Solutions, 311-XXII

April 16, 2003

21(4/ 8) Introduction to Divergence and Stokes theorems Sec. 4.(9) p.262(7,8*,12*,13,14*,16*)

1 Problem 4.9.8

Use Stokes' theorem to solve 4.1.6:
Find the integral $\oint \mathbf{F}\cdot\mathbf{R}$ around the circumference of the circle $x^2-2x+y^2=2$, $z=1$, where $\mathbf{F}=y\mathbf{i}+x\mathbf{j}+xyz^2\mathbf{k}$.
Solution:
By Stokes theorem we have

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \int\int_\Sigma (\nabla\times\mathbf{F})\cdot\mathbf{n}dS$$

where Σ is the interior of the circle $(x-1)^2+y^2=2$, $z=1$ with C its boundary, oriented clockwise as seen from the positive z-axis.

The curl of $\mathbf{F}$ is needed for the surface integral:

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & x & xyz^2 \end{vmatrix} = z^2\left(x\mathbf{i}-y\mathbf{j}\right)$$

while

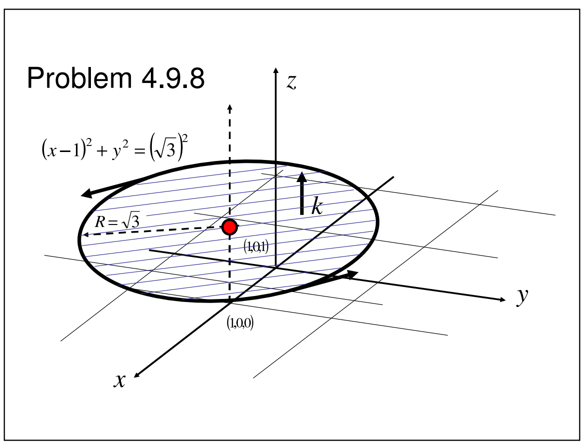
$$\mathbf{n}dS = \mathbf{k}dxdy \ ,$$

since the circle lies on the plane $z=1$, parallel to the xy plane. Then

$$\int\int_\Sigma (\nabla\times\mathbf{F})\cdot\mathbf{n}dS = 0 \ .$$

To get the same result without using Stokes' them, we evaluate the line integral directly. On the circle $z = 1$ and $d\mathbf{R} = 2(x-1)\mathbf{i}dx + 2y\mathbf{j}dy$, while $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + xy\mathbf{k}$. Converting to polars, $x = 1 + \sqrt{2}\cos\theta$, $y = \sqrt{2}\sin\theta$, with $dx = -\sqrt{2}\sin\theta d\theta$ and $dy = \sqrt{2}\cos\theta d\theta$.

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{R} &= \oint_C (y\mathbf{i} + x\mathbf{j} + xy\mathbf{k})\cdot(2(x-1)\mathbf{i}dx + 2y\mathbf{j}dy) \\
&= \oint_C (2y(x-1)dx + 2xydy) \\
&= \int_{\theta=0}^{2\pi} \left(-4\sqrt{2}\sin^2\theta\cos\theta + 4\sin\theta\cos\theta\left(1+\sqrt{2}\cos\theta\right)\right) d\theta \\
&= \int_{\theta=0}^{2\pi} \left(-4\sqrt{2}\cos\theta\sin^2\theta + 4\sqrt{2}\cos^2\theta\sin\theta + 4\cos\theta\sin\theta\right) d\theta \\
&= 0
\end{aligned}$$



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2 Problem 4.9.12

Let S be the portion of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 0$, and let $\mathbf{F} = (y-z)\mathbf{i} - (x+z)\mathbf{j} + (x+y)\mathbf{k}$. Find $\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$.
Solution:
By Stokes theorem we have

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \oint_C \mathbf{F} \cdot d\mathbf{R} = \int\int_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

where Σ is the interior of the circle $x^2 + y^2 = 9$, $z = 1$ on the x-y plane and C is the common boundary of S and Σ oriented clockwise as seen from the positive z-axis.

The curl of $\mathbf{F}$ is needed:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ (y-z) & -(x+z) & (x+y) \end{vmatrix} = 2(\mathbf{i} - \mathbf{j})$$

Any one of these integrals will give the desired answer. For practice I show all three:

1. The original integral $\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$.
   On S, we have that $z = f(x, y) = 9 - x^2 - y^2$. Then

$$\begin{aligned} \mathbf{n} dS &= \partial_x \mathbf{R} \times \partial_y \mathbf{R} dx dy = (\mathbf{i} - 2x\mathbf{k}) \times (\mathbf{j} - 2y\mathbf{k}) \, dx dy \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k} \end{aligned}$$

$$\begin{aligned} \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS &= \int\int_\Sigma 2(\mathbf{i} - \mathbf{j}) \cdot (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}) \, dx dy \\ &= 4 \int\int_\Sigma (x - y) dx dy = 0 \end{aligned}$$

   by the symmetry of the domain about $(x, y) = (0, 0)$ and the oddness of the integrand.

