Math. 311
#20

p.169, 3.10 (1,2,4)

3.10.1 > Derive the equs. of transformation between cylindrical & spherical coordinates.

Cylindrical	Spherical
$x = \rho\cos\vartheta$	$x = r\sin\phi\cos\vartheta$
$y = \rho\sin\vartheta$	$y = r\sin\phi\sin\vartheta$
$z = z$	$z = r\cos\phi$

$$\vartheta = \vartheta, \quad \rho = r\sin\phi, \quad z = r\cos\phi \quad \longrightarrow \quad r = \sqrt{\rho^2 + z^2}, \quad \phi = \tan^{-1}(\rho/z)$$

3.10.2 > Use:

$$\left.\begin{array}{l} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \\ z = z \end{array}\right\} \to \left\{\begin{array}{l} \rho = \sqrt{x^2+y^2} \\ \vartheta = \tan^{-1}(y/x) \\ z = z \end{array}\right.$$

and ~~to~~ der

$$\vec{e}_z = \nabla z / |\nabla z|$$
$$\vec{e}_\rho = \nabla\rho / |\nabla\rho|$$
$$\vec{e}_\vartheta = \nabla\vartheta / |\nabla\vartheta|$$

to derive: $\vec{e}_z = \vec{k}$,

$$\vec{e}_\rho = \frac{x\vec{i} + y\vec{j}}{(x^2+y^2)^{1/2}}, \quad \vec{e}_\vartheta = \frac{-y\vec{i} + x\vec{j}}{(x^2+y^2)^{1/2}}$$

$$\vec{\nabla} z = \vec{i}\underbrace{\frac{\partial z}{\partial x}}_{0} + \vec{j}\underbrace{\frac{\partial z}{\partial y}}_{0} + \vec{k}\underbrace{\frac{\partial z}{\partial z}}_{1} = \vec{k} \quad ; |\nabla z| = |\vec{k}| = 1$$

$$\vec{\nabla}\rho = \vec{i}\,\partial_x\rho + \vec{j}\,\partial_y\rho + \vec{k}\,\partial_z\rho = \vec{i}\frac{x}{(x^2+y^2)^{1/2}} + \vec{j}\frac{y}{(x^2+y^2)^{1/2}} \quad ; |\nabla\rho| = 1$$

$$\vec{\nabla}\vartheta = \vec{i}\,\frac{1}{1+(y/x)^2}\cdot\left(-\frac{y}{x^2}\right) + \vec{j}\,\frac{1}{1+(y/x)^2}\cdot\left(\frac{1}{x}\right) = \left(-\frac{iy}{\sqrt{x^2+y^2}} + \frac{\vec{j}x}{\sqrt{x^2+y^2}}\right)\frac{1}{\sqrt{x^2+y^2}}$$

$$|\nabla\vartheta| = \frac{1}{\sqrt{x^2+y^2}} \quad ; \quad \vec{e}_\vartheta = \nabla\vartheta / |\nabla\vartheta|$$

3.10.4 Use

$$\left.\begin{aligned} x&=r\sin\phi\cos\vartheta\\ y&=r\sin\phi\sin\vartheta\\ z&=r\cos\phi\end{aligned}\right\} \to \begin{aligned} r&=(x^2+y^2+z^2)^{1/2}\\ \phi&=\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\\ \vartheta&=\tan^{-1}(y/x)\end{aligned} \quad \text{and} \quad \begin{cases} \vec{e}_r = \nabla r/|\nabla r| \\ \vec{e}_\vartheta = \nabla\vartheta/|\nabla\vartheta| \\ \vec{e}_\phi = \nabla\phi/|\nabla\phi| \end{cases}$$

to derive: $\vec{e}_r = \dfrac{\vec{R}}{|\vec{R}|}$, $\vec{e}_\phi = \dfrac{z(x\vec{i}+y\vec{j})-(x^2+y^2)\vec{k}}{(x^2+y^2)^{1/2}(x^2+y^2+z^2)^{1/2}}$

$\vec{e}_\vartheta = \dfrac{-y\vec{i}+x\vec{j}}{(x^2+y^2)^{1/2}}$.

$$\nabla r = \frac{x\vec{i}+y\vec{j}+z\vec{k}}{(x^2+y^2+z^2)^{1/2}} \;;\; |\nabla r| = 1$$

$$\nabla\vartheta = (-y\vec{i}+x\vec{j})/(x^2+y^2)^{1/2}\cdot\frac{1}{(x^2+y^2)^{1/2}} = \vec{e}_\vartheta\,|\nabla\vartheta|$$

(like 3.10.2)

$$\nabla\phi = \nabla\left(\cos^{-1}(z/r)\right) = -\frac{1}{\sqrt{1-(z/r)^2}}\,\nabla(z/r)$$

$$= -\frac{r}{\sqrt{r^2-z^2}}\cdot\left(\frac{1}{r}\nabla z - \frac{z}{r^2}\nabla r\right)$$

$$= -\frac{\vec{k}}{\sqrt{x^2+y^2}} + \frac{z(x\vec{i}+y\vec{j}+z\vec{k})}{\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2+z^2}\right)^2} = \frac{z(x\vec{i}+y\vec{j})-(x^2+y^2)\vec{k}}{(x^2+y^2)^{1/2}(x^2+y^2+z^2)^{1/2}}\cdot\frac{1}{r}$$

$$\begin{cases} |\nabla\phi| = 1/r = (x^2+y^2+z^2)^{1/2} \\ |\nabla\vartheta| = \dfrac{1}{(x^2+y^2)^{1/2}} = \dfrac{1}{r\sin\phi} \\ |\nabla r| = 1 \end{cases}$$

(i.e. $ds^2 = dr^2 + r^2\,d\phi^2 + r^2\sin^2\phi\,d\vartheta^2$)