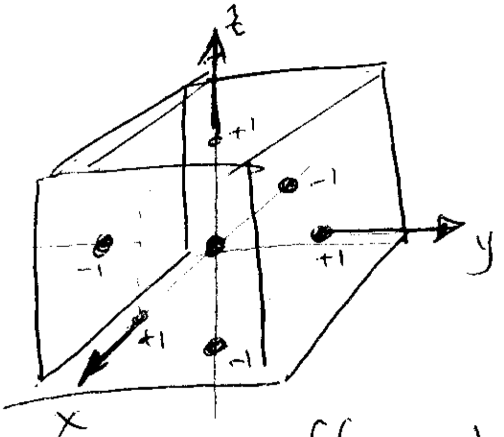
Math. 311
#26
p. 246: (2cd, 4, 11, 14, 18) Sec. 4.7

p. 246, 4.7.2cd Compute $\iint \vec F\cdot d\vec S$, where S is the surface of the cube bounded by the planes $x=\pm1$, $y=\pm1$, $z=\pm1$, if

(c) $\vec F = x\vec i + y\vec j + z\vec k$ (d) $\vec F = x^2\vec i + y^2\vec j + z^2\vec k$

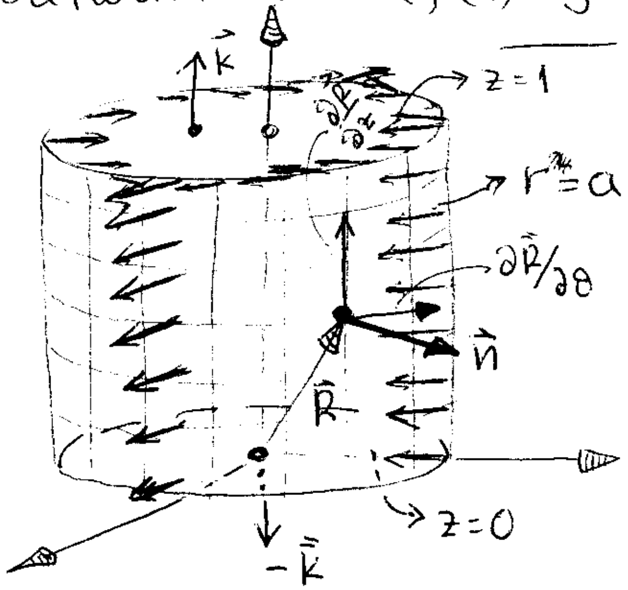


Face	Normal	$F_d\cdot\vec n$	$F_c\cdot\vec n$	$\iint F_d\cdot\vec n\,ds$	$\iint F_c\cdot\vec n\,ds$
$x=+1$	$\vec i$	1	1	4	4
$x=-1$	$-\vec i$	-1	1	-4	4
$y=+1$	$+\vec j$	1	1	4	4
$y=-1$	$-\vec j$	-1	1	-4	4
$z=+1$	$+\vec k$	1	1	4	4
$z=-1$	$-\vec k$	-1	1	-4	4

$\oiint \vec F\cdot\vec n\,dS =$ (d) 0 (c) 24

Verification by divergence theorem:
(c) $\nabla\cdot F = 3$; $\iiint \nabla\cdot F\,dV = 3\iiint dV = 3\cdot 8 = 24$
(d) $\nabla\cdot F = 2(x+y+z)$; $\iiint \nabla\cdot F\,dV = 6\iiint x\,dV = 0$

4.7.4 Given $\vec F = x\vec i - y\vec j$, find the values of $\iint \vec F\cdot\vec n\,dS$ over the closed surface bounded by the planes $z=0$, $z=1$ and the cylinder $x^2+y^2=a^2$ where $\vec n$ is the unit outward normal, (a) by direct calculation, (b) by symmetry



No flux on top/bottom planes since $\vec F\cdot\vec k = 0$; this leaves only side surface, where

$r=a:\quad \vec R = x\vec i + y\vec j + z\vec k$

$$d\vec S = \frac{\partial\vec R}{\partial\theta}\times\frac{\partial\vec R}{\partial z}\,d\theta\,dz$$

$$\frac{\partial\vec R}{\partial\theta} = -a\sin\theta\,\vec i + a\cos\theta\,\vec j,\quad \frac{\partial\vec R}{\partial z} = \vec k \Longrightarrow$$

$d\vec{S} = a(\cos\vartheta\,\vec{i} + \sin\vartheta\,\vec{j})\,dz\,d\vartheta \;, \quad \vec{F} = x\vec{i} - y\vec{j} = a\cos\vartheta\,\vec{i} - y\sin\vartheta\,\vec{j}$

$$\iint_{\substack{r=a \\ 0\le z\le 1}} F\cdot d\vec{S} = a\int_{\vartheta=0}^{2\pi} d\vartheta \int_{z=0}^{1} dz\,(\cos^2\vartheta - \sin^2\vartheta) = a\int_0^{2\pi}\cos 2\vartheta\,d\vartheta = 0$$

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(b) Arguing by symmetry, note that $\iint_{\text{side}} F\cdot n\,d\vec{S} =$

$= \left(\oint F\cdot n\,dr\right)\cdot$height, since integrand is independent of z.

