Math. 311 | (1)

\# 32

p. 299 (1, 2, 3, 4)

1. Given the vector field $\mathbf{F} = 3y\mathbf{i} + (5 - 2x)\mathbf{j} + (z^2 - 2)\mathbf{k}$, find

(a) div **F**.

(b) curl **F**.

(c) the surface integral of the normal component of curl **F** over the open hemispherical surface $x^2 + y^2 + z^2 = 4$ above the xy plane.

[*Hint:* By a double application of Stokes' theorem, part (c) can be reduced to a triviality.]

2. Given that curl $\mathbf{F} = 2y\mathbf{i} - 2z\mathbf{j} + 3\mathbf{k}$, find the surface integral of the normal component of curl **F** (*not* **F**) over

(a) the open hemispherical surface $x^2 + y^2 + z^2 = 9$, $z > 0$.

(b) the sphere $x^2 + y^2 + z^2 = 9$.

(In both parts, you should be able to write the answer down by inspection.)

3. Prove $\iint_S \nabla\phi \times \nabla\psi \cdot d\mathbf{S} = \oint_C \phi\nabla\psi \cdot d\mathbf{R}$.

4. Show why the positive orientations on C and Γ, in the proof of Stokes' theorem, correspond. (*Hint:* Reread the beginning of section 4.6.)

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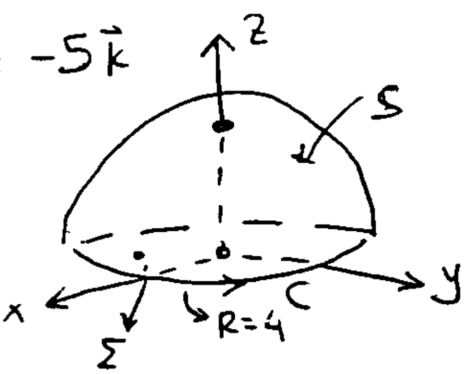
(1) $\vec{F} = 3y\vec{i} + (5-2x)\vec{j} + (z^2-2)\vec{k}$

$\nabla\cdot\vec{F} = 2z$, $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 3y & 5-2x & z^2-2 \end{vmatrix} = \vec{k}(-2-3) = -5\vec{k}$



(1) $I = \iint_S \nabla\times F\cdot d\vec{S} = \oint_C F\cdot d\vec{R} = \iint_\Sigma \nabla\times F\cdot d\vec{S}$

But $\nabla\times F = -5\vec{k}$, $d\vec{S} = \vec{k}\,dS$ on Σ,

so $I = -5\iint dS = -5\cdot\pi 4 = \underline{\underline{-20\pi}}$

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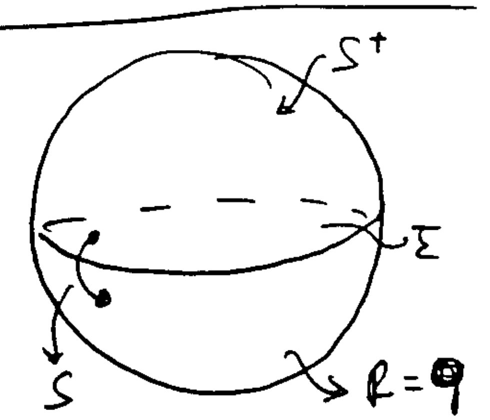
(2) $\nabla\times F = 2y\vec{i} - 2z\vec{j} + 3\vec{k}$

Over the sphere: $\oiint_{\text{sphere } S} \nabla\times F\cdot dS = \iiint \overbrace{\nabla\cdot(\nabla\times F)}^{0}\,dV = 0$



For the hemisphere:

$\iint_{\substack{\text{hemi}\\ S^+}} \nabla\times F\cdot d\vec{S} = \iint_{\substack{\Sigma\\ \text{circle}}} \nabla\times F\cdot\vec{k}\,dS = 3\iint_{\substack{\Sigma,\\ \text{radius } 3}} dS = 3\cdot\pi 9$

$= \underline{\underline{27\pi}}$

$$\iint_S \nabla\phi\times\nabla\psi\cdot dS = \oint \phi \cdot$$ by Stokes

Since $\nabla\times(\phi\nabla\psi) = \nabla\phi\times\nabla\psi$ (since $\nabla\times\nabla\psi = 0$)

(in general: $\nabla(\phi\vec{F}) = \nabla\phi\times\vec{F} + \phi\,\nabla\times\vec{F}$)

