2/18/05

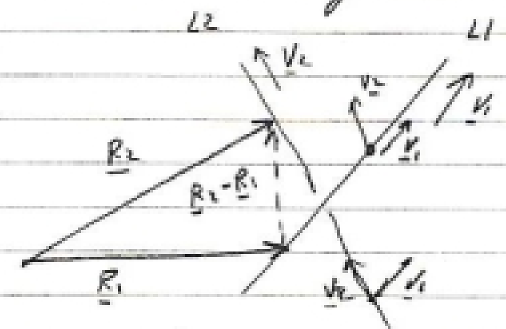
Solution to HW # 23 p.52

23 p.52 (1st solution)

Line 1 $\frac{1}{3}x = \frac{1}{2}y = \frac{1}{2}z$ Line 2 $\frac{1}{5}x = \frac{1}{3}y = \frac{1}{2}(z-4)$

a) For them to intersect we must have $\frac{1}{3}x = \frac{1}{2}y$ and $\frac{1}{5}x = \frac{1}{3}y \Rightarrow x = y = 0$ but for Line 1 this means $z = 0$ and for Line 2 $z = 4$, so they can't intersect.

b) & c) Consider the general case of 2 oblique lines



$$\underline{R}_1 = \underline{R}_{10} + t\,\underline{V}_1$$
$$\underline{R}_2 = \underline{R}_{20} + s\,\underline{V}_2$$

Note that the lines lie in two ∥ planes defined by $\underline{V}_1, \underline{V}_2$ and a pt on each line, and the distance between the lines is given by the distance between the two planes.

To find a line ⊥ to both lines consider the vector $\underline{R}_2 - \underline{R}_1$. If we can find t and s so that $\underline{R}_2 - \underline{R}_1$ is ⊥ to $\underline{V}_1$ and $\underline{V}_2$ then we have solved both parts b) & c)

$$\underline{V}_1 \cdot (\underline{R}_2 - \underline{R}_1) = \underline{V}_1 \cdot (\underline{R}_{20} - \underline{R}_{10}) + s\, V_1 \cdot V_2 - t\, V_1 \cdot V_1 = 0$$

$$\underline{V}_2 \cdot (\underline{R}_2 - \underline{R}_1) = V_2 \cdot (R_{20} - R_{10}) + s\, V_2 \cdot V_2 - t\, V_2 \cdot V_1 = 0$$

Now choose $R_{10} = (0,0,0)$ & $R_{20} = (0,0,4)$ also we have $\underline{V}_1 = (3,2,2)$ & $\underline{V}_2 = (5,3,2)$

$\therefore (R_{20} - R_{10}) \cdot V_1 = 8 = (R_{20} - R_{10}) \cdot V_2$ and $V_1 \cdot V_1 = 17$, $V_1 \cdot V_2 = 25$ $V_2 \cdot V_2 = 38$

$$\frac{1}{2}x - \frac{52}{7} = -\frac{1}{4}y + \frac{52}{21} = z - \frac{208}{21}$$

$$\frac{x - \frac{104}{7}}{2} = \frac{y - \frac{52\cdot 4}{21}}{-4} = \frac{z - \frac{208}{21}}{1}$$

$$25s - 17t = 8$$
$$38s - 25t = 8$$

$$s = \frac{\begin{vmatrix} -8 & -17 \\ -8 & -25 \end{vmatrix}}{\begin{vmatrix} 25 & -17 \\ 38 & -25 \end{vmatrix}} = \frac{64}{21} \qquad t = \frac{\begin{vmatrix} 25 & -8 \\ 38 & -8 \end{vmatrix}}{21} = \frac{104}{21}$$

$\therefore$ a line $\perp$ to both $L1$ & $L2$ is a line thru the points

$$P_1 \equiv R_{10} + \frac{104}{21}(3,2,2) = \frac{104}{21}(3,2,2)$$
$$P_2 \equiv R_{20} + \frac{64}{21}(5,3,2) = (0,0,4) + \frac{64}{21}(5,3,2)$$

the line is given by $\dfrac{x - \frac{104\cdot 3}{21}}{a} = \dfrac{y - \frac{104\cdot 2}{21}}{b} = \dfrac{z - \frac{104}{21}\cdot 2}{c}$

which is the answer in the book if $a=2$, $b=-4$ and $c=1$

$$21(P_2 - P_1) = (0,0,84) + (\overset{320}{64\cdot 5}, \overset{128}{64\cdot 3}, 64\cdot 2) - (\overset{312}{104\cdot 3}, 208, 208)$$