Now $\oint x\,(\vec{i}\cdot\hat{n})\,dr = \oint y\,(\vec{j}\cdot\hat{n})\,dr$ since these integrals look the same under a ~~rot~~ reflection about the main diagonal $\Rightarrow \oint (x\vec{i} - y\vec{j})\cdot\hat{n}\,dr = 0.$

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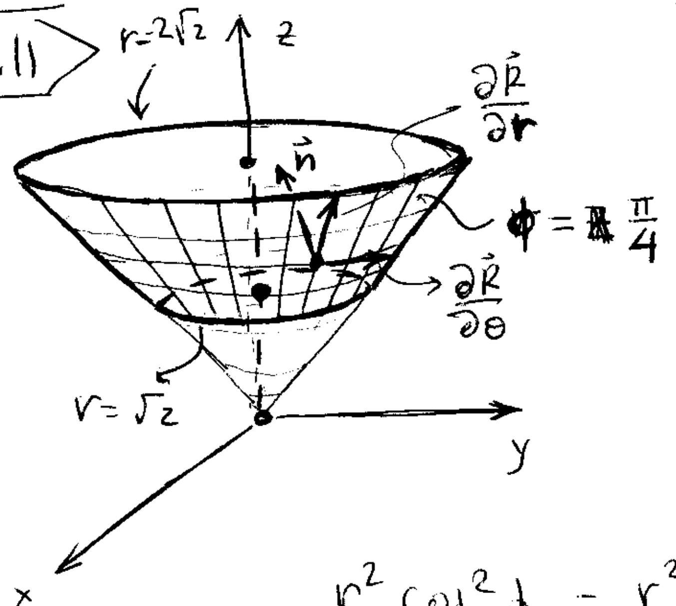


The equation of the cone is

$$z^2 = x^2 + y^2$$

In spherical coordinates:

$$\left.\begin{aligned} z &= r\cos\phi \\ x &= r\sin\phi\cos\vartheta \\ y &= r\sin\phi\sin\vartheta \end{aligned}\right\}$$

$$r^2\cos^2\phi = r^2\sin^2\phi\,(\cos^2\vartheta + \sin^2\vartheta) \Rightarrow \cos^2\phi = \sin^2\phi$$
$$\Rightarrow \phi = \pi/4$$

Then

$$z = 1 \Rightarrow \boxed{r = \sqrt{2}}\;; \quad z = 2 \Rightarrow \boxed{r = 2\sqrt{2}}$$

$$\frac{\partial\vec{R}}{\partial\vartheta} = -r\sin\phi\sin\vartheta\,\vec{i} + r\sin\phi\cos\vartheta\,\vec{j} = \frac{r}{\sqrt{2}}(-\sin\vartheta\,\vec{i} + \cos\vartheta\,\vec{j})\;;$$

$$\frac{\partial\vec{R}}{\partial r} = \sin\phi\cos\vartheta\,\vec{i} + \sin\phi\sin\vartheta\,\vec{j} + \cos\phi\,\vec{k} = \frac{1}{\sqrt{2}}(\cos\vartheta\,\vec{i} + \sin\vartheta\,\vec{j} + \vec{k})$$

→

$$d\vec{S} = \frac{\partial \vec{R}}{\partial \vartheta} \times \frac{\partial \vec{R}}{\partial r}\, d\vartheta\, dr = \frac{r}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\vartheta & \cos\vartheta & 0 \\ \cos\vartheta & \sin\vartheta & 1 \end{vmatrix} d\vartheta\, dr$$

$$\Rightarrow \quad d\vec{S} = \frac{r}{2}\left(\cos\vartheta\, \vec{i} + \sin\vartheta\, \vec{j} - \vec{k}\right) dr\, d\vartheta$$

$$\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$$

$$x^2 = r^2\cos^2\vartheta \cdot \tfrac{1}{2} \ (\cos\phi) \qquad z^2 = r^2\cos^2\phi = \tfrac{1}{2} r^2 \qquad y^2 = r^2\sin^2\vartheta \cdot \tfrac{1}{2}$$

