Math. 311
Spring 99
Set 5

(p.95, 1, 3*, 4, 5*, 6*, (Sec. 2.3) 10)

(1,3,4): (a) Find speed (b) $a_{\parallel}, a_{\perp}$ (c) $\vec{T}$ (d) k

p.95, 2.3.1 $x = e^t \cos t,\ y = e^t \sin t,\ z = 0$

$$\vec{R} = e^t \cos t\, \vec{i} + e^t \sin t\, \vec{j}$$

$$\vec{V} = \frac{d\vec{R}}{dt} = e^t(\cos t - \sin t)\vec{i} + e^t(\sin t + \cos t)\vec{j}$$

(a) $$|V| = e^t\left((\cos t - \sin t)^2 + (\sin t + \cos t)^2\right)^{1/2}$$
$$= e^t\Big[\underbrace{\cos^2 t + \sin^2 t}_{=1} - \cancel{2\cos t \sin t} + \underbrace{\sin^2 t + \cos^2 t}_{=1} + \underbrace{2\cos t \sin t}_{=0}\Big]^{1/2} = e^t\sqrt{2} = \frac{ds}{dt}$$

(c) $$\vec{T} = \frac{\vec{V}}{V} = \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\vec{i} + \left(\frac{\sin t + \cos t}{\sqrt{2}}\right)\vec{j}$$

(d) $$\frac{d\vec{T}}{ds} = \frac{1}{V}\frac{d\vec{T}}{dt} = \frac{e^{-t}}{\sqrt{2}}\left\{-\frac{(\sin t + \cos t)}{\sqrt{2}}\vec{i} + \frac{(\cos t - \sin t)}{\sqrt{2}}\vec{j}\right\} = k\vec{N}$$

$$\left|\frac{d\vec{T}}{ds}\right| = \frac{e^{-t}}{2}\Big(\underbrace{(\sin t + \cos t)^2 + (\cos t - \sin t)^2}_{1 + \cancel{2\cos t \sin t} + 1 - \cancel{2\cos t \sin t}}\Big)^{1/2} = \boxed{\frac{e^{-t}}{\sqrt{2}} = k(t)}$$

(b) $$a_{\parallel} = \frac{d^2 s}{dt^2} = e^t\sqrt{2}$$
$$a_{\perp} = k\left(\frac{ds}{dt}\right)^2 = \frac{e^{-t}}{\sqrt{2}} \cdot 2e^{2t} = \sqrt{2}\, e^t$$

p.95, 2.3.3* > $\vec{R}(t) = e^t\left(\cos t\,\vec{i} + \sin t\,\vec{j} + k\right)$

$\left(\frac{5}{2/4}\right)$ $\vec{V} = \frac{d\vec{R}}{dt} = e^t\left\{(\cos t - \sin t)\vec{i} + (\sin t + \cos t)\vec{j} + \vec{k}\right\}$

$$v = \frac{ds}{dt} = e^t\Big(\underbrace{(\cos t - \sin t)^2 + (\sin t + \cos t)^2}_{\overset{\shortparallel}{2}} + 1\Big)^{1/2} = \sqrt{3}\,e^t$$

$$\left.\begin{array}{l}\cos^2 t + \sin^2 t - 2\sin t\cos t \\ + \cos^2 t + \sin^2 t + 2\sin t\cos t\end{array}\right\} = 2$$

$$\vec{T} = \frac{\vec{V}}{v} = \frac{1}{\sqrt{3}}\left\{(\cos t - \sin t)\vec{i} + (\sin t + \cos t)\vec{j} + \vec{k}\right\}$$

$$\frac{1}{v}\frac{d\vec{T}}{dt} = \frac{e^{-t}}{\sqrt{3}} \circ \frac{1}{\sqrt{3}}\left\{-(\sin t + \cos t)\vec{i} + (\cos t - \sin t)\vec{j}\right\} = k\vec{N}$$

