Useful information: In spherical coordinates,

$$\hat{\mathbf{t}} = \hat{\mathbf{e}}_r \frac{dr}{ds} + \hat{\mathbf{e}}_\theta r \frac{d\theta}{ds} + \hat{\mathbf{e}}_\phi r \sin\theta \frac{d\phi}{ds}.$$

(b) Calculate the flux of the dipole field through a sphere of radius R centered at the origin.

(c) What is the flux of the dipole field over *any* closed surface which does not pass through the origin?

IV–7 Here is a "proof" that there is no such thing as magnetism. One of Maxwell's equations tells us that

$$\nabla \cdot \mathbf{B} = 0,$$

where $\mathbf{B}$ is any magnetic field. Then using the divergence theorem, we find

$$\iint_S \mathbf{B} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{B}\, dV = 0.$$

Because $\mathbf{B}$ has zero divergence, we know (see Problem III–24) there exists a vector function, call it $\mathbf{A}$, such that

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Combining these last two equations, we get

$$\iint_S \hat{\mathbf{n}} \cdot \nabla \times \mathbf{A}\, dS = 0.$$

Next we apply Stokes' theorem and the above result to find

$$\oint_C \mathbf{A} \cdot \hat{\mathbf{t}}\, ds = \iint_S \hat{\mathbf{n}} \cdot \nabla \times \mathbf{A}\, dS = 0.$$

Thus we have shown that the circulation of $\mathbf{A}$ is path-independent. It follows that we can write $\mathbf{A} = \nabla\psi$ where ψ is some scalar function. Since the curl of the gradient of a function is zero, we arrive at the remarkable fact that

$$\mathbf{B} = \nabla \times \nabla\psi = 0;$$

that is, all magnetic fields are zero! Where did we go wrong? [Taken from G. Arfken, *Amer. J. Phys.*, **27**, 526 (1959).]

IV–8 Fick's law states that in certain diffusion processes the current density $\mathbf{J}$ is proportional to the negative of the gradient of the density ρ; that is, $\mathbf{J} = -k\nabla\rho$, where k is a positive constant. If a substance of density $\rho(x, y, z, t)$ and velocity $\mathbf{v}(x, y, z, t)$ diffuses according to Fick's law, show that the flow is *irrotational* (that is, $\nabla \times \mathbf{v} = 0$).

IV–9 (a) A substance diffuses according to Fick's law (see Problem IV–8). Assuming the diffusing matter is conserved, derive the diffusion equation

$$\frac{\partial \rho}{\partial t} = k\nabla^2\rho.$$

(b) Bacteria of density ρ diffuse in a medium according to Fick's law and reproduce at a rate $\lambda\rho$ per unit volume (λ is a positive constant). Show that

$$\frac{\partial \rho}{\partial t} = k\nabla^2\rho + \lambda\rho.$$

IV–10 (a) A fluid is said to be *incompressible* if its density ρ is a constant (that is, is independent of x, y, z, and t). Use the continuity equation to show that the velocity $\mathbf{v}$ of an incompressible fluid satisfies the equation $\nabla \cdot \mathbf{v} = 0$.
(b) If $\nabla \times \mathbf{v} = 0$, the fluid flow is said to be *irrotational.* Show that for an incompressible fluid undergoing irrotational flow,

$$\nabla^2\phi = 0,$$

where ϕ, a scalar function called the *velocity potential,* is so defined that $\mathbf{v} = \nabla\phi$.

IV–11 The heat Q in a body of volume V is given by

$$Q = c\iiint_V T\rho\, dV,$$

where c is a constant called the specific heat of the body, and $T(x, y, z, t)$ and $\rho(x, y, z)$ are, respectively, the temperature and density of the body. (Note that we are assuming the density to be independent of time.) The rate at which heat flows through S, the bounding surface of the body, is given by

$$\frac{dQ}{dt} = k\iint_S \hat{\mathbf{n}} \cdot \nabla T\, dS,$$

where k (assumed constant) is the thermal conductivity of the body, and the integral is taken over the surface S bounding the body. Use these facts to derive the heat flow equation

$$\nabla^2 T = \alpha\frac{\partial T}{\partial t},$$

where $\alpha = c\rho/k$.

IV–12 In nonrelativistic quantum mechanics a particle of mass m moving in a potential $V(x, y, z)$ is described by the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial \psi}{\partial t},$$

where $\hbar$ is Planck's constant divided by 2π and $\psi(x, y, z, t)$, which is complex, is called the wave function. The quantity $\rho = \psi^*\psi$ is interpreted as the probability density.

(a) Use the Schrödinger equation to derive an equation of the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

and obtain thereby an expression for $\mathbf{J}$ in terms of ψ, ψ^*, m, and $\hbar$.

(b) Give an interpretation of $\mathbf{J}$ and of the equation derived in (a).

IV–13 (a) Find the charge density $\rho(x, y, z)$ which produces the electric field

$$\mathbf{E} = g(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z),$$

where g is a constant.

(b) Find an electrostatic potential Φ such that $-\nabla\Phi$ is the field $\mathbf{E}$ given in (a).

(c) Verify that $\nabla^2\Phi = -\rho/\epsilon_0$.

IV–14 (a) Starting with the divergence theorem, derive the equation

$$\iint_S \hat{\mathbf{n}} \cdot (u\nabla v)\, dS = \iiint_V [u\nabla^2 v + (\nabla u) \cdot (\nabla v)]\, dV,$$

where u and v are scalar functions of position and S is a closed surface enclosing the volume V. This is sometimes called the first form of Green's theorem.

(b) If $\nabla^2 u = 0$ use the first form of Green's theorem to show that

$$\iint_S \hat{\mathbf{n}} \cdot (u\nabla u)\, dS = \iiint_V |\nabla u|^2\, dV,$$

where $|\nabla u|^2 = (\nabla u) \cdot (\nabla u)$.

(c) Use the first form of Green's theorem to show that

$$\iint_S \hat{\mathbf{n}} \cdot (u\nabla v - v\nabla u)\, dS = \iiint_V (u\nabla^2 v - v\nabla^2 u)\, dV.$$

This is the second form of Green's theorem.

IV–15 An equation of the form

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2},$$

where f is a differentiable function of position and time, is called a wave equation. It describes a wave propagating in space with velocity v. Use Maxwell's equations (Problem III–20) to show that in the absence of charges and currents (that is, ρ and $\mathbf{J}$ both zero), all three