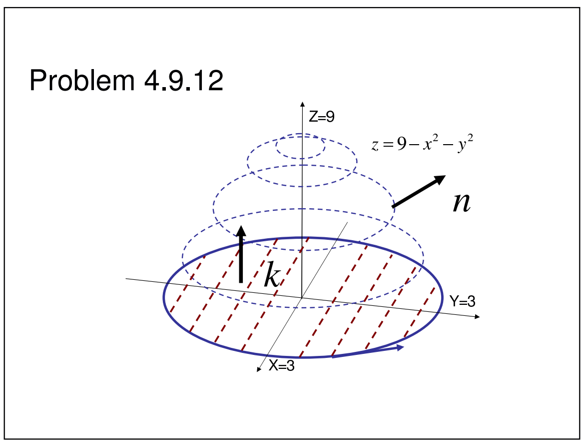
2. The integral over the circle $\int\int_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$.
   On the circle $\mathbf{n} dS = \mathbf{k} dx dw$, so that $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = 2(\mathbf{i} - \mathbf{j}) \cdot \mathbf{k} = 0$ so that

$$\int\int_\Sigma (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = 0 \ .$$

3. The line integral $\oint_C \mathbf{F} \cdot d\mathbf{R}$.
Now, since again $z = 0$ and $d\mathbf{R} = 2x\mathbf{i}dx + 2y\mathbf{j}dy$, we have (we will use polar coordinates, $x = 3\cos\theta$ and $y = 3\sin\theta$):

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{R} &= \oint_C ((y-z)\mathbf{i} - (x+z)\mathbf{j} + (x+y)\mathbf{k}) \cdot (2x\mathbf{i}dx + 2y\mathbf{j}dy) \\
&= \oint_C 2xy\,(dx - dy) = \int_{\theta=0}^{2\pi} 18\cos\theta\sin\theta\,(-3\sin\theta + 3\cos\theta)\,d\theta \\
&= 54\int_{\theta=0}^{2\pi} \left(\cos^2\theta\sin\theta - \cos\theta\sin^2\theta\right) d\theta \\
&= -18\left(\cos^3\theta + \sin^3\theta\right)\Big|_{\theta=0}^{2\pi} = 0
\end{aligned}$$

Thus, all versions produce the same answer, 0, and in this case the easiest computation was the surface integral over the planar surface (circle Σ).



3 Problem 4.9.14

Use Stokes' theorem to evaluate

$$\int_C \left[x \sin y \mathbf{i} - y \sin x \mathbf{j} + (x+y) z^2 \mathbf{k} \right] \cdot d\mathbf{R}$$

along the path consisting of straight-line segments successively joining the points $P_0 = (0,0,0)$ to $P_1 = (\pi/2, 0, 0)$ to $P_2 = (\pi/2, 0, 1)$ to $P_3 = (0,0,1)$ to $P_4 = (0, \pi/2, 1)$ to $P_5 = (0, \pi/2, 0)$ and back to $P_0 = (0,0,0)$.

Solution:

We will use Stokes' theorem so we can convert to a surface integral over the two planar rectangles one with vertices P_0, P_1, P_2, P_3 on the $y = 0$ plane with unit normal $\mathbf{j}$, the other with vertices P_3, P_4, P_5, P_0 on the $x = 0$ plane with unit normal $\mathbf{i}$. We need the curl of $\mathbf{F} := x \sin y \mathbf{i} - y \sin x \mathbf{j} + (x+y) z^2 \mathbf{k}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x \sin y & -y \sin x & (x+y) z^2 \end{vmatrix} = z^2 (\mathbf{i} - \mathbf{j}) - (y \cos x + x \cos y) \mathbf{k} \ .$$

Then

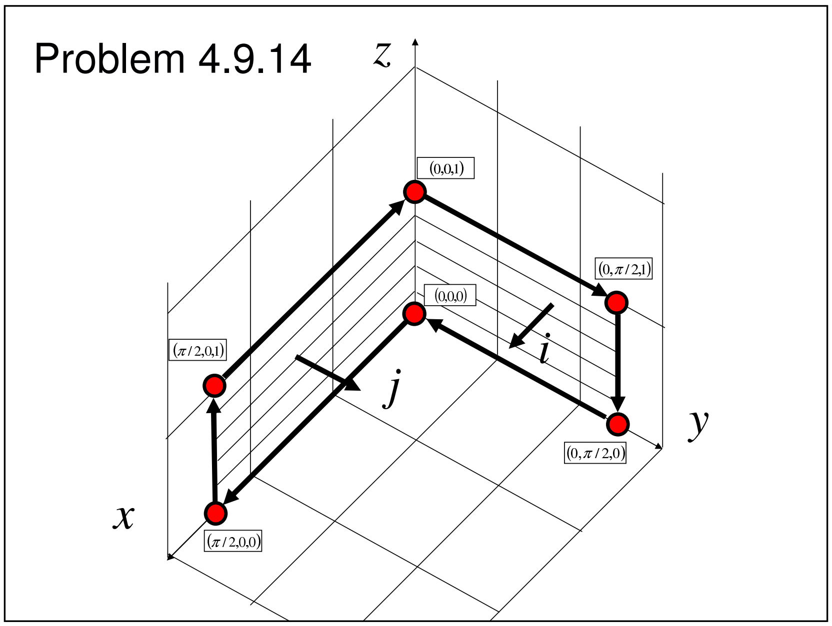
$$\nabla \times \mathbf{F} \cdot \mathbf{j}\big|_{y=0} = -z^2 \ ,$$