$$k = \frac{e^{-t}}{3}\left((\sin t + \cos t)^2 + (\cos t - \sin t)^2\right)^{1/2} = e^{-t}\frac{\sqrt{2}}{3}$$

$$a_{\parallel} = \frac{d^2 s}{dt^2} = \sqrt{3}\,e^t$$

$$a_{\perp} = \cancel{\#}\, k\left(\frac{ds}{dt}\right)^2 = e^{-t}\frac{\sqrt{2}}{3} \cdot 3e^{2t} = \sqrt{2}\,e^t$$

p.95, 2.3.4 > $\vec{R} = 5\sin 4t\,\vec{i} + 5\cos 4t\,\vec{j} + 10t\,\vec{k}$

$\vec{v} = 20\cos 4t\,\vec{i} - 20\sin 4t\,\vec{j} + 10\vec{k}$

$v = \left(400\cos^2 4t + 400\sin^2 4t + 100\right)^{1/2} = 10\sqrt{5}$

$\vec{T} = \frac{4}{\sqrt{5}}\left(\frac{\cos 4t}{2}\vec{i} - \frac{\sin 4t}{2}\vec{j} + \frac{\vec{k}}{4}\right)$

$\frac{1}{v}\frac{dT}{dt} = \frac{1}{10\sqrt{5}} \cdot \frac{8}{\sqrt{5}}\left(-2\sin 4t\,\vec{i} - 2\cos 4t\,\vec{j}\right) = -\frac{8}{50}\left(\sin 4t\,\vec{i} + \cos 4t\,\vec{j}\right)$

$k = \frac{4}{25}$; $\quad a_{\parallel} = 0$, $\quad a_{\perp} = \frac{8}{50}\cdot\left(10\sqrt{5}\right)^2 = 80$

p.96, 2.3.5* The position vector of a moving particle is $\vec{R} = \cos t\,(\vec{i}-\vec{j}) + \sin t\,(\vec{i}+\vec{j}) + \frac{1}{2} t\vec{k}$

(a) Determine the velocity and the speed of the particle (b) Find $\vec{T}$ (c) Find $\vec{a}$ (d) Find k, constant? (e) Show helix:

$\left(5\ 3/4\right)$

$$\vec{v} = \frac{d\vec{R}}{dt} = -\sin t\,(\vec{i}-\vec{j}) + \cos t\,(\vec{i}+\vec{j}) + \frac{1}{2}\vec{k}$$
$$= (\cos t - \sin t)\,\vec{i} + (\cos t + \sin t)\,\vec{j} + \frac{1}{2}\vec{k}$$

$$v = \Big(\underbrace{(\cos t - \sin t)^2}_{1-2\cancel{\cos t}\sin t} + \underbrace{(\cos t + \sin t)^2}_{1+2\cancel{\cos t}\sin t} + \frac{1}{4}\Big)^{1/2} = \left(\frac{9}{4}\right)^{1/2} = \frac{3}{2}$$

constant speed

$$\vec{T} = \frac{1}{v}\vec{v} = -\frac{2}{3}\sin t\,(\vec{i}-\vec{j}) + \frac{2}{3}\cos t\,(\vec{i}+\vec{j}) + \frac{1}{3}\vec{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -(\cos t + \sin t)\,\vec{i} + (\cos t - \sin t)\,\vec{j}$$

