Math. 311
# 22

3.10 (6*, 8(a, b*), 11*, 12*)

3.11 (6, 12*, 14)

3.10.6

Cylindrical: $ds^2 = d\rho^2 + \rho^2 d\vartheta^2 + dz^2$

Spherical $ds^2 = dr^2 + r^2 d\phi^2 + r^2 \sin^2\phi \, d\vartheta^2$

$$\Delta h = \nabla \cdot \nabla h$$

Gradient

Cylindrical

$$\vec{F} = \nabla h = \frac{\partial h}{\partial \rho}\vec{e}_\rho + \frac{1}{\rho}\frac{\partial h}{\partial \vartheta}\vec{e}_\vartheta + \frac{\partial h}{\partial z}\vec{e}_z$$

$$\nabla \cdot \vec{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\vartheta}{\partial \vartheta} + \frac{\partial F_z}{\partial z}$$

$$\Downarrow$$

$$\Delta h = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial h}{\partial \rho}\right) + \frac{1}{\rho}\frac{\partial}{\partial \vartheta}\left(\frac{1}{\rho}\frac{\partial h}{\partial \vartheta}\right) + \frac{\partial^2 h}{\partial z^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial h}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 h}{\partial \vartheta^2} + \frac{\partial^2 h}{\partial z^2}$$

Spherical

$$\vec{F} = \nabla h = \frac{\partial h}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial h}{\partial \phi}\vec{e}_\phi + \frac{1}{r\sin\phi}\frac{\partial h}{\partial \vartheta}\vec{e}_\vartheta$$

$$\nabla \cdot \vec{F} = \frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial F_\vartheta}{\partial \vartheta} + \frac{1}{r\sin\phi}\frac{\partial(F_\phi \sin\phi)}{\partial \phi}$$

$$\Downarrow$$

$$\Delta h = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial h}{\partial r}\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial \vartheta}\left(\frac{1}{r\sin\phi}\frac{\partial h}{\partial \vartheta}\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}\left(\frac{\sin\phi}{r}\frac{\partial h}{\partial \phi}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial h}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 h}{\partial \vartheta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial h}{\partial \phi}\right)$$

$h_1 = 1$, $h_2 = \rho$, $h_3 = 1$
$u_1 = \rho$, $u_2 = \vartheta$, $u_3 = z$

$h_1 = 1$, $h_2 = r$, $h_3 = r\sin\phi$
$u_1 = r$, $u_2 = \phi$, $u_3 = \vartheta$

$$\Delta K = \frac{1}{h_1 h_2 h_3}\left[\partial_{u_1}\left(\frac{h_2 h_3}{h_1}\partial_{u_1} K\right) + \partial_{u_2}\left(\frac{h_1 h_3}{h_2}\partial_{u_2} K\right) + \partial_{u_3}\left(\frac{h_1 h_2}{h_3}\partial_{u_3} K\right)\right]$$

3.10.8(b) Find $\nabla\cdot$ and $\nabla\times$ in cylindrical coords of $\vec{F} = (-y\hat{i} + x\hat{j})/x^2+y^2$

$$\vec{F} = \frac{1}{\rho}\vec{e}_\theta$$

$$\nabla\cdot\vec{F} = \frac{1}{\rho}\,\partial_\theta\left(\frac{1}{\rho}\right) = 0$$

$$\nabla\times\left(\frac{1}{\rho}\vec{e}_\theta\right) = \frac{1}{\rho}\begin{vmatrix} \vec{e}_\rho & \rho\vec{e}_\theta & \vec{e}_z \\ \partial_\rho & \partial_\theta & \partial_z \\ 0 & \rho\cdot\frac{1}{\rho} & 0 \end{vmatrix} = 0$$

3.10.11 Compute $\nabla f$, in sphericals, of $f = \cos\phi/r^2$

$$\nabla\left(\frac{\cos\phi}{r^2}\right) = \vec{e}_r\partial_r\left(\frac{\cos\phi}{r}\right) + \frac{\vec{e}_\phi}{r}\,\partial_\phi\left(\frac{\cos\phi}{r}\right)$$

$$= -\frac{\cos\phi}{r^2}\vec{e}_r - \frac{\sin\phi}{r^2}\vec{e}_\phi$$

3.10.12 Compute divergence & curl (sphericals) of $\vec{F}(r,\phi,\theta) = \vec{e}_r + r\vec{e}_\phi + r\cos\phi\,\vec{e}_\theta$

$$\nabla\cdot F = \frac{1}{r^2}\partial_r(r^2) + \frac{1}{r\sin\phi}\partial_\phi(r\sin\phi) + \frac{1}{r\sin\phi}\underbrace{\partial_\theta(r\cos\phi)}_{=0}$$

$$= \frac{2}{r} + \cot\phi$$

$$\nabla\times F = \frac{1}{r^2\sin\phi}\begin{vmatrix} \vec{e}_r & r\vec{e}_\phi & r\sin\phi\,\vec{e}_\theta \\ \partial_r & \partial_\phi & \partial_\theta \\ 1 & r^2 & r^2\sin\phi\cos\phi \end{vmatrix} = \frac{1}{\cancel{r^2}\sin\phi}\left(\cancel{r^2}\cos 2\phi\right)\vec{e}_r + \frac{-1}{r\cancel{\sin\phi}}\left(2r\cancel{\sin\phi}\cos\phi\right)\vec{e}_\phi + \frac{1}{\cancel{r}}\cdot 2\cancel{r}\,\vec{e}_\theta$$

