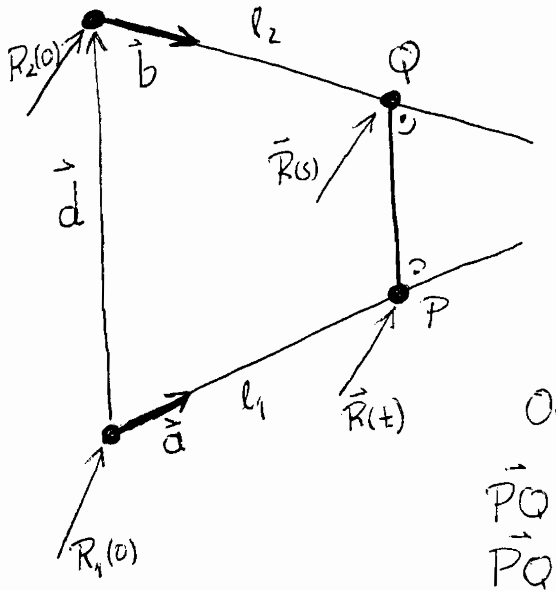
# "Oblique lines"




$$\vec P = \vec R_1(t) = \vec R_1 + \vec a t$$
$$\vec Q = \vec R_2(s) = \vec R_2(0) + \vec b(t)$$
$$\text{Let } \vec d = \vec R_2(0) - \vec R_1(0)$$
$$\vec{PQ} = \vec R_2(s) - \vec R_1(t) = \vec d + \vec b s - \vec a t$$

**PROBLEM**

Given two oblique lines $l_1, l_2$; find points $P, Q$ on them such that $PQ$ is the shortest distance between $l_1, l_2$

Orthogonality:

$$\vec{PQ}\cdot\vec a = 0 \Rightarrow \vec d\cdot\vec a + s\,\vec b\cdot\vec a - t\,\vec a\cdot\vec a = 0$$
$$\vec{PQ}\cdot\vec b = 0 \Rightarrow \vec d\cdot\vec b + s\,\vec b\cdot\vec b - t\,\vec a\cdot\vec b = 0$$

Get system of two equations in two unknowns, $t$ and $s$:

$$\left.\begin{aligned}(\vec a\cdot\vec a)t - (\vec a\cdot\vec b)s &= \vec a\cdot\vec d\\ (\vec a\cdot\vec b)t - (\vec b\cdot\vec b)s &= \vec b\cdot\vec d\end{aligned}\right\}\ \text{solve by Cramer's rule:}$$

$$t = \frac{\begin{vmatrix} a\cdot d & -(a\cdot b)\\ b\cdot d & -(b\cdot b)\end{vmatrix}}{\begin{vmatrix} a\cdot a & -(a\cdot b)\\ a\cdot b & -(b\cdot b)\end{vmatrix}} \overset{(*_1)}{=} \frac{\begin{vmatrix} a\cdot d & a\cdot b\\ b\cdot d & b\cdot b\end{vmatrix}}{\begin{vmatrix} a\cdot a & a\cdot b\\ a\cdot b & b\cdot b\end{vmatrix}} \overset{(*_2)}{=} \frac{(a\times b)\cdot(d\times b)}{(a\times b)\cdot(a\times b)}$$

$$S = \frac{\begin{vmatrix} a\cdot a & a\cdot d\\ a\cdot b & b\cdot d\end{vmatrix}}{\begin{vmatrix} a\cdot a & -b\cdot a\\ a\cdot b & -b\cdot b\end{vmatrix}} \overset{(*_1)}{=} \frac{\begin{vmatrix} a\cdot d & a\cdot a\\ b\cdot d & a\cdot b\end{vmatrix}}{\begin{vmatrix} a\cdot a & b\cdot a\\ a\cdot b & b\cdot b\end{vmatrix}} \overset{(*_2)}{=} \frac{(a\times b)\cdot(d\times a)}{(a\times b)\cdot(a\times b)}$$

Example

If $l_1: \frac{x-1}{3} = \frac{y}{2} = z$ : $\vec R_1(t) = (1,0,0) + (3,2,1)t$

$l_2: \frac{x}{2} = y = -z$ : $\vec R_2(s) = (0,0,0) + (2,1,-1)s$

$$\boxed{\begin{aligned}\vec a &= (3,2,1)\\ \vec b &= (2,1,-1)\\ \vec d &= (-1,0,0)\end{aligned}}$$

$a\times b = (-3,5,-1)$, $|a\times b| = \sqrt{35}$

$d\times a = (0,+1,-2)$, $d\times b = (0,-1,-1)$

$$\begin{cases} t = \dfrac{(a\times b)\cdot(d\times b)}{|a\times b|^2} = -4/35\\[2ex] S = \dfrac{(a\times b)\cdot(d\times a)}{|a\times b|^2} = +7/35 = +1/5\end{cases}$$

$$\begin{cases} P(4/35) = (1,0,0) + (3,2,1)\,4/35 = (23/35, -8/35, -4/35)\\ Q(-1/5) = (2,1,-1)(+1/5) = (+2/5, +1/5, -1/5)\end{cases} \Rightarrow (PQ)^2 = 9/35$$

NOTE: $|\vec{PQ}| = \dfrac{|\vec d\cdot\vec a\times\vec b|}{|\vec a\times\vec b|}$ gives shortest distance

Here $|\vec{PQ}| = \dfrac{|-3|}{\sqrt{35}} = \dfrac{3}{\sqrt{35}}$

$(*_1)$: using facts that $\begin{vmatrix} r\alpha & \beta\\ r\gamma & \delta\end{vmatrix} = r\begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix}$, $\begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix} = -\begin{vmatrix}\beta & \alpha\\ \delta & \gamma\end{vmatrix}$

$(*_2)$: using identity $\begin{vmatrix} a\cdot c & b\cdot c\\ a\cdot d & b\cdot d\end{vmatrix} = (a\times b)\cdot(c\times d)$