Math. 311
11 – Solutions

p. 140 (4*, 5) Sec. 3.6
p. 150, Sec. 3.8 (6*, 9*, 10(a, b*, e, g, h*, i), 11, 12*.

p. 140, 3.6.4 $\Delta f = 0$?

(a) $f = e^{z} \sin y$

$f_{xx} = 0, \quad f_{yy} = -e^{z} \sin y, \quad f_{zz} = e^{z} \sin y$

$\Rightarrow \Delta f = 0$

(b) $f = \sin x \sinh y + \cos x \cosh y$

$f_{xx} = -\sin x \sinh y - \cos x \cosh y, \quad f_{yy} = \sin x \sinh y + \cos x \cosh y, \quad f_{zz}$

$\Rightarrow \Delta f = 0 \quad$ (since $(\sin x)_{xx} = -\sin x$, $(\sinh y)_{yy} = \sinh y$ ct

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(c) $f = \sin px \sinh qy$: $\quad f_{xx} = -p^2 \sin px \sinh qy \qquad f_{zz} =$

$f_{yy} = q^2 \sin px \sinh qy$

$\Rightarrow \Delta f = (-p^2 + q^2) f = 0$ only if $\boxed{q^2 - p^2 = 0}$

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p. 150, 3.8.6 $\vec{A} = a_1 \vec{i} + a_2 \hat{j} + a_3 \vec{k}$; show $\nabla \cdot \left(\frac{A \times \bar{R}}{R} \right) = 0$:

$$\nabla \cdot \left(\frac{1}{R} (A \times R) \right) = \nabla \left(\frac{1}{R} \right) \cdot (A \times R) + \frac{1}{R} \nabla \cdot (A \times R)$$

$$= -\underbrace{\frac{\vec{R}}{R^3} \cdot (A \times R)}_{\equiv 0} + \frac{1}{R} A \cdot (\underbrace{\nabla \times R}_{\equiv 0}) - \frac{1}{R} R \cdot (\underbrace{\nabla \times A}_{\equiv 0})$$

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p. 150, 3.8.9 Evaluate $A \cdot \nabla R + \nabla (A \cdot R) + (A \times \nabla) \times R$

(note parentheses!) $\nabla \times R \equiv 0$, so we chose the other possibility. They are not equivalent, just like $(A \times B) \times C \neq A \times (B \times C)$!

$A \cdot \nabla R = (A_1 \partial_x + A_2 \partial_y + A_3 \partial_z)(x \vec{i} + y \vec{j} + z \vec{k})$

$= A_1 \vec{i} + A_2 \vec{j} + A_3 \vec{k} = \vec{A}$

$\nabla (A \cdot R) = (\vec{i} \partial_x + \vec{j} \partial_y + \vec{k} \partial_z)(A_1 x + A_2 y + A_3 z) = \vec{A}$

$(A \times \nabla) \times R = [\vec{i} (A_2 \partial_z - A_3 \partial_y) - \vec{j} (A_1 \partial_z - A_3 \partial_x) + \vec{k} (A_1 \partial_y - A_2 \partial_x)] \times \vec{R}$

over

$$(A\times\nabla)\times\vec{R} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_2\partial_z - A_3\partial_y & A_3\partial_x - A_1\partial_z & A_1\partial_y - A_2\partial_x \\ x & y & z \end{vmatrix}$$

$$= \vec{i}\left((A_3\partial_x - A_1\partial_z)z - (A_1\partial_y - A_2\partial_x)y \right)$$

$$- \vec{j}\left((A_2\partial_z - A_3\partial_y)z - (A_1\partial_y - A_2\partial_x)x \right)$$

$$+ \vec{k}\left((A_2\partial_z - A_3\partial_y)y - (A_3\partial_x - A_1\partial_z)x \right)$$

$$= \vec{i}\left(-A_1 - A_1\right) - \vec{j}\left(A_2 + A_2\right) + \vec{k}\left(-A_3 - A_3\right)$$

$$= -2\vec{A}$$

$$\therefore\ (A\cdot\nabla)R + \nabla(A\cdot R) + (A\times\nabla)\times R = 0$$

Using <u>tensor notation</u> (if you manage, do read through it!):

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad ; \quad \varepsilon_{ijk} = \begin{cases} 1 & ; \ ijk \text{ cyclic permutation of } (1,2,3) \\ 0 & ; \text{ repeated indices} \\ -1 & ; \text{ non-cyclic permutation} \end{cases}$$

Verify: $\varepsilon_{ijk} = -\varepsilon_{ikj}$; $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$; $\delta_{ij} = \delta_{ji}$

<u>Summation convention:</u> sum over repeated indices!