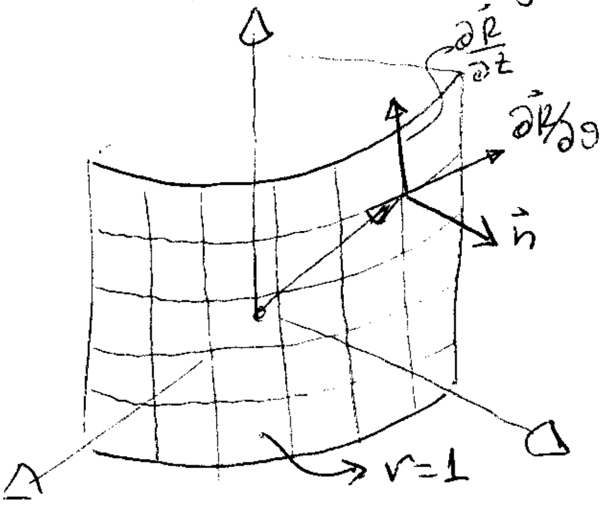
$$\iint F \cdot d\vec{S} = \int_{r=\sqrt{2}}^{2\sqrt{2}} \int_{\vartheta=0}^{2\pi} \frac{r}{2}\left(x^2\cos\vartheta + y^2\sin\vartheta - z^2\right) dr\, d\vartheta$$

$$= \int_{r=\sqrt{2}}^{2\sqrt{2}} \int_{\vartheta=0}^{2\pi} \left\{ \frac{r^3}{4}\cos^3\vartheta + \frac{r^3}{4}\sin^3\vartheta - \frac{r^2}{4} \right\} dr\, d\vartheta$$

odd ϕ
by symmetry

$$= \frac{2\pi}{4} \cdot \frac{r^4}{4}\Bigg|_{\sqrt{2}}^{2\sqrt{2}} = \frac{\pi}{8}\left(64 - 4\right) = \frac{15\pi}{2}$$

4.7.14 Let $\vec{F} = y\vec{i} + (x+2)\vec{j} + x^3\sin(yz)\vec{k}$, and let S be the portion of the cylinder $x^2+y^2=1$ that lies in the first octant, below $z=1$.



$$d\vec{S} = \frac{\partial \vec{R}}{\partial \theta} \times \frac{\partial \vec{R}}{\partial z}\, dz\, d\vartheta$$

$$\frac{\partial \vec{R}}{\partial \theta} = -\sin\vartheta\, \vec{i} + \cos\vartheta\, \vec{j} \ ; \quad \frac{\partial \vec{R}}{\partial z} = \vec{k}$$

$$d\vec{S} = \left(\cos\vartheta\, \vec{i} + \sin\vartheta\, \vec{j}\right) dz\, d\vartheta$$

$$\vec{F}\cdot d\vec{S} = \left(y\cos\vartheta + (x+2)\sin\vartheta\right) dz\, d\vartheta$$

$$= \left(2\cos\vartheta\sin\vartheta + 2\sin\vartheta\right) dz\, d\vartheta$$

$$\iint_S \vec{F}\cdot d\vec{S} = 2\int_{\theta=0}^{\pi/2} d\theta \int_{z=0}^{1} dz\,(\cos\theta\sin\theta + \sin\theta)$$

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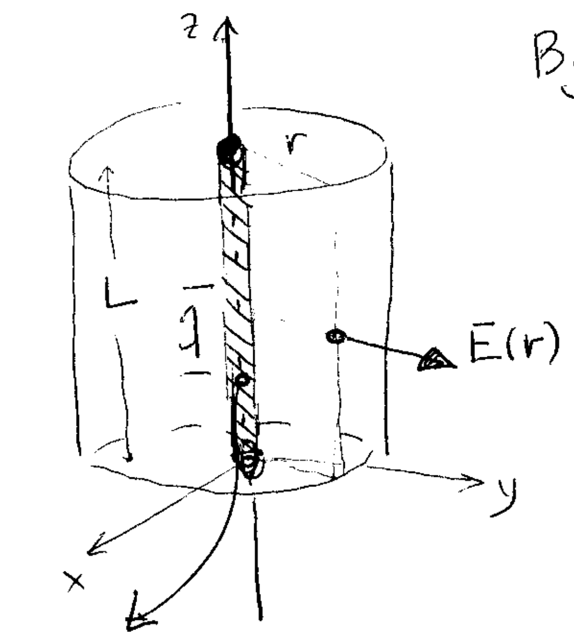
$$= \cancel{2}\int_0^{\pi/2} (\sin 2\theta + 2\sin\theta)\,d\theta = -\frac{1}{2}\cos 2\theta - 2\cos\theta\Big|_0^{\pi/2}$$

$$= \cancel{\tfrac{1}{2}}\,1 + 2 = 3$$

4.7.18 > Use Gauss' law to determine the magnitude of the electric field intensity at a point r units away from an infinitely long thin wire carrying a charge of λ units per unit length



Total charge:

$$Q = L\lambda$$

By symmetry, $\hat{E}$ field has the form

$$\vec{E} = E(r)\,\hat{e}_r$$

Consider a cylinder of length l, radius r, concentric with wire, which lies along the z-axis.

Then

$$\cancel{4\pi}\,Q = \iint \hat{E}\cdot d\vec{S} = \int_{z=0}^{L}\int_{\theta=0}^{2\pi} E(r)\,r\,d\theta\,dz$$

$$\cancel{4\pi}\,\lambda L$$

$$= r\,E(r)\cdot L\cdot 2\pi$$

$$\Rightarrow \boxed{E(r) = \frac{\cancel{4\pi}\,\lambda}{2\pi r}}$$