Math. 311 # 31

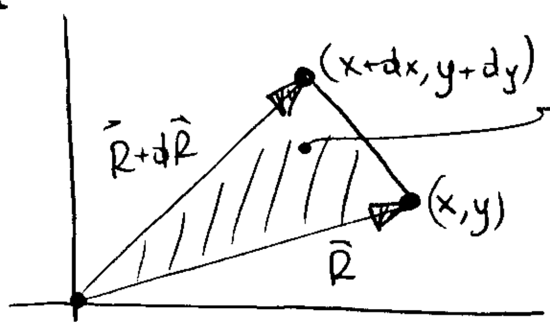
5.4(5,6,7,10)

(5) Let $\vec{R} = x\vec{i} + y\vec{j}$ and $d\vec{R} = dx\,\vec{i} + dy\,\vec{j}$

(a) Compute the mag. of the cross product $\vec{R} \times (\vec{R} + d\vec{R})$

(b) Thus interpret $\int_C \frac{1}{2}(x\,dy - y\,dx)$ geometrically.

(a)



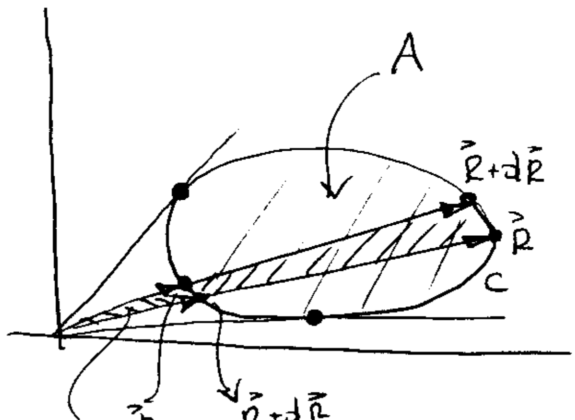
$$\vec{\text{Area}} = \frac{1}{2}\vec{R} \times (\vec{R} + d\vec{R}) = \frac{1}{2}\vec{R} \times d\vec{R}$$

$$= \frac{1}{2}(x\,dy - y\,dx)\,\vec{k}$$

Thus

$$\frac{1}{2}\oint_C \vec{R} \times d\vec{R} = A\vec{k}$$ (oriented area enclosed by C.

$\therefore$ $\frac{1}{2}\oint(x\,dy - y\,dx)$ gives area enclosed by curve.



this segment gets subtracted when we work on lower part of the contour

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5.4.6) $\vec{F} = x\vec{i} + y\vec{j}$, C oriented closed curve enclosing A.
what is $\oint_C F \cdot T\,ds$?

$$\oint F \cdot d\vec{R} = \oint F \cdot \frac{d\vec{R}}{ds}\,ds = \boxed{\oint F \cdot \vec{T}\,ds = 0}$$

$$\| $$

$$\int F_1\,dx + F_2\,dy = \iint_A \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy = 0$$

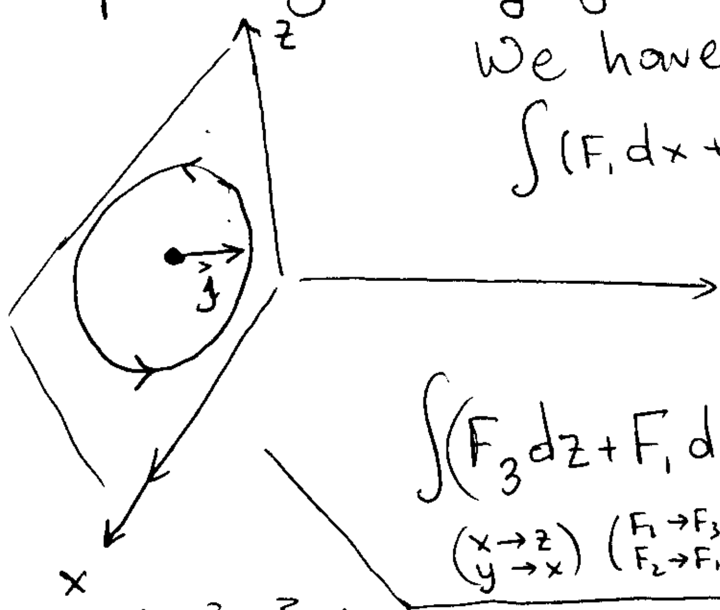
$$\frac{\partial F_2}{\partial x} = 0, \quad \frac{\partial F_1}{\partial y} = 0$$

$\boxed{5.4.7}$ Let C denote the circle $x^2+y^2=9$, let $\vec{F} = y\vec{i} - 3x\vec{j}$. What is the line integral of the tangential component of $\vec{F}$ around C ($\circlearrowleft$).

$$\oint F \cdot d\vec{R} = \iint_A \Big(\underbrace{\frac{\partial F_2}{\partial x}}_{=-3} - \underbrace{\frac{\partial F_1}{\partial y}}_{=1}\Big) dx\,dy = -4 \iint_A dx\,dy = -36\pi$$

$\hookrightarrow$ area $= 9\pi$.

$\boxed{5.4.10}$ $\vec{F} = 4z\vec{i} - 3x\vec{k}$. Compute the line integral of the tangential component of $\vec{F}$ about the circle $(x-5)^2 + (\cancel{z}-\cancel{7})^2 = 4$ in xz plane. Orient the plane by taking $\vec{j}$ to be unit normal.



We have

$$\int (F_1\,dx + F_2\,dy) = \iint \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big) dx\,dy$$

Cyclic: permutation: $x, F_1 \to z, F_3$; y, F_2

$$\int (F_3\,dz + F_1\,dx) = \iint \Big(\underbrace{\frac{\partial F_1}{\partial z}}_{=4} - \underbrace{\frac{\partial F_3}{\partial x}}_{=3}\Big) dz\,dx = 28\pi$$

$\binom{x \to z}{y \to x}$ $\binom{F_1 \to F_3}{F_2 \to F_1}$

$$\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \cancel{\partial_y} & \partial_z \\ F_1 & \underset{=0}{F_2} & F_3 \end{vmatrix} = \vec{j}\Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)$$

Or:

From Stokes thm: $\oint F \cdot dR = \iint (\nabla \times F) \cdot \vec{n}\, dS = \iint (\nabla \times F) \cdot \vec{j}\, dS$

$$= \iint \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big) dx\,dz = \iint_{A = 4\pi} (4 + 3)\, dS = 28\pi$$