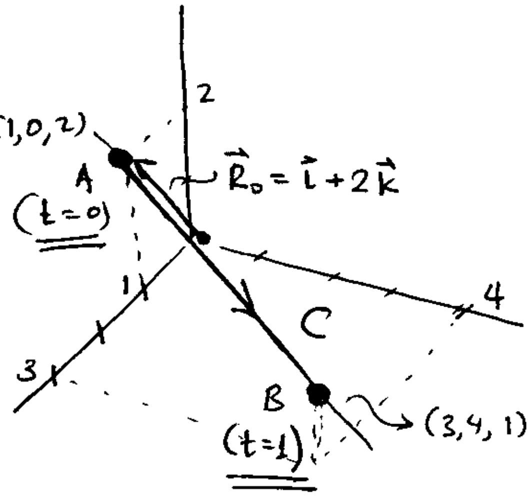
Set 13
Math. 311
p.190, Sec. 4.1
5*, 6*, 7*, 8*

4.1.5 $R_0 = i + 2j$

Line: $\vec{AB} = (3-1, 4-0, 1-2)$
$= 2\vec{i} + 4\vec{j} - \vec{k}$

$\vec{R}(t) = \vec{R}_0 + t\,\vec{AB}$
$= (\vec{i} + 2\vec{j}) + t(2\vec{i} + 4\vec{j} - \vec{k})$



Then $0 \le t \le 1$ $\begin{cases} x = 2t+1 \\ y = 4t \\ z = -t+2 \end{cases}$ $\dfrac{d\vec{R}}{dt} = \vec{A}B = (2\vec{i} + 4\vec{j} - \vec{k})$

Given $\vec{F} = 2xy\,\vec{i} + (x^2+z)\,\vec{j} + y\,\vec{k}$

On C: $\vec{F}(t) \cdot \vec{AB} = 4xy + 4(x^2+z) - y$

$$\int_C \vec{F}\cdot d\vec{R} = \int_{t=0}^{1} \{4(2t+1)4t + 4[\underbrace{(2t+1)^2 + (2-t)}_{4t^2+4t+1+2-t}] - 4t\}\,dt$$

$$= \int_{t=0}^{1} \{32t^2 + 16t + 16t^2 + 12t + 12 - 4t\}\,dt$$

$$= \int_{t=0}^{1} \{48t^2 + 24t + 12\}\,dt = 12\int_0^1 (4t^2 + 2t + 1)\,dt$$

$$= 12\left(\tfrac{4}{3}t^3 + t^2 + t\right)\Big|_0^1 = 12 \cdot \frac{10}{3} = 40$$

Another way: $\vec{F} = \vec{\nabla}\phi$ with $\phi(x,y,z) = (x^2+z)y$

Then $\displaystyle\int_{t=0}^{1} \vec{F}\cdot d\vec{R} = \int_{t=0}^{1} \left(\frac{d\vec{R}}{dt} \cdot \nabla\phi\right) dt \equiv \int_{t=0}^{1} \frac{d\phi}{dt}\,dt =$

$$= \int_{t=0}^{1} d\phi = \phi\big|_{t=1} - \phi\big|_{t=0} = \phi(3,4,1) - \phi(1,0,2)$$

$$= (9+1)\cdot 4 - (1+2)\cdot 0 = 40$$

We say that ϕ is a potential for $\vec{F}$

(note that $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy & x^2+z & y \end{vmatrix} = \begin{matrix} \vec{i}\left(\partial_y y - \partial_z(x^2+z)\right) = 0 \\ -\vec{j}\left(\partial_x y - \partial_z(2xy)\right) = 0 \\ +\vec{k}\left(\partial_x(x^2+z) - \partial_y(2xy)\right) = 0 \end{matrix}$)

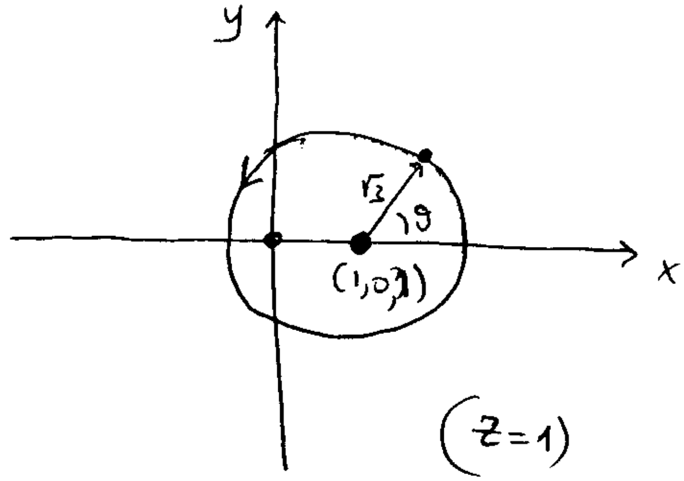
natural: $\nabla \times \nabla\phi \equiv 0$

