Solutions, 311-XXI

April 13, 2003

21(4/ 8) Introduction to Divergence and Stokes theorems Sec. 4.(9) p.262(3(a,b*),5,9,10*,11*,17*)

1 Problem 4.9.3b

Use the divergence theorem to solve 4.7.5:
Given $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (z^2 - 1)\mathbf{k}$ find $\int\int \mathbf{F} \cdot \mathbf{n} dS$ over the closed surface bounded by the planes $z = 0$, $z = 1$, and the cylinder $x^2 + y^2 = a^2$, where $\mathbf{n}$ is the unit outward normal.
Solution:
By the divergence theorem,

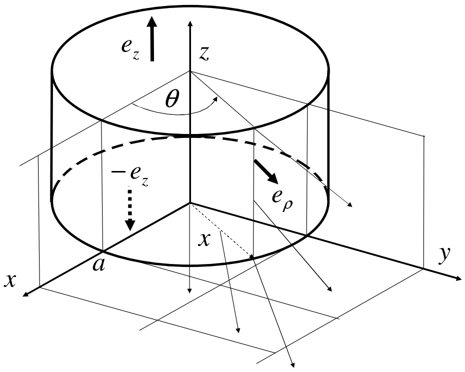
$$\int\int \mathbf{F} \cdot \mathbf{n} dS = \int\int\int \nabla \cdot \mathbf{F} dV$$

so we compute the divergence of $\mathbf{F}$:

$$\nabla \cdot \mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial (z^2 - 1)}{\partial z} = 2 + 2z \ .$$

Integrating:

$$\begin{aligned}
\int\int \mathbf{F} \cdot \mathbf{n} dS &= \int\int\int \nabla \cdot \mathbf{F} dV \\
&= \int_{z=0}^{1} \int_{\rho=0}^{a} \int_{\theta=0}^{2\pi} 2(1+z)\rho d\theta d\rho dz \\
&= 2\int_{z=0}^{1} (1+z) dz \int_{\rho=0}^{a} \rho d\rho \int_{\theta=0}^{2\pi} d\theta \\
&= 2(z + z^2/2)_{z=1} (\rho^2/2)_{\rho=a} 2\pi \\
&= 3\pi a^2 \ .
\end{aligned}$$



Geometry for problem 4.9.3b

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2 Problem 4.9.10

By means of Stokes' theorem, find $\int \mathbf{F} \cdot d\mathbf{r}$ around the ellipse $x^2 + y^2 = 1$, $z = y$, where $\mathbf{F} = x\mathbf{i} + (x+y)\mathbf{j} + (x+y+z)\mathbf{k}$.

Solution:

We assume a counterclockwise sense of integration (else change the sign of the result). By Stokes we have that

$$\int \mathbf{F} \cdot d\mathbf{r} = \int\int (\nabla \times_S \mathbf{F}) \cdot d\mathbf{S}$$

where S is the region on the plane $z = y$ cut-off by the cylinder. Then, since $z = f(x, y) = y$ we can use the standard formula

$$d\mathbf{S} = (\mathbf{i} + f_x\mathbf{k}) \times (\mathbf{j} + f_y\mathbf{k})\, dxdy = (\mathbf{k} - \mathbf{j})\, dxdy$$

while

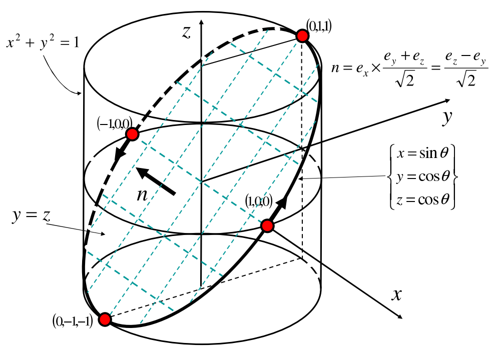
$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x & (x+y) & (x+y+z) \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

so that

$$\nabla \times \mathbf{F} \cdot d\mathbf{S} = (\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (\mathbf{k} - \mathbf{j})\, dxdy = 2dxdy$$

and the integral becomes

$$\begin{aligned} \int \mathbf{F} \cdot d\mathbf{r} &= \int\int_\Sigma 2dxdy \\ &= 2\text{Area of } \Sigma \\ &= 2\pi\ . \end{aligned}$$



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3 Problem 4.9.11

Evaluate $\int\int (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where $\mathbf{F} = y\mathbf{i} + (x - 2x^3z)\mathbf{j} + xy^3\mathbf{k}$ and S is the surface of a sphere $x^2 + y^2 + z^2 = a^2$ above the xy plane.

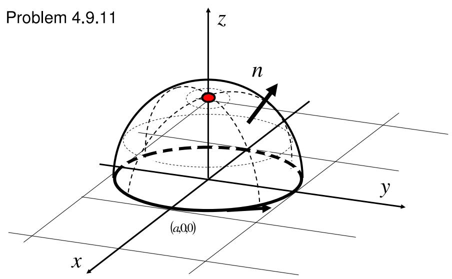
Solution:

By Stokes' theorem we have

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_\Sigma \mathbf{F} \cdot d\mathbf{R}$$

where Σ is the circle $x^2 + y^2 = a^2$ oriented counterclockwise i.e. $d\mathbf{R} = a\mathbf{e}_\theta d\theta = a(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\, d\theta$. On the circle we have that $z = 0$, $x = a\cos\theta$, $y = a\sin\theta$. So the desired integral can be computed as

$$\begin{aligned}
\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \oint_\Sigma \mathbf{F} \cdot d\mathbf{R} \\
&= \int_{\theta=0}^{2\pi} (a\sin\theta\mathbf{i} + a\cos\theta\mathbf{j}) \cdot a(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\, d\theta \\
&= a^2 \int_{\theta=0}^{2\pi} (\cos\theta - \sin\theta)\, d\theta \\
&= a^2 \int_{\theta=0}^{2\pi} \cos 2\theta d\theta = 0
\end{aligned}$$



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4 Problem 4.9.17

Given $\phi(x, y, z) = xyz + 5$, find the surface integral of the normal component of $\nabla\phi$ over $x^2 + y^2 + z^2 = 9$.
Solution:
By the divergence theorem

$$\int\int_S \nabla\phi \cdot d\mathbf{S} = \int\int\int_V \nabla \cdot \nabla\phi dV = \int\int\int_V \triangle\phi dV \ .$$

The Laplacian of ϕ is

$$\nabla \cdot \nabla\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial x^2} = 0 \ ,$$

so that the integral vanishes.