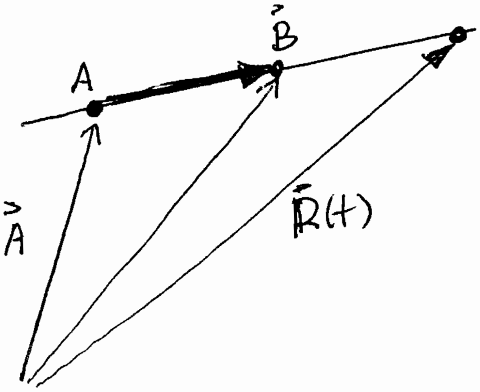
Math 311
1.8.12, 1.10.28
1.13.17, 1.14.15
(4.5(1, 6, 7, 8))

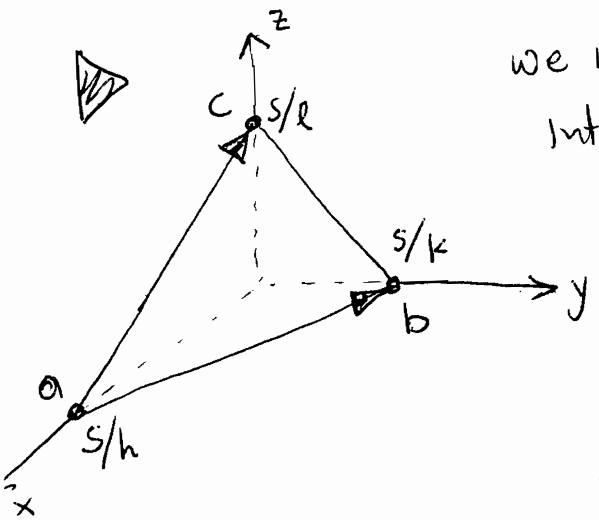
1.8.12 Let A and B be two points with position vectors $\vec{A}$ and $\vec{B}$, respectively. Show that the line passing through A, B may be represented by the vector equation $\vec{R} = s\vec{A} + t\vec{B}$ $(s + t = 1)$



We have

$$\vec{R}(t) = \vec{A} + (\vec{B} - \vec{A})t = (1-t)\vec{A} + t\vec{B}$$
$$= s\vec{A} + t\vec{B}, \quad s = 1 - t$$

1.10.28 In the study of crystals with cubic symmetry the "Miller indices" of a plane are numbers (h, k, l) proportional to the reciprocals of the intercepts of the plane with the x, y, z axes. Write down a normal vector to a plane with **M. ind.** (h, k, l).



we need the normal to a plane with intercepts $s\left(\frac{1}{h}, \frac{1}{k}, \frac{1}{l}\right)$

The normal is parallel to the vector

$$(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -s/h & s/k & 0 \\ -s/h & 0 & s/l \end{vmatrix}$$

$$= \vec{i}\left(\frac{s^2}{kl}\right) + \vec{j}\left(\frac{s^2}{hl}\right) + \vec{k}\left(\frac{s^2}{hk}\right)$$

$$\Rightarrow \vec{N} \parallel \frac{1}{kl}\vec{i} + \frac{1}{hl}\vec{j} + \frac{1}{hk}\vec{k}$$

1.13.17 Show that an arbitrary vector $\vec{V}$ can be expressed in terms of any three non-coplanar vectors $\vec{A}, \vec{B}$ and $\vec{C}$ by

$$\vec{V} = \frac{[V,B,C]}{[A,B,C]}\vec{A} + \frac{[A,V,C]}{[A,B,C]}\vec{B} + \frac{[A,B,V]}{[A,B,C]}\vec{C}$$

Let $\vec{V} = a\vec{A} + b\vec{B} + c\vec{C}$

Then $\vec{V} \cdot B \times C = a\, A \cdot B \times C \Rightarrow a = \dfrac{V \cdot B \times C}{A \cdot B \times C} = \dfrac{[V,B,C]}{[A,B,C]}$

$V \cdot A \times C = b\, B \cdot A \times C \Rightarrow V \times A \cdot C = b\, B \times A \cdot C \Rightarrow$