4.1.6 $x^2 - 2x + y^2 = 2$: complete square:

(13 2/4) $(x^2 - 2x + 1) + y^2 = 3 \Rightarrow (x-1)^2 + y^2 = 3$

Circle, center at $(x,y,z) = (1,0,1)$; radius $\sqrt{3}$

Assume counterclockwise



$(z=1)$

$$\left.\begin{aligned} x &= 1 + \sqrt{3}\cos\theta \\ y &= \sqrt{3}\sin\theta \end{aligned}\right\}$$

$$\frac{d\vec{R}}{d\theta} = -\sqrt{3}\sin\theta\,\vec{i} + \sqrt{3}\cos\theta\,\vec{j}$$

$$\vec{F}(\theta) = y\vec{i} + x\vec{j} + xyz^2\vec{k}$$
$$= \sqrt{3}\sin\theta\,\vec{i} + (1+\sqrt{3}\cos\theta)\vec{j} + (1+\sqrt{3}\cos\theta)\sqrt{3}\sin\theta\,\vec{k}$$

$$F\cdot\frac{dR}{d\theta} = -3\sin^2\theta + (1+\sqrt{3}\cos\theta)\sqrt{3}\cos\theta$$
$$= \sqrt{3}\cos\theta + 3\cos^2\theta - 3\sin^2\theta = \sqrt{3}\cos\theta + 3\cos 2\theta$$

Then $\oint F\cdot d\vec{R} = \int_{\theta=0}^{2\pi} (\sqrt{3}\cos\theta + 3\cos 2\theta)\,d\theta = 0$

4.1.7 Let $\vec{F} = \frac{y}{x^2+y^2}\vec{i} - \frac{x}{x^2+y^2}\vec{j}$

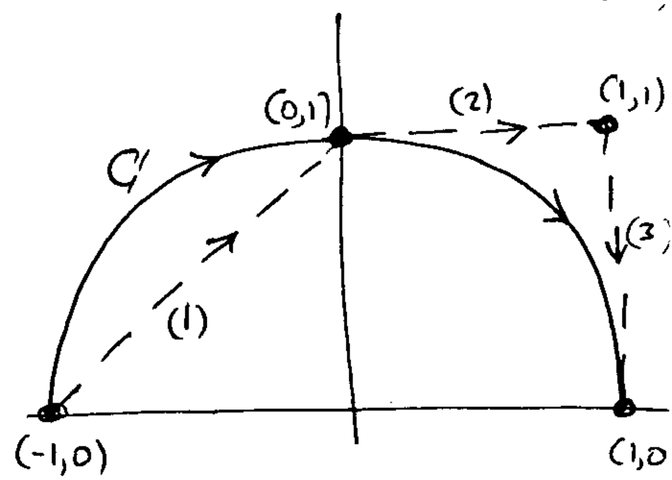
Find the line integral of the tangential component of $\vec{F}$, from $(-1,0)$ to $(1,0)$, (a) along the semicircle $y = \sqrt{1-x^2}$

(b) along the dotted polygonal path shown in figure 4.4.

4.1.8 Change to polar coordinates, to find the answer in (7) by inspection.

over →

(a) $C: y=\sqrt{1-x^2}$; $x^2+y^2=1$



$\vec{R} = x\vec{i} + y\vec{j} = x\vec{i} + \sqrt{1-x^2}\,\vec{j}$

$\dfrac{d\vec{R}}{dx} = \vec{i} - \dfrac{x}{\sqrt{1-x^2}}\vec{j}$

$\vec{F} = \dfrac{y}{1}\vec{i} - \dfrac{x}{1}\vec{j} = \sqrt{1-x^2}\,\vec{i} - x\vec{j}$

$\vec{F}\cdot\dfrac{d\vec{R}}{dx} = \sqrt{1-x^2} + \dfrac{x^2}{\sqrt{1-x^2}} =$

$= \dfrac{(1-x^2)+x^2}{\sqrt{1-x^2}} = \dfrac{1}{\sqrt{1-x^2}}$;

$\int_C F\cdot dR = \int_{-1}^{1}\dfrac{dx}{\sqrt{1-x^2}} = \int_{-\pi/2}^{\pi/2}\dfrac{\cos\vartheta\, d\vartheta}{\cos\vartheta} = \int_{-\pi/2}^{\pi/2} d\vartheta = \pi$