$$= (8, -16, 4) = 4(2,-4,1)$$

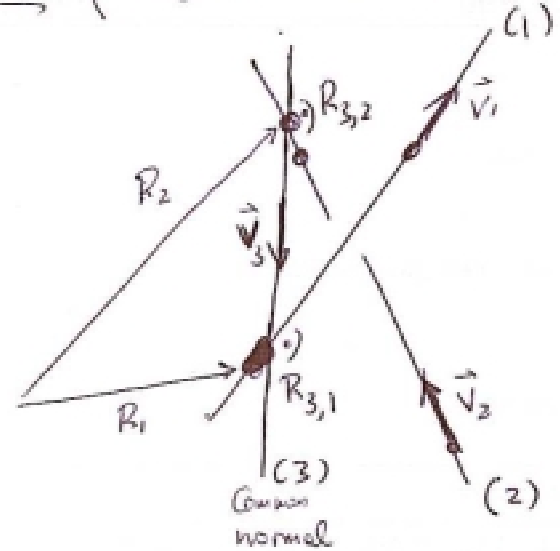
$\therefore (2,-4,1)$ works

To find the distance we simple evaluate $|\underline{R_2} - \underline{R_1}|$ at $t = \frac{64}{21}$, $s = \frac{104}{21}$. At this point we have

$$R_2 - R_1 = \left(\frac{64\cdot 5 - 104\cdot 3}{21}, \frac{64\cdot 3 - 208}{21}, 4 + \frac{64\cdot 2 - 208}{21} \right)$$
$$= \left(\frac{8}{21}, -\frac{16}{21}, \frac{4}{21} \right)$$

$$|R_2 - R_1| = \frac{\sqrt{64 + 256 + 16}}{21} = \frac{\sqrt{16\cdot 21}}{21} = \frac{4}{\sqrt{21}}$$

#23 (Second solution)



line 1

$\frac{1}{3}x = \frac{1}{2}y = \frac{1}{2}z \;:\; \vec{R}_1 = \vec{R}_{1,0} + t\,\vec{v}_1$

$R_{1,0} = \vec{0}, \quad \vec{v}_1 = (3,2,2)$

line 2

$\frac{1}{5}x = \frac{1}{3}y = \frac{1}{2}(z-4) \;:\; \vec{R}_2 = \vec{R}_{2,0} + s\vec{V}_2$

$R_{2,0} = (0,0,4), \quad \vec{v}_2 = (5,3,2)$

Common normal vector:

$\vec{V}_3 = \vec{V}_1 \times \vec{V}_2$

line 3: $\vec{R}_3 = \vec{R}_{3,0} + u\,\vec{V}_3$

To find common normal, let $\vec{R}_{3,0}$ be an arbitrary point on line 1, $\vec{R}_{3,0} = \vec{R}_{1,0} + t\vec{v}_1$ and find $t = t_3$ such that line 3 intersects line 2: that is

$\therefore$ let $\vec{R}_3 = (R_{1,0} + t\vec{v}_1) + u\vec{V}_3 = \vec{R}_{2,0} + s\,\vec{V}_2$

Seek values of t, u, s that make equation true. This will give 3 equations in the 3 unknowns.

We have: $\vec{V}_3 = \begin{vmatrix} i & j & k \\ 3 & 2 & 2 \\ 5 & 3 & 2 \end{vmatrix} = \vec{i}(4-6) + \vec{j}(10-6) + \vec{k}(9-10)$
$= -2\vec{i} + 4\vec{j} - \vec{k}$

Then: $\underbrace{(0,0,0)}_{R_{1,0}} + t(3,2,2) + u(-2,4,-1) = (0,0,4) + s(5,3,2)$

$$\Rightarrow \left.\begin{aligned} 3t - 2u - 5s &= 0 \\ 2t + 4u - 3s &= 0 \\ 2t - u - 2s &= 4 \end{aligned}\right\} \Rightarrow \left(\begin{array}{ccc|c} 3 & -2 & -5 & 0 \\ 2 & 4 & -3 & 0 \\ 2 & -1 & -2 & 4 \end{array}\right). \quad \text{(using Gaussian elimination)}$$