and

$$\nabla \times \mathbf{F} \cdot \mathbf{i}\big|_{x=0} = z^2 \ .$$

so that

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{R} &= -\int\int_{xz} z^2 dx dz + \int\int_{yz} z^2 dy dz \\ &= -\int_{x=0}^{2} dx \int_{z=0}^{1} z^2 dz + \int_{y=0}^{\pi/2} \int_{z=0}^{1} z^2 dz \\ &= -x\big|_0^{\pi/2} \frac{z^3}{3}\bigg|_0^1 + y\big|_0^{\pi/2} \frac{z^3}{3}\bigg|_0^1 \\ &= -\frac{\pi}{6} + \frac{\pi}{6} = 0 \ . \end{aligned}$$


Problem 4.9.14
z
(0,0,1)
(0,π/2,1)
(0,0,0)
i
(π/2,0,1)
j
y
(0,π/2,0)
x
(π/2,0,0)
1

4 Problem 4.9.16

If $\mathbf{F} = xz\mathbf{i} - y\mathbf{j} + x^2y\mathbf{k}$, use Stokes' theorem to evaluate $\int_C \mathbf{F} \cdot \mathbf{R}$, where C is the closed path consisting of the edges of the triangle with vertices at the points $P_1 = (1,0,0)$, $P_2 = (0,0,1)$, $P_3 = (0,0,0)$ transversed from P_1 to P_2 to P_3, and back to P_1.

Solution:

We will use Stokes' theorem so we can convert to a surface integral over the surface of the triangle with vertices P_0, P_1, P_2 on the $y = 0$ plane with unit normal $\mathbf{j}$. We need the curl of $\mathbf{F}$:

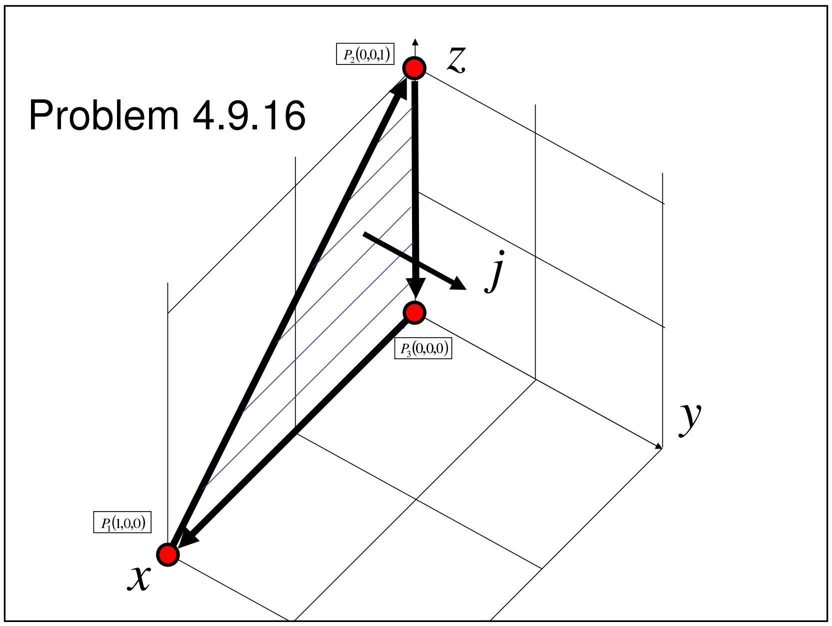
$$\nabla \times \mathbf{F} \quad = \quad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ xz & -y & x^2y \end{vmatrix} = x^2\mathbf{i} + x(1-2y)\mathbf{j} \ .$$

Then

$$\nabla \times \mathbf{F} \cdot \mathbf{j}\big|_{y=0} = x \ ,$$

and we have

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{R} &= \int\int_{xz} x dx dz \\ &= \int_{x=0}^{1} dx x \int_{z=0}^{1-x} dz \\ &= \int_{x=0}^{1} dx x(1-x) \\ &= \frac{x^2}{2} - \frac{x^3}{3}\bigg|_0^1 \\ &= \frac{1}{6} \end{aligned}$$



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