$x=\sin\vartheta$, $-\pi/2<\vartheta<\pi/2$, $\sqrt{1-x^2}=\cos\vartheta$, $dx=\cos\vartheta\, d\vartheta$

(b) $\int_{(1)}\vec{F}\cdot d\vec{R}$: $\vec{R}(t) = (-1,0) + t\,\overrightarrow{(1,1)} = (-1+t)\vec{i} + t\vec{j}$, $0\le t\le 1$

$(x=t-1,\ y=t)$

$\vec{F}(t) = \dfrac{y}{x^2+y^2}\vec{i} - \dfrac{x}{x^2+y^2}\vec{j}$

$= \dfrac{t}{t^2+(t-1)^2}\vec{i} - \dfrac{(t-1)}{t^2+(t-1)^2}\vec{j}$

$\dfrac{d\vec{R}}{dt} = \vec{i}+\vec{j}$

$\vec{F}\cdot\dfrac{d\vec{R}}{dt} = \dfrac{t}{t^2+(t-1)^2} - \dfrac{t-1}{t^2+(t-1)^2} = \dfrac{1}{t^2+(t-1)^2} = \dfrac{1}{2t^2-2t+1}$

$t = \dfrac{2\pm\sqrt{4-8}}{2} = 1\pm i$ complex roots

$\int_{(1)}\vec{F}\cdot d\vec{R} = \int_{t=0}^{1}\dfrac{dt}{2t^2-2t+1} = \tan^{-1}\left(\dfrac{4t-1}{2}\right)\Big|_0^1$

over $\longrightarrow$

here: $4ac-b^2 = 4\cdot2\cdot1-2^2 = 4$

Tables: $\int\dfrac{dx}{ax^2+bx+c} = \dfrac{2}{\sqrt{4ac-b^2}}\tan^{-1}\dfrac{2ax+b}{\sqrt{4ac-b^2}}$ if $4ac>b^2$ (Dwight)

So: $\displaystyle\int_{(1)} \vec F\cdot d\vec R = \tan^{-1}\left(\frac{4t-2}{2}\right)\Big|_{t=0} = \underset{\pi/4}{\tan^{-1}(1)} - \underset{-(-\pi/4)}{\tan^{-1}(-1)} = \pi/2$

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$F = \frac{1}{1+t^2}\, i - \frac{}{1+t^2}\, j \quad , \quad \frac{}{dt} = \frac{}{1+t^2}$

$$\int_{(2)} \vec F\cdot d\vec R = \int_{t=0}^{1} \frac{dt}{1+t^2} = \tan^{-1} t\,\Big|_0^1 = \overset{\pi/4}{\tan^{-1}(1)} - \overset{0}{\tan^{-1}(0)}$$

Along (1): $x=1,\ y = 1-t,\ 0\le t\le 1$:

$$\frac{d\vec R}{dt} = -\vec j \ , \quad \vec F = \frac{1-t}{1+(1-t)^2}\,\vec i - \frac{1}{1+(1-t)^2}\,\vec j$$

$$\vec F\cdot\frac{d\vec R}{dt} = \frac{1}{1+(1-t)^2}$$

$$\int_{(3)} \vec F\cdot\frac{d\vec R}{dt}\,dt = \int_{t=0}^{1} \frac{dt}{1+(1-t)^2} = -\int_{u=1}^{0} \frac{du}{u^2+1} = \tan^{-1}(1) - \tan^{-1}(0) = \pi/4$$

$u = 1-t$

So: $\displaystyle\int_{(1)} + \int_{(2)} + \int_{(3)} = \frac{\pi}{2} + \frac{\pi}{4} + \frac{\pi}{4} = \pi$ (same as in (a))

Now, use polars: $x = r\cos\theta,\ y = r\sin\theta$

$$F = \frac{r\sin\theta}{r^2}\,\vec i - \frac{r\cos\theta}{r^2}\,\vec j = \frac{1}{r}\left(\sin\theta\,\vec i - \cos\theta\,\vec j\right).$$

On the semicircle: $\vec R = \cos\theta\,\vec i + \sin\theta\,\vec j$, $\frac{d\vec R}{dt} = -\sin\theta\,\vec i + \cos\theta\,\vec j$

$$\vec F\cdot\frac{d\vec R}{d\theta} = \frac{1}{r}\left(-\sin^2\theta - \cos^2\theta\right) = -\frac{1}{r}$$

$$\int_{\theta=\pi}^{0} \left(-\frac{1}{r}\right) d\theta = -\theta\,\Big|_{\pi}^{0} = \pi.$$

$r = 1$