Math. 313
Set 16

p. 212, (3*, 9*)

4.4.3 Show that the scalar field $\phi = -1/|\vec{R}|$ which is defined everywhere except at the origin, is a potential function for the vector field $\vec{R}/|\vec{R}|^3$, where $\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$.

(a) by writing ϕ in terms of (x, y, z) and computing its gradient:

$$\phi = -\left(x^2+y^2+z^2\right)^{-1/2}$$

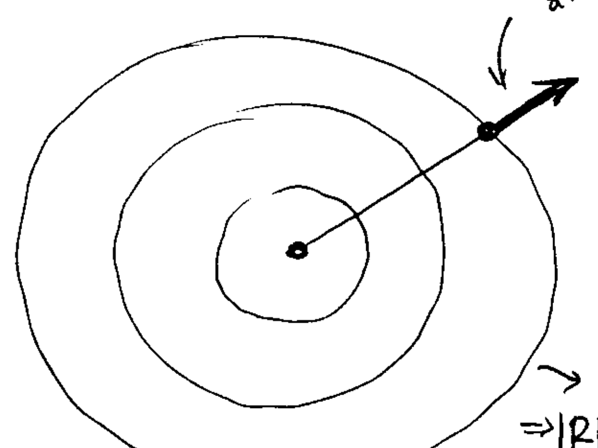
$$\partial_x \phi = -(-1/2)\left(x^2+y^2+z^2\right)^{-3/2} \cdot (2x) = x/|\vec{R}|^3$$

similarly: $\partial_y \phi = y/|\vec{R}|^3$, $\partial_z \phi = z/|\vec{R}|^3$

So: $\nabla \phi = -\vec{R}/|R|^3$.

(b) $\phi = -1/|R|$ increases (from $-\infty$ at $|R| = 0$ to 0 as $|R| \to \infty$), so

$$\boxed{\nabla \phi = f(x,y,z) \cdot \frac{\vec{R}}{|R|}}$$



This is so, because $\vec{\nabla}\phi$ points in a direction perpendicular to the surfaces of constant ϕ (i.e. spheres), towards increasing values of ϕ (i.e. outward). Then, $|\nabla\phi| = f(x,y,z)$ is the maximum rate of change of ϕ. Since ϕ depends only on the distance from the origin, R, then its maximum rate of change must be $\dfrac{d\phi}{dR} = \dfrac{d}{dR}\left(-\dfrac{1}{R}\right) = \dfrac{1}{R^2}$ (we write $R = |\vec{R}|$)

i.e. $\vec{\nabla}\phi = \dfrac{1}{R^2} \cdot \dfrac{\vec{R}}{R} = \vec{R}/R^3$

4.4.9 Find potential for

(a) $\vec{F} = (2xyz + z^2 - 2y^2 + 1)\vec{i} + (x^2 z - 4xy)\vec{j} + (x^2 y + 2xz - 2)\vec{k}$

$$\int_0^x F_1(t,0,0)\,dt = \int_0^x dt = x$$

$$\int_0^y F_2(x,t,0)\,dt = \int_0^y (-4xt)\,dt = -2xy^2$$

$$\int_0^z F_3(x,y,t)\,dt = \int_0^z (x^2 y + 2xt - 2)\,dt = x^2 yz + xz^2 - 2z$$

$$\Rightarrow \boxed{\phi(x,y,z) = x - 2xy^2 + x^2 yz + xz^2 - 2z + C}$$

(b) $\vec{G} = \dfrac{x}{(x^2+z^2)}\vec{i} + \dfrac{z}{(x^2+z^2)}\vec{k}$ satisfies $\nabla \times G = 0$ except for points on the y-axis (where $x^2+z^2=0$).

Is $\vec{G}$ conservative? ___ . ___

If we look for a potential, we find:

$$\phi_x = \frac{x}{x^2+z^2} = \frac{1}{2}\frac{d}{dx}\left(\ln(x^2+z^2)\right)$$

This works for z as well: $\phi_z = \frac{1}{2}\frac{d}{dz}\ln(x^2+z^2)$

i.e. $\phi(x,y,z) = \frac{1}{2}\ln(x^2+z^2)$ gives

$$\nabla\phi = \vec{G}$$

So, $\vec{G}$ is in fact conservative, although the theorem 4.3 (p. 205) does not apply to any region including segments of the y-axis. Still $\oint_C G \cdot d\vec{R} = 0$, on any closed path that does not go thru y-axis.