6\vec{k}$

So $$V = \begin{vmatrix} 1 & 0 & 0 \\ -3 & -1 & 3 \\ -3 & -2 & 6 \end{vmatrix} = 1\cdot\begin{vmatrix} -1 & 3 \\ -2 & 6 \end{vmatrix} = -6+6 = 0$$

The vanishing of the volume means that the three vectors are coplanar (i.e. the four points A, B, C, D lie in the same plane!).

p. 57, 1.13.6 Find the equation of the plane passing through the origin parallel to the vectors $\vec{A} = 3\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{B} = \vec{i} - \vec{j} + 5\vec{k}$.

$\left(\frac{3}{3/4}\right)$




For the plane we need a point $P_0$ (here $(0,0,0)$) and a normal. Find $\vec{n}$ as

$$\vec{n} = \vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -2 \\ 1 & -1 & 5 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 1 & -2 \\ -1 & 5 \end{vmatrix} + \vec{j} \begin{vmatrix} -2 & 3 \\ 5 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix}$$

$$\Rightarrow \vec{n} = 3\vec{i} - 17\vec{j} - 4\vec{k}$$

So: $\vec{n} \cdot (\vec{R} - \vec{R}_0) = 0 \Rightarrow \boxed{3x - 17y - 4z = 0}$

p. 60, 1.14(11) Simplify $[\vec{A} \times (\vec{A} \times \vec{B})] \times \vec{A} \cdot \vec{C}$

$$\vec{A} \times (\vec{A} \times \vec{B}) = (\vec{A} \cdot \vec{B})\vec{A} - (\vec{A} \cdot \vec{A})\vec{B}$$

$$[A \times (A \times B)] \times A = (A \cdot B) \underbrace{A \times A}_{=0} - (A \cdot A) B \times A = (A \cdot A) \vec{A} \times \vec{B}$$

$$[\vec{A} \times (A \times B)] \times \vec{A} \cdot \vec{C} = (A \cdot A)\, A \times B \cdot C = (A \cdot A)\, [A, B, C]$$

over $\rightarrow$ (some extra solutions) p.51, 1.12.5 $N = \begin{vmatrix} i & j & k \\ 3 & 1 & 0 \\ 2 & -1 & -5 \end{vmatrix} = -5\vec{i} + 15\vec{j} - 5k$

(3/4)

$|N| = \sqrt{5^2+15^2+5^2} = 5\sqrt{1+9+1} = 5\sqrt{11}$

$\vec{n} = \vec{N}/|N| = \frac{1}{\sqrt{11}}(-\vec{i}+3\vec{j}-\vec{k})$

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p.52, 1.12.19 $A \circ B = 0 \Rightarrow |A||B|\cos\theta = 0$

$A \times B = 0 \Rightarrow |A||B|\sin\theta = 0$

since $\cos^2\theta + \sin^2\theta = 1$, must have $|A| = 0$ or $|B| = 0$.

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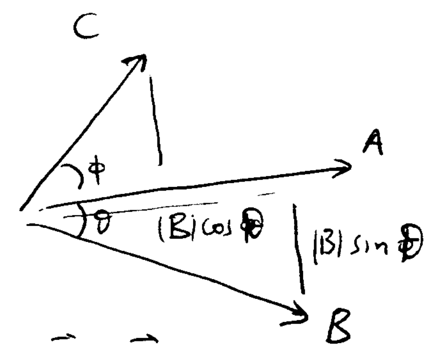
p.52, 1.12.20 $A \neq 0$; $A \cdot B = A \cdot C$ (1), $A \times B = A \times C$ (2)

$AB\cos\theta = A\cancel{B}\cos\phi$ $\quad |A||B|\sin\theta = |A||C|\sin\phi$

(1) $A \cdot (B \times C) = (A \times B) \cdot C = (A \times C) \cdot C = A \cdot (C \times C) = 0$
(by (2))

$\Rightarrow$ A, B, C coplanar.




(2) Then $|B|\cos\theta = |C|\cos\phi$

and $|B|\sin\theta = |C|\sin\phi$

mean that B and C have equal projections onto A and normal to A $\Rightarrow \vec{B} = \vec{C}$.

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p.60, 1.14.6 (a) $A \times B = -B \times A$ ; (b) $(A \times B) \times C = (A \cdot C)B - (B \cdot C)A$

$\neq$

$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$

(c) $A \times B = A \times C \Rightarrow A \times (B - C) = 0$

$\Rightarrow (B - C)$ parallel to A (not $\vec{0}$)

(d) $A \times B = 0$ if $A = 0$ or $B = 0$ or $\sin\theta = 0$ (ie. if $\vec{A} \parallel \vec{B}$)

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p.60, 1.14.7 $|A \times B|^2 = |A|^2|B|^2\sin^2\theta$ ; $(A \cdot B)^2 = |A|^2 \cdot |B|^2\cos^2\theta$

$\Rightarrow |A \times B|^2 + (A \cdot B)^2 = |A|^2|B|^2(\sin^2\theta + \cos^2\theta) = |A|^2|B|^2$