Math. 311
29

p. 277 (6*, 7*, 8*, 9*, 10)

5.1.6 What is the flux output per unit volume at (3,1,-2) if $\vec{F} = x^3\vec{i} + xy\vec{j} - x^3\vec{k}$?

$$\Phi = \frac{\lim_{V\to 0}}{V}\iiint \nabla\cdot\vec{F}\,dV = \nabla\cdot\vec{F}\Big|_{(3,1,-2)} = 3x^2+x = 30$$

5.1.7 What is the flux output from an ellipsoid of volume V if $\vec{F} = 3x\vec{i} + y\vec{j} + z\vec{k}$?

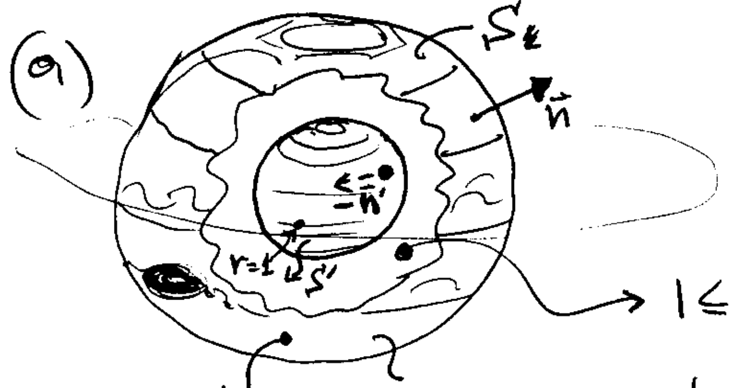
$$\Phi = \frac{1}{V}\oiint \vec{F}\cdot d\vec{s} = \frac{1}{V}\iiint \nabla\cdot F\,dV = \frac{1}{V}\iiint (3+1+1)\,dV = 5$$

5.1.8 If $\vec{F} = 3x^2\vec{i} + y\vec{j} + z\vec{k}$, would the flux output from an ellipsoid depend on the location of the ellipsoid as well as its volume?

YES: $\Phi = \frac{1}{V}\oiint F\cdot d\vec{s} = \frac{1}{V}\iiint \nabla\cdot F\,dV = \frac{1}{V}\iiint (6x+2)\,dV$

$= 2 + \frac{6}{V}\boxed{\iiint_V x\,dV} \to$ the integral depends on the location of V.

5.1.9 (see

(a) 

$$1 \le \overbrace{x^2+y^2+z^2}^{r^2} \le 4$$

Surface has two components,

$$\left.\begin{array}{l} \text{sphere } r=4, \text{ normal } \dfrac{\vec{R}}{R} \text{ (outwards)}, \\ \text{sphere } r=1, \text{ normal } -\dfrac{\vec{R}}{R} \text{ (inwards)} \end{array}\right\} \text{both}$$

normals point away from region which is a spherical shell of thickness 3, inner radius 1, outer radius 4.

(b)

$$\boxed{\iiint_V \nabla\cdot F\,dV = \iint_S F\cdot\vec{n}\,dS - \iint_{S'} F\cdot\vec{n}'\,dS}$$

$\vec{n}'$ is outwards norm hence (−)

$$= \iint_{\substack{S\\ r=4}} \vec{F}\cdot\frac{\vec{R}}{R}\cdot R^2\sin\phi\,d\phi\,d\theta = 4\iint_{r=4} \overbrace{\vec{F}\cdot\vec{R}}^{4F_r}\sin\phi\,d\phi\,d\theta$$

$$- \iint_{r=1} \vec{F}\cdot\frac{\vec{R}}{R}\cdot R^2\sin\phi\,d\phi\,d\theta = -\iint \underbrace{\vec{F}\cdot\vec{R}}_{F_r}\sin\phi\,d\phi\,d\theta$$

for example, if we use spherical coords.

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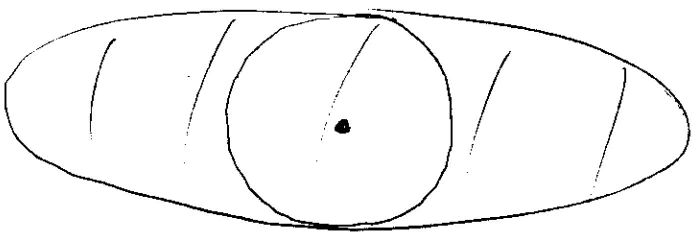
(c) $\nabla \cdot F = 0$, $R \neq 0$:

$$\Rightarrow \iiint \nabla \cdot F \, dV = 0 = \oiint_S F \cdot d\vec{S} - \oiint_{S'} F \cdot d\vec{S}$$

$$\Rightarrow \oiint_S F \cdot dS = \oiint_{S'} F \cdot dS$$, fluxes are equal.

(d) 

Answers should still be the same, since $\nabla \cdot F$ is zero in region between two surfaces.

e) $$\underset{\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=1}{\oiint_S} F \cdot d\vec{S} = \underset{x^2+y^2+z^2=1}{\oiint_{S'}} \vec{F} \cdot d\vec{S}$$

For any field such that $\nabla \cdot \vec{F} = 0$ between sphere & ellipsoid.

Here $\vec{F} = \frac{\vec{R}}{R^3} = -\nabla\left(\frac{1}{R}\right)$ and we know that

$\nabla \cdot \vec{F} = 0$, $R \neq 0$: then

$$\oiint_{\text{ellipsoid}} F \cdot d\vec{S} = \iint_{\substack{\text{sphere} \\ r=1}} F \cdot d\vec{S} = \int_{\theta=0}^{2\pi} d\theta \int_{\phi=0}^{\pi} d\phi \left(\frac{\vec{R}}{R^3} \cdot \cancel{\frac{\vec{R}}{R}}\right) R^2 \sin\phi$$

$$= 2\pi \cdot \int_0^{\pi} \sin\phi \, d\phi = 4\pi$$