$$r^2\sin\phi\cos\phi = \tfrac{1}{2}r^2\sin 2\phi$$

$$= \cos 2\phi/\sin\phi\,\vec{e}_r - 2\cos\phi\,\vec{e}_\phi + 2\vec{e}_\theta$$

3.11.12 > Parabolic cylindrical coordinates $(u, v, z)$ are defined by $x = \frac{1}{2}(u^2 - v^2)$, $y = uv$, $z = z$, where $-\infty < u, z < \infty$, $v \geq 0$. To make use of the formulas in 3.11, find the scale factors $h_u, h_v, h_z$.

$$h_u = \left|\frac{\partial \vec{R}}{\partial u}\right| = |u\vec{i} + v\vec{j}| = \sqrt{u^2+v^2}$$

$$h_v = \left|\frac{\partial \vec{R}}{\partial v}\right| = |-v\vec{i} + u\vec{j}| = \sqrt{u^2+v^2}$$

$$h_z = \left|\frac{\partial \vec{R}}{\partial z}\right| = |\vec{k}| = 1$$

3.11.6 > Let $u_1 = x+y$, $u_2 = x-y$, $u_3 = 2z$

(a) Is this an orthogonal system?

$\nabla u_1 = \vec{i} + \vec{j}$, $\nabla u_2 = \vec{i} - \vec{j}$, $\nabla u_3 = 2\hat{k}$ : orthogonal

(b) Solve for $x, y$ & $z$:

$$x = \frac{u_1 + u_2}{2}, \quad y = \frac{u_1 - u_2}{2}, \quad z = \frac{u_3}{2}$$

(c)
$$\left.\begin{aligned} h_{u_1} &= \left|\frac{\partial R}{\partial u_1}\right| = \left|\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}\right| = 1/\sqrt{2} \\ h_{u_2} &= \left|\frac{\partial R}{\partial u_2}\right| = \left|\frac{1}{2}\vec{i} - \frac{1}{2}\vec{j}\right| = 1/\sqrt{2} \\ h_{u_3} &= \left|\frac{\partial R}{\partial u_3}\right| = \left|\frac{1}{2}\vec{k}\right| = \frac{1}{2} \end{aligned}\right\} ds^2 = \frac{1}{2}du_1^2 + \frac{1}{2}du_2^2 + \frac{1}{4}du_3^2$$

(d) What is the laplacian in $(u_1, u_2, u_3)$:

$$\Delta\psi = \frac{1}{\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{1}{2}}\left[\partial_{u_1}\left(\frac{\frac{1}{\sqrt{2}}\frac{1}{2}}{\frac{1}{\sqrt{2}}}\partial_{u_1}\psi\right) + \partial_{u_2}\left(\frac{\frac{1}{\sqrt{2}}\frac{1}{2}}{\frac{1}{\sqrt{2}}}\partial_{u_2}\psi\right) + \partial_{u_3}\left(\frac{\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}}{\frac{1}{2}}\partial_{u_3}\psi\right)\right]$$

$$= 4\left[\frac{1}{2}\partial_{u_1}^2\psi + \frac{1}{2}\partial_{u_2}^2\psi + \partial_{u_3}^2\psi\right]$$

(e) Let $f(u_1,u_2,u_3) = u_1 + u_2 + 2u_3$; Find grad $f$:

$$\nabla f = \frac{\vec{e}_1}{\sqrt{2}}\frac{\partial}{\partial u_1}(u_1+u_2+2u_3) + \frac{\vec{e}_2}{\sqrt{2}}\frac{\partial}{\partial u_2}(u_1+u_2+2u_3) + \frac{\vec{e}_3}{2}\frac{\partial}{\partial u_3}(u_1+u_2+2u_3)$$

$$= \frac{1}{\sqrt{2}}\vec{e}_1 + \frac{1}{\sqrt{2}}\vec{e}_2 + \vec{e}_3$$

3.11.14 $\nabla \cdot A$ Let $p = \sqrt{u^2+v^2}$ ; $h_u = h_v = p$, $h_z = 1$

$$\nabla\cdot A = \frac{1}{u^2+v^2}\left[\partial_u\left(F_u\sqrt{u^2+v^2}\right) + \partial_v\left(F_v\sqrt{u^2+v^2}\right) + \partial_z\left(F_z(u^2+v^2)\right)\right]$$

$$= \frac{1}{p^2}\left[\partial_u(pF_u) + \partial_v(pF_v)\right] + \partial_z F_z$$

$$\Delta\phi = \frac{1}{u^2+v^2}\left[\partial_u^2\phi + \partial_v^2\phi + \partial_z\left((u^2+v^2)\partial_z\phi\right)\right]$$

$$= \frac{1}{p^2}\left(\partial_u^2\phi + \partial_v^2\phi\right) + \partial_z^2\phi$$