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

Let $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$

Then $\partial_{x_i} x_j \equiv \dfrac{\partial x_j}{\partial x_i} = \delta_{ij}$

$$\frac{\partial x_i}{\partial x_i} \equiv \sum_{i=1}^{3} \frac{\partial x_i}{\partial x_i} = 1+1+1 = 3$$

This technology is very useful if you are ever going to do very complicated calculations. It is a life saver! B... you do not need it to get the basic ideas, so we usually don't teach it at this level!...

Then, defining $\vec{e}_1 = \vec{i}$, $\vec{e}_2 = \vec{j}$, $\vec{e}_3 = \vec{k}$

$$\vec{e}_i \partial_{x_i} \equiv \vec{e}_1 \frac{\partial}{\partial x_1} + \vec{e}_2 \frac{\partial}{\partial x_2} + \vec{e}_3 \frac{\partial}{\partial x_3} = \nabla$$

$$\partial_{x_i} F_i = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} = \nabla \cdot \vec{F}$$

$$\vec{e}_i \varepsilon_{ijk} A_j B_k = \vec{e}_1 (\overset{+1}{\varepsilon_{123}} A_2 B_3 + \overset{-1}{\varepsilon_{132}} A_3 B_2) \quad \text{(other combinations vanish}$$
$$+ \vec{e}_2 (\overset{-1}{\varepsilon_{213}} A_1 B_3 + \overset{+1}{\varepsilon_{231}} A_3 B_1)$$
$$+ \vec{e}_3 (\overset{+1}{\varepsilon_{312}} A_1 B_2 + \overset{-1}{\varepsilon_{321}} A_2 B_1)$$
$$= \vec{e}_1 (A_2 B_3 - A_3 B_2) + \vec{e}_2 (A_3 B_1 - A_1 B_3) + \vec{e}_3 (A_1 B_2 - A_2 B_1)$$
$$= \vec{A} \times \vec{B}$$

$$\vec{e}_i \varepsilon_{ijk} \partial_{x_j} F_k = \nabla \times F \quad \text{(show similarly)}$$

So:

$$\nabla \phi = \vec{e}_i \partial_{x_i} \phi$$
$$\nabla \cdot F = \partial_{x_i} F_i$$
$$A \times B = \vec{e}_i \varepsilon_{ijk} A_j B_k$$
$$\nabla \times F = \vec{e}_i \varepsilon_{ijk} \partial_{x_j} F_k$$
$$\delta_{ii} = 3$$
$$A \cdot B = A_i B_i$$

Can show

$$\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

very important identity!

Summation convention: when an index appears twice, sum over it:

$$A_i B_i \equiv \sum_{i=1}^{3} A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$$

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Tensor notation

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$$
$$\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$$

$$\varepsilon_{112} = \varepsilon_{113} = \varepsilon_{122} = \cdots = \varepsilon_{333} = 0$$

Then:

$$A\times(B\times C) = \vec{e}_i\left\{\varepsilon_{ijk}A_j\left(\varepsilon_{klm}B_\ell C_m\right)\right\}$$

$$= \vec{e}_i\ \underbrace{\left(\varepsilon_{ijk}\varepsilon_{klm}\right)}_{\left(\delta_{i\ell}\delta_{jm}-\delta_{im}\delta_{j\ell}\right)} A_j B_\ell C_m$$

$$= \vec{e}_i\ \delta_{i\ell}\delta_{jm}A_jB_\ell C_m - \vec{e}_i\,\delta_{im}\delta_{j\ell}A_jB_\ell C_m$$