$\Rightarrow A \times V \cdot C = b\, A \cdot B \times C \Rightarrow b = \dfrac{[A,V,C]}{[A,B,C]}$

and $\cancel{V}\, A \times B \cdot V = c\, A \times B \cdot C \Rightarrow c = \dfrac{[A,B,V]}{[A,B,C]}$

1.14.15 Write $(u \times v) \cdot (u \times v)$ as a determinant involving only scalar products

Using $(A \times B) \times C = (A \cdot C) B - (B \cdot C) A$:

$(u \times v) \cdot (u \times v) = (u \times v) \times u \cdot v = [(u \cdot u) v - (u \cdot v) u] \cdot v$

$= (u \cdot u)(v \cdot v) - (u \cdot v)^2 = \begin{vmatrix} u \cdot u & u \cdot v \\ u \cdot v & v \cdot v \end{vmatrix}$

Math. 311
4.5 (1, 6, 7, 8)

4.5.1 Verify that $F = \nabla\times G$

(a) $\vec F = \vec A$, $\quad \vec G = \dfrac{\vec A\times\vec R}{2}$

(b) $\vec F = \vec A\times\vec R$, $\quad \vec G = \frac{1}{3}(\vec A\times\vec R)\times\vec R$

use identity: $\nabla\times(F\times G) = (\nabla\cdot G)F + (G\cdot\nabla)F - (\nabla\cdot F)G - (F\cdot\nabla)G$; $\quad A\cdot\nabla R = A$; $\quad \nabla\cdot R = 3$

(a) $\frac{1}{2}\nabla\times(A\times R) = \frac{1}{2}\left((\nabla\cdot R)A + \underbrace{\cancel{(R\cdot\nabla)A}}_{=0} - \underbrace{\cancel{(\nabla\cdot A)}R}_{=0} - (A\cdot\nabla)R\right) = \frac{1}{2}(3A - A) = A$

(b) $\frac{1}{3}\nabla\times[(A\times R)\times R] = \frac{1}{3}\Big[\underbrace{(\nabla\cdot R)(A\times R)}_{3(A\times R)} + \underbrace{(R\cdot\nabla)(A\times R)}_{\underbrace{\underbrace{A}_{\text{constant}}\times(\underbrace{R\cdot\nabla R}_{R})}_{=(A\times R)}} - \underbrace{\nabla\cdot(A\times R)}_{A\cdot\nabla\times R\,=\,0}\, R - \underbrace{(A\times R)\cdot\nabla R}_{(A\times R)}\Big] = \frac{1}{\cancel{3}}(\cancel{3}A\times R) = A\times R$

4.5.6 Derive the following expression for the divergence of the vector potential defined by $\vec G := \int_0^1 t\, F(\vec r)\times\frac{dr}{dt}\,dt$; $\vec r(t) = \vec R_0 + t(\vec R - \vec R_0)$
(where G is defined for $\vec R\in D(\vec R_0)$, a domain D that is starshaped about $\vec R_0$:

$$\nabla_{\vec R}\cdot\vec G(\vec R) = (\vec R - \vec R_0)\cdot\int_0^1 t^2[\operatorname{curl}\vec F](t\vec R)\,dt$$

Here I believe there is a misprint. Either $\vec R_0$ is zero, or the argument for $\operatorname{curl}\vec F$ in the integrand should be $(\vec R_0 + t(\vec R - \vec R_0))$, like in eq.(4.13). I'll operate under the second assumption, → but I will set $\vec R_0 = 0$ for ease. In any case, we need to compute the R-divergence ($\vec\nabla_{\vec R}$) of $\vec G(\vec R)$:

$$\nabla_R\cdot\vec G(\vec R) = \nabla_R\cdot\int_0^1 t\,\vec F(\vec r)\times\vec R\,dt$$

while we have seen (text) that

$$\vec\nabla_R\times(\vec F(\vec r)\times\vec R) = \underbrace{R\cdot\nabla_R F}_{tR\cdot\nabla_r F} - \underbrace{F\cdot\nabla_R R}_{\vec F} + \underbrace{(\nabla_R\cdot R)}_{3}F - \underbrace{(\nabla_R\cdot F)}_{t\nabla_r\cdot F\,=\,0}\vec R = t\,R\cdot\nabla_r F + 2\vec F,$$

with $+(\nabla_R\cdot R)F = +3\vec F$.

