(*a*) straight line segments from (1,0) to (1,1), then to (−1,1), then to (−1,0).
(*b*) straight line segments from (1,0) to (1,−1), then to (−1,−1), then to (−1,0).

Show that although $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the line integral *is dependent* on the path joining (1,0) to (−1,0) and explain.

Ans. (*a*) π (*b*) $-\pi$

51. By changing variables from (x,y) to (u,v) according to the transformation $x = x(u,v)$, $y = y(u,v)$, show that the area A of a region R bounded by a simple closed curve C is given by

$$A = \iint_R \left| J\left(\frac{x,y}{u,v}\right) \right| du\, dv \qquad \text{where} \qquad J\left(\frac{x,y}{u,v}\right) \equiv \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\[2mm] \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

is the Jacobian of x and y with respect to u and v. What restrictions should you make ? Illustrate the result where u and v are polar coordinates.

Hint: Use the result $A = \frac{1}{2}\int x\,dy - y\,dx$, transform to u,v coordinates and then use Green's theorem.

52. Evaluate $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$, where $\mathbf{F} = 2xy\,\mathbf{i} + yz^2\,\mathbf{j} + xz\,\mathbf{k}$ and S is:

(*a*) the surface of the parallelepiped bounded by $x=0$, $y=0$, $z=0$, $x=2$, $y=1$ and $z=3$,
(*b*) the surface of the region bounded by $x=0$, $y=0$, $y=3$, $z=0$ and $x+2z=6$.

Ans. (*a*) 30 (*b*) 351/2

53. Verify the divergence theorem for $\mathbf{A} = 2x^2y\,\mathbf{i} - y^2\,\mathbf{j} + 4xz^2\,\mathbf{k}$ taken over the region in the first octant bounded by $y^2+z^2=9$ and $x=2$. *Ans.* 180

54. Evaluate $\displaystyle\iint_S \mathbf{r}\cdot\mathbf{n}\, dS$ where (*a*) S is the sphere of radius 2 with center at (0,0,0), (*b*) S is the surface of the cube bounded by $x=-1$, $y=-1$, $z=-1$, $x=1$, $y=1$, $z=1$, (*c*) S is the surface bounded by the paraboloid $z = 4-(x^2+y^2)$ and the xy plane. *Ans.* (*a*) 32π (*b*) 24 (*c*) 24π

55. If S is any closed surface enclosing a volume V and $\mathbf{A} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$, prove that $\displaystyle\iint_S \mathbf{A}\cdot\mathbf{n}\, dS = (a+b+c)V$.

56. If $\mathbf{H} = \text{curl}\,\mathbf{A}$, prove that $\displaystyle\iint_S \mathbf{H}\cdot\mathbf{n}\, dS = 0$ for any closed surface S.

57. If $\mathbf{n}$ is the unit outward drawn normal to any closed surface of area S, show that $\displaystyle\iiint_V \text{div}\,\mathbf{n}\, dV = S$.

58. Prove $\displaystyle\iiint_V \frac{dV}{r^2} = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\, dS$.

59. Prove $\displaystyle\iint_S r^5\mathbf{n}\, dS = \iiint_V 5r^3\,\mathbf{r}\, dV$.

60. Prove $\displaystyle\iint_S \mathbf{n}\, dS = \mathbf{0}$ for any closed surface S.

61. Show that Green's second identity can be written $\displaystyle\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dV = \iint_S \left(\phi\frac{d\psi}{dn} - \psi\frac{d\phi}{dn}\right)dS$

62. Prove $\displaystyle\iint \mathbf{r}\times d\mathbf{S} = \mathbf{0}$ for any closed surface S.