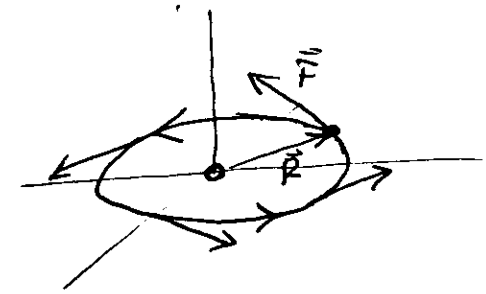
Math. 313
Set 15

4.3.4 Compute $\oint F \cdot d\vec{R}$ around the closed path consisting of a circle of radius r, centered at the origin, in the xy plane, taking

$$\vec{F} = \frac{(-y\hat{i} + x\hat{j})}{x^2+y^2} \qquad \text{(use polar)}$$

In polar:



path: $x = r\cos\theta,\ y = r\sin\theta$

$$\vec{R}(\theta) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$$

$$\frac{d\vec{R}}{d\theta} = -r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}$$

$$\vec{F}(\theta) = \frac{1}{r^2}\cdot\left(-r\sin\theta\,\vec{i} + r\cos\theta\,\vec{j}\right) = \frac{1}{r}\left(-\sin\theta\,\vec{i} + \cos\theta\,\vec{j}\right)$$

(i.e. $\vec{F}(\theta)$ is tangential to C, pointing along $\frac{d\vec{R}}{d\theta}$)

then
$$\oint_C \vec{F}\cdot d\vec{R} = \int_{\theta=0}^{2\pi} \frac{1}{r}(-\sin\theta\,\vec{i} + \cos\theta\,\vec{j})\cdot r(-\sin\theta\,\vec{i} + \cos\theta\,\vec{j})\,d\theta$$

$$= \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\,d\theta = \int_0^{2\pi} d\theta = 2\pi \neq 0$$

4.3.5 Show $\vec{F}$ in #4 is $\vec{F} = \nabla\phi$, $\phi = \tan^{-1}(y/x)$. How can it be that $\vec{F}$ is not conservative?

Recall $\frac{d}{du}(\tan^{-1}u) = \frac{1}{1+u^2}$; so, let $u = (y/x)$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\tan^{-1}u = \frac{1}{1+u^2}\frac{\partial u}{\partial x} = -\frac{y}{x^2}\,\frac{1}{1+(y/x)^2} = \frac{-y}{x^2+y^2} = F_1$$

$$\frac{\partial\phi}{\partial y} = \ldots = \frac{x}{x^2+y^2} = F_2.$$

So $\vec{F} = \nabla\phi$ but ϕ (and $\vec{F}$) <u>not differentiable</u> at $(x,y) = (0,0)$. So theorem not violated!

4.3.6 Find a potential for the force field
$$\vec{F} = (y + z\cos xz)\vec{i} + x\vec{j} + (x\cos xz)\vec{k}$$

$$\phi(x,y,z) = \phi_0 + \int_0^x F_1(t,0,0)\,dt + \int_0^y F_2(x,t,0)\,dt + \int_0^z F_3(x,y,t)\,dt$$

$$\int_0^x F_1(t,0,0)\,dt = 0 \;;\; \int_0^y F_2(x,t,0)\,dt = x\int_0^y dt = xy$$

$$\int_0^z F_3(x,y,t)\,dt = x\int_0^z \cos(xt)\,dt = \sin xz$$

i.e. $\boxed{\phi(x,y,z) = C + xy + \sin xz}$

4.4.1 Use the $(\nabla\times F = 0)$ test to determine if the following fields are conservative:

(a) $\vec{F} = (12xy + yz)\vec{i} + (6x^2 + xz)\vec{j} + xy\,\vec{k}$

$$\left.\begin{aligned} \partial_y F_1 &= 12x + z = \partial_x F_2 \;; \\ \partial_z F_1 &= y = \frac{\partial F_3}{\partial x} \;; \\ \partial_z F_2 &= x = \partial_y F_3 \end{aligned}\right\} \Rightarrow \vec{F} \text{ conservative since } \nabla\times F = 0$$

(c) $\vec{F} = \sin x\,\vec{i} + y^2\vec{j} + e^z\vec{k}$: automatic! $\begin{cases} F_1 = F_1(x) \\ F_2 = F_2(y) \\ F_3 = F_3(z) \end{cases}$

$\nabla\times F = 0$ conservative

(e) $\vec{F} = \dfrac{2x}{x^2+y^2}\vec{i} + \dfrac{2y}{x^2+y^2}\vec{j} + 2z\vec{k}$

$$\left.\begin{aligned} \partial_y F_1 &= -\frac{4xy}{(x^2+y^2)^2} = \partial_x F_2 \\ \partial_z F_1 &= 0 = \partial_x F_3 \\ \partial_z F_2 &= 0 = \partial_y F_3 \end{aligned}\right\}$$

Field is conservative in any ~~domain~~ simply connected domain that excludes the z axis ~~origin~~ (since it is not defined, let alone differentiable, at the z axis)