Math. 311
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chaum's p. 134
(67, 68, 69)

#67 A vector $\vec{B}$ is always normal to a given closed surface S. Show that $\iiint_V \nabla\times B\, dV = 0$ where V is the region bound by S.

In the divergence theorem, $\iiint \nabla\cdot F\, dV = \oiint F\cdot d\vec{S}$ (*)

let $\vec{F} = \vec{B}\times\vec{C}$ with $\vec{C}$ arbitrary constant vector. Then

$\nabla\cdot F = \nabla\cdot(\vec{B}\times\vec{C}) = C\cdot\vec{\nabla}\times\vec{B}$, and $\vec{F}\cdot\hat{n} = (B\times C)\cdot n =$

$= -C\times B\cdot n = -C\cdot(B\times n)$. Substitution in (*) gives

$$\vec{C}\cdot\iiint \nabla\times B\, dV = -\vec{C}\cdot\oiint \vec{B}\times\vec{n}\, dS$$

. Since $\vec{C}$ is arbitrary

it follows $\vec{C}\cdot\left\{\iiint \nabla\times B\, dV + \oiint \vec{B}\times\hat{n}\, dS\right\} = 0$ and

the expression in brackets is orthogonal to any vector $\vec{C}$,

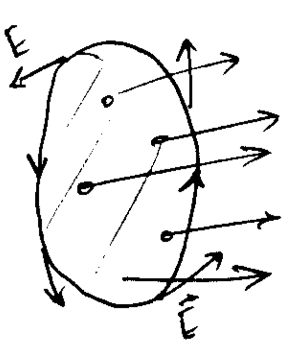
. it vanishes:

$$\iiint_V \nabla\times B\, dV = \oiint_S \hat{n}\times\vec{B}\, dS$$

(proven in class)

But if $\vec{B}\perp S$ everywhere $\Rightarrow \vec{B}\parallel\hat{n} \Rightarrow \overline{\vec{B}\times\hat{n} \equiv 0}$

#68 If $\oint_C E\cdot d\vec{r} = -\frac{1}{c}\partial_t\iint_S H\cdot dS$, where S is any surface bounded by the closed curve C, show that $\nabla\times E = -\frac{1}{c}\partial_t H$.



By Stokes, $\oint_C E\cdot d\vec{r} = \iint \nabla\times E\cdot d\vec{S}$,

so that $\iint \nabla\times E\cdot d\vec{S} = -\frac{1}{c}\partial_t\iint H\cdot d\vec{S}$

$= \iint\left(-\frac{1}{c}\partial_t H\right)\cdot d\vec{S}$

$\Rightarrow \iint_S\left\{\nabla\times E + \frac{1}{c}\partial_t H\right\}\cdot d\vec{S} = 0$. Now if we can assume that this is true for any C, then quantity in brackets vanishes.

#69) $\oint_C \phi \, d\vec{r} = \iint d\vec{S} \times \nabla\phi$

Consider $\vec{C}$ = constant, arbitrary. Then

$$\vec{C}\cdot\oint \phi \, d\vec{r} = \oint (\phi\vec{C})\cdot d\vec{r} = \iint \nabla\times(\phi\vec{C})\cdot d\vec{S}$$

$$= \iint (\nabla\phi\times\vec{C}\cdot\vec{n})\, d\vec{S} = C\cdot\iint \vec{n}\times\nabla\phi \, dS$$

$$\Rightarrow C\cdot\left\{\oint \phi \, d\vec{r} - \iint d\vec{S}\times\nabla\phi\right\} = 0$$; true for any $\vec{C}$

~~#2~~ $\oint$