here $\vec\nabla_R\cdot(\vec F(\vec r)\times\vec R) = \vec R\cdot\nabla_R\times F - F\cdot\underbrace{\cancel{\nabla_R}\times R}_{=0} = t\vec R\cdot[\nabla_r\times F(r)] = t\vec R\cdot\operatorname{curl}\vec F(\vec r)$

Key calculation

so: $\vec\nabla_R\cdot G = \vec R\cdot\int_0^1 t^2\operatorname{curl}\vec F(\underbrace{\vec r}_{t\vec R})\,dt$

4.5.7 Prove: if $\vec{F}$ and $\vec{G}$ are irrotational, then $\vec{F}\times\vec{G}$ is solenoidal. Can you find the vector potential for $F\times G$?

$\vec{F}, \vec{G}$ have scalar potentials: $\vec{F} = \nabla\phi,\ \vec{G} = \nabla\psi$.

Then $\vec{F}\times\vec{G} = \nabla\phi \times \nabla\psi$

Use the identity $\nabla\times(\phi\vec{F}) = \nabla\phi\times\vec{F} + \phi\,\vec{\nabla}\times\vec{F}$ here:

$$\nabla\times(\phi\nabla\psi) = \nabla\phi\times\nabla\psi + \phi\underbrace{\nabla\times(\nabla\psi)}_{\equiv\, 0 \text{ curl of a gradient}} = \nabla\phi\times\nabla\psi$$

similarly $-\nabla\times(\psi\nabla\phi) = -\nabla\psi\times\nabla\phi = \nabla\phi\times\nabla\psi$

$$\left.\begin{array}{l} H_1 = \phi\vec{\nabla}\psi \\ H_2 = -\psi\vec{\nabla}\phi \end{array}\right\}$$

So $$\left.\begin{aligned} F\times G &= \nabla\times\vec{H}_1 = \nabla\times(\phi\nabla\psi) \\ &= \nabla\times\vec{H}_2 = \nabla\times(-\psi\nabla\phi) \end{aligned}\right\}\ \text{equivalent potentials}$$

In fact $\nabla\times(H_1 - H_2) = 0$ since $H_1 - H_2 = \phi\nabla\psi + \psi\nabla\phi = \nabla(\phi\psi)$

i.e. the two potentials differ by the gradient of a scalar field.

4.5.8 Construct a vector potential $\vec{G}$ for the rotating fluid field $\vec{F} = \vec{A}\times\vec{R}$ with $G_3 = 0$

$\vec{F} = \vec{A}\times\vec{R} = (zA_2 - yA_3)\vec{i} + (xA_3 - zA_1)\vec{j} + (yA_1 - xA_2)\vec{k} = \nabla\times\vec{G} =$

$(\vec{G} = G_1\vec{i} + G_2\vec{j}\ :)\quad = -\frac{\partial G_2}{\partial z}\vec{i} + \frac{\partial G_1}{\partial z}\vec{j} + \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\vec{k}$

Already know
$\nabla\times G = A\times R$
$G = \frac{1}{3}(A\times R)\times R$

(i) $$\left.\begin{aligned} \frac{\partial G_2}{\partial z} &= -zA_2 + yA_3 \Rightarrow G_2(x,y,z) = -\frac{z^2}{2}A_2 + yzA_3 + f_2(x,y) \\ \frac{\partial G_1}{\partial z} &= xA_3 - zA_1 \Rightarrow G_1(x,y,z) = xzA_3 - \frac{z^2}{2}A_1 + f_1(x,y) \end{aligned}\right\}$$

$\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) = f_{2,x} - f_{1,y} = yA_1 - xA_2$ which can have many solutions

say, $f_1 \equiv 0$; $f_2 = xyA_1 - \frac{x^2}{2}A_2$

Show that these differ by the gradient of a scalar.

i.e. $\boxed{\vec{G} = \left(xzA_3 - \frac{z^2}{2}A_1\right)\vec{i} + \left(\left(-\frac{z^2}{2}A_2 + yzA_3\right) + \left(xyA_1 - \frac{x^2}{2}A_2\right)\right)\vec{j}}$