Solve this system of 3 equations in 3 unknowns by Cramers rule:

$$\text{if } \det = \begin{vmatrix} 3 & -2 & -5 \\ 2 & 4 & -3 \\ 2 & -1 & -2 \end{vmatrix} = 3\begin{vmatrix} 4 & -3 \\ -1 & -2 \end{vmatrix} + 2\begin{vmatrix} 2 & -3 \\ 2 & -2 \end{vmatrix} - 5\begin{vmatrix} 2 & 4 \\ 2 & -1 \end{vmatrix}$$

$$= 3(-11) + 2(2) - 5(-10) = 21$$

$$t = \begin{vmatrix} 0 & -2 & -5 \\ 0 & 4 & -3 \\ 4 & -1 & -2 \end{vmatrix} / \det = 4\begin{vmatrix} -2 & -5 \\ 4 & -3 \end{vmatrix} / 21 = \frac{104}{21}$$

$$u = \begin{vmatrix} 3 & 0 & -5 \\ 2 & 0 & -3 \\ 2 & 4 & -2 \end{vmatrix} / \det = -4\begin{vmatrix} 3 & -5 \\ 2 & -3 \end{vmatrix} / 21 = -4/21$$

$$s = \begin{vmatrix} 3 & -2 & 0 \\ 2 & 4 & 0 \\ 2 & -1 & 4 \end{vmatrix} / \det = 4\begin{vmatrix} 3 & -2 \\ 2 & 4 \end{vmatrix} / 21 = 64/21$$

$$\therefore\ t = 104/21, \quad u = -4/21, \quad s = 64/21$$

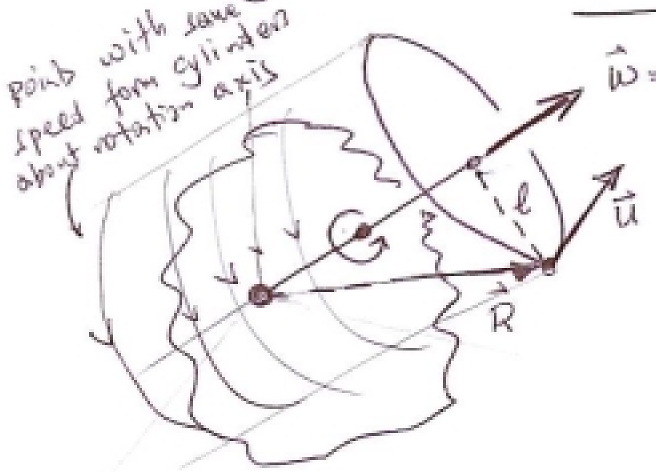
The equation of the common normal line is

$$\vec{R}_3(u) = \frac{104}{21}\vec{v}_1 + u\vec{v}_3 = \underbrace{\frac{104}{21}(3,2,2)}_{\equiv \vec{R}_{3,1} \equiv \vec{R}_{3,0}} + u(-2,4,-1)$$

This line crosses line (1) for $u=0$ at $R_3(0) \equiv R_{3,1}$ while it crosses line (2) for $u = -4/21$ at $\vec{R}_{3,2} = R_3(-4/21) = R_{3,1} - 4/21\,(-2,4,-1)$. The distance between these is $d = \|\bar{R}_{3,2} - \bar{R}_{3,1}\| = \|-4/21\,(-2,4,-1)\|$

$$= \frac{4}{21}\sqrt{4+16+1} = 4/\sqrt{21}$$

1.12.24 If ω point in the direction of $\vec{i}+\vec{j}+\vec{k}$ and the body rotates about an axis through the origin with angular velocity $10\sqrt{3}$ rad/sec, find the locus of points having speed 20 ft/sec. What does this locus represent?



$\vec{\omega} = 10(\vec{i}+\vec{j}+\vec{k})$

since $|\vec{\omega}| = 10\sqrt{3}$ while $\frac{\vec{\omega}}{\cancel{|\omega|}} \parallel (\vec{i}+\vec{j}+\vec{k})$, it follows

that $\vec{\omega} = 10\sqrt{3}\,\underbrace{\left(\frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}\right)}_{\text{unit vector in given direction}} = 10(\vec{i}+\vec{j}+\vec{k})$

∴ The velocity of a point at location $\vec{R}$ is

$$\vec{u} = \omega \times R = 10\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = 10\left[(z-y)\vec{i} + (x-z)\vec{j} + (y-x)\vec{k}\right]$$

or $$|u| = \cancel{10}\sqrt{(z-y)^2+(x-z)^2+(y-x)^2} = \cancel{20}^{\,2} \Rightarrow$$

$$(z-y)^2+(x-z)^2+(y-x)^2 = 4 \Rightarrow 2x^2+2y^2+2z^2-2xy-2yz-2zx = 4$$

$$\Rightarrow \boxed{x^2+y^2+z^2-xy-yz-zx = 2}$$

We know from geometry that this is a cylinder about rotation axis.

To find radius ℓ of cylinder directly, consider the relationship $u = \omega\ell \Rightarrow 20$ ft/sec $= \ell \cdot 10\sqrt{3}$ rad/sec

$$\Rightarrow \boxed{\ell = 10/\sqrt{3}\ \text{ft}}$$