But
$$\begin{cases}\delta_{i\ell}B_\ell = \delta_{i1}B_1+\delta_{i2}B_2+\delta_{i3}B_3 = B_i\\ \delta_{jm}A_jC_m = A_mC_m = A\circ C\\ \delta_{im}C_m = C_i\\ \delta_{j\ell}A_jB_\ell = A_\ell B_\ell = A\circ B\end{cases}$$

$$\Rightarrow\ A\times(B\times C) = \vec{B}(A\circ C)-\vec{C}(A\circ B)$$

Now, for problem (9): $\vec{A}=\vec{e}_iA_i,\ \vec{R}=\vec{e}_i x_i,\ \vec{\nabla}=\vec{e}_j\partial_{x_j}$

$$(A\circ\nabla)\vec{R} = \left(A_i\frac{\partial}{\partial x_i}\right)\vec{e}_j x_j = \vec{e}_j\,A_i\left(\frac{\partial x_j}{\partial x_i}\right) = \vec{e}_jA_i\delta_{ij}$$

$$= \vec{e}_iA_i = \vec{A}$$

$$\nabla(A\circ R) = \vec{e}_i\partial_{x_i}(A_jx_j) = \vec{e}_iA_j\frac{\partial x_j}{\partial x_i} = \vec{e}_iA_j\delta_{ij} = \vec{e}_iA_i$$

$$= \vec{A}$$

$$(A\times\nabla)\times R = \vec{e}_i\ \varepsilon_{ijk}\left(\varepsilon_{j\ell m}A_\ell\partial_{x_m}\right)x_k$$

$$= \vec{e}_i\left(\varepsilon_{ijk}\varepsilon_{j\ell m}\right)A_\ell\frac{\partial x_k}{\partial x_m} = -\vec{e}_i\,\delta_{i\ell}\delta_{km}A_\ell\delta_{km}$$

$$\varepsilon_{ijk}\varepsilon_{j\ell m} = -\varepsilon_{ikj}\varepsilon_{j\ell m} = -\delta_{i\ell}\delta_{km}+\delta_{im}\delta_{k\ell}, \qquad \frac{\partial x_k}{\partial x_m} = \delta_{km}$$

$$+\vec{e}_i\delta_{im}\delta_{k\ell}A_\ell\delta_{km} =$$

$$= -\vec{e}_iA_i\delta_{mm}+\vec{e}_i\delta_{ki}A_k$$

$$= -3\vec{A}+\vec{A} = -2\vec{A}$$

p. 150, 3.8.10(b,h) (a) $\nabla\times(R^2\vec{A}) = \nabla R^2 \times \vec{A} + R^2 \underbrace{\cancel{\nabla\times A}}_{=0}$

$(\nabla R^2 = 2\vec{R})$ $\qquad \nabla R^2 \times \vec{A} = 2\vec{R}\times\vec{A}$ $\qquad$ ($\vec{A}$ constant)

(h) $\nabla\times(A\times\vec{R}) = \vec{A}\,\nabla\cdot\vec{R} - (A\cdot\nabla)\vec{R}$

$= 3\vec{A} - \vec{A} = 2\vec{A}$

(note that it is true that: **Note**

$A\times\nabla = -\nabla\times A$

if we treat both of these expressions as operators, that is

(1) $A\times\nabla\phi = -\nabla\times(A\phi)$ (see identity 3.29)

(2) $(A\times\nabla)\cdot\vec{F} = -\nabla\times A\cdot F = -\nabla\cdot(A\times F)$ (see 3.36)

$(A\times\nabla)\cdot\vec{F} = A\cdot(\nabla\times F)$ $\qquad -\nabla\cdot(A\times F) = -F\cdot\underbrace{(\cancel{\nabla\times A})}_{=0} + A\cdot(\nabla\times F)$

(3) $(A\times\nabla)\times\vec{F} = -(\nabla\times A)\times\vec{F}$ but now ∇ must be thought as acting on $\vec{F}$ as well, and parentheses must be kept — we get different ~~exp~~ results if we move them!

p. 150, 3.8.12 For what value of the constant C is the field $\vec{v} = (x+4y)\hat{i} + (y-3z)\hat{j} + Cz\hat{k} = \nabla\times\vec{F}$ for some $\vec{F}$?

If $\vec{v} = \nabla\times\vec{F} \Rightarrow \nabla\cdot\vec{v} = 0$; Chose C to make it so!

$\vec{\nabla}\cdot\vec{v} = \partial_x(x+4y) + \partial_y(y-3z) + \partial_z(Cz)$

$= 1 + 1 + C = 0 \Rightarrow \underline{C = -2}$

$\vec{v} = (x+4y)\hat{i} + (y-3z)\hat{j} - 2z\hat{k} = \nabla\times\vec{F}$ for some $\vec{F}$