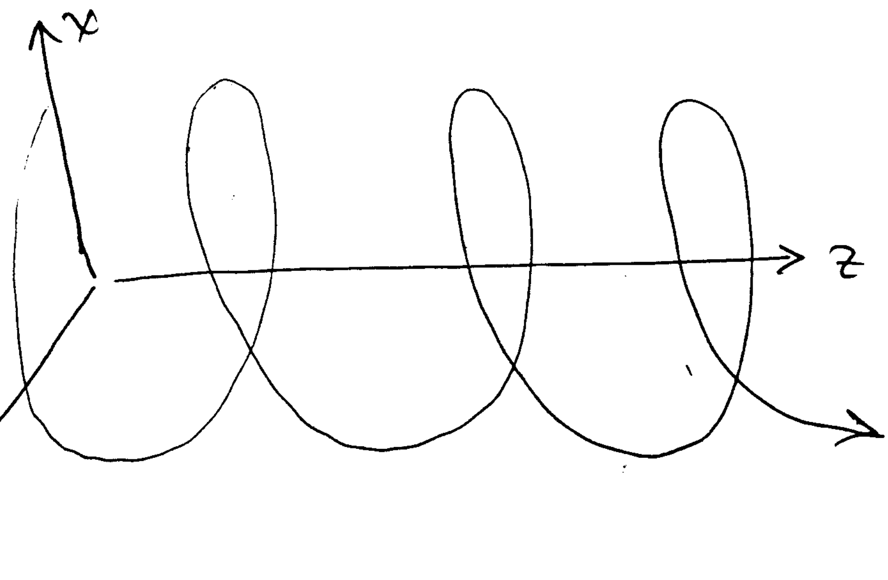
$$\frac{1}{v}\frac{d\vec{T}}{dt} = \frac{2}{3}\cdot\frac{2}{3}\Big[-\cos t\,(\vec{i}-\vec{j}) - \sin t\,(\vec{i}+\vec{j})\Big]$$
$$-(\cos t + \sin t)\,\vec{i} + (\cos t - \sin t)\,\vec{j}$$

$$K = \frac{4}{9}\left((\cos t + \sin t)^2 + (\cos t - \sin t)^2\right)^{1/2} = \frac{4\sqrt{2}}{9} \quad \text{constant}$$

Constant curvature

constant speed in z-direction

Helix



p.96, 2.3.6*) Find the curvature of the space curve $x=3t^2-t^3$, $y=3t^2$, $z=3t+t^3$

(5 4/4)

$$\vec{R}=(3t^2-t^3)\vec{i}+3t^2\vec{j}+(3t+t^3)\vec{k}$$

$$\frac{d\vec{R}}{dt}=(6t-3t^2)\vec{i}+6t\vec{j}+(3+3t^2)\vec{k}=3$$

$$v=\left((6t-3t^2)^2+36t^2+9(1+t^2)^2\right)^{1/2}$$

$$=\left(36t^2+9t^4-36t^3+36t^2+9+9t^4+18t^2\right)^{1/2}$$

$$=\left(18t^4-36t^3+90t^2+9\right)^{1/2}$$

$$\boxed{|\vec{R}'|=3\left(2t^4-4t^3+10t^2+1\right)^{1/2}}$$

$\frac{1}{16}$ $\frac{1}{8}$ $\frac{1}{4}$

$$\frac{d^2R}{dt^2}=\tfrac{1}{8}(6-6t)\vec{i}+6\vec{j}+6t\vec{k}=6\left[(1-t)\vec{i}+\vec{j}+t\vec{k}\right]$$

$$\frac{d\vec{R}}{dt}=3\left[t(2-t)\vec{i}+2t\vec{j}+\cancel{8}(1+t^2)\vec{k}\right]$$

$$R'\times R''=18\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2t-t^2 & 2t & 1+t^2\\ 1-t & 1 & t\end{vmatrix}=\begin{matrix}\vec{i}(2t^2-1-t^2)\\ -\vec{j}\left(2t^2-t^3-(1-t+t^2-t^3)\right)\\ +\vec{k}(2t-t^2-2t+2t^2)\end{matrix}$$

$$=18\left[(t^2-1)\vec{i}+(t^2+t-1)\vec{j}+t^2\vec{k}\right]$$

$$|R'\times R''|=18\left((t^2-1)^2+(t^2+t-1)^2+t^4\right)^{1/2}$$

$$=18\left(t^4-2t^2+1+t^4+2t^3+t^2-2t^2-2t+1+t^4\right)^{1/2}$$

$$=18\left(3t^4+2t^3-3t^2-2t+2\right)^{1/2}$$

$$k=|R'\times R''|\,/\,|R'|^3$$