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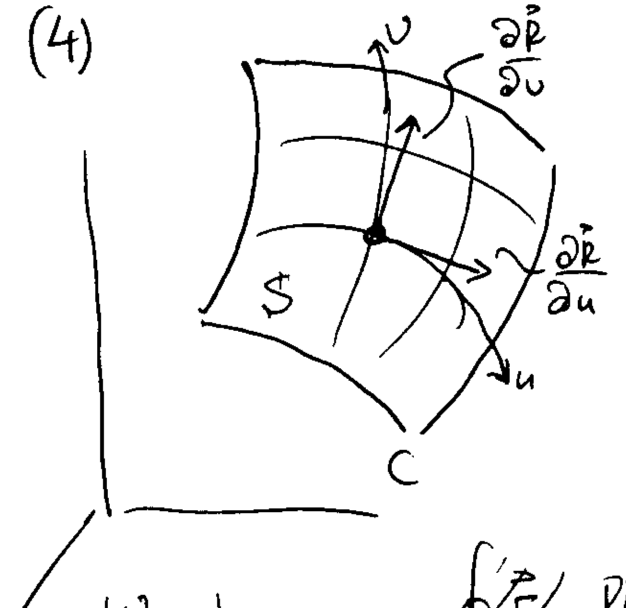
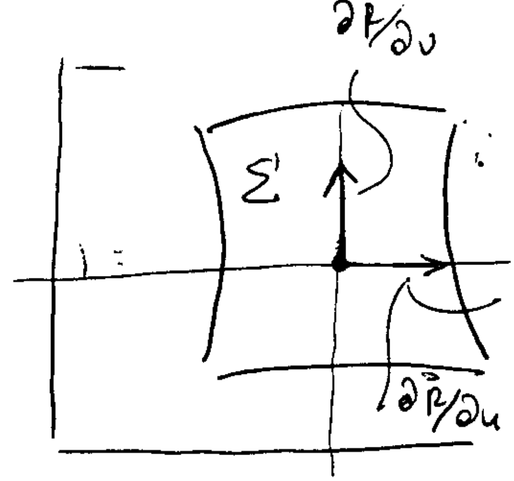
(4)





We have $\oint_C \vec{F}\cdot d\vec{R}$ on orientation chosen on

(any $\vec{F}$) C by parametrizing enclosed surface S

via u, v: $\vec{R} = R(u,v)$. Orientation is imposed

by choosing the normal $\vec{N} = \frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v}$.

This orientation is carried over to Σ', which

is the image of S under the mapping $\vec{R} = \vec{R}(u,v)$

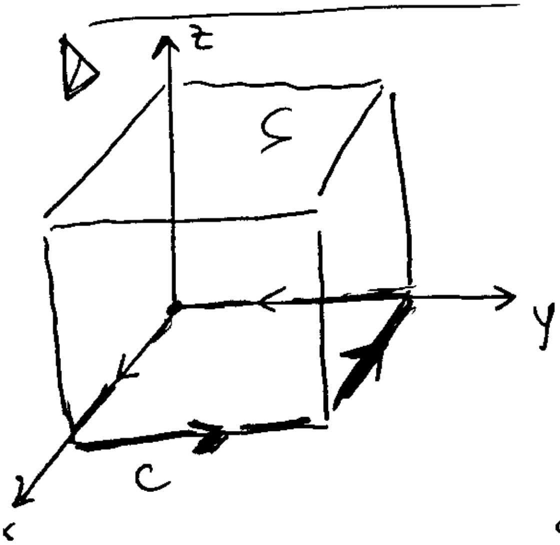
($\vec{R} \in R^3$ while $\binom{u}{v} \in R^2$). In that sense,

~~that~~ the two orientations are the same.

Math. 311
\# 33

Schaum's p. 134
63, 65

p. 134, 63 Verify Stokes' thm. for $\vec{A} = (y-z+2)\vec{i} + (yz+4)\vec{j} - xz\vec{k}$ where S is the surface of the cube $x=y=z=2$, $x=y=z=0$, over the xy plane



$$\nabla \times A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ y-z+2 & yz+4 & -xz \end{vmatrix} = \begin{matrix} \vec{i}(y) \\ -\vec{j}(-z+1) \\ +\vec{k}(-1) \end{matrix}$$

$$= y\vec{i} + (1-z)\vec{j} - \vec{k}$$

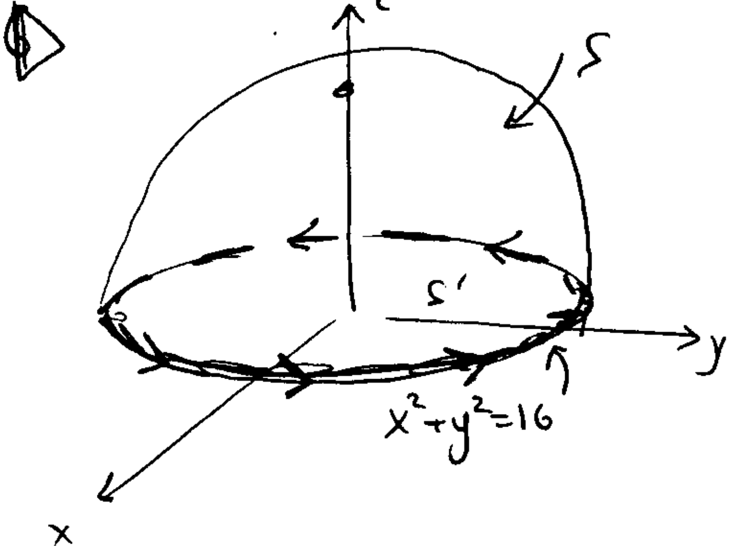
$$\underset{\substack{\text{surf. } S \\ \text{over } xy \text{ plane}}}{\iint} \nabla \times A \cdot dS = \underset{\substack{\text{square} \\ \text{on } xy \text{ plane}}}{\iint} \nabla \times A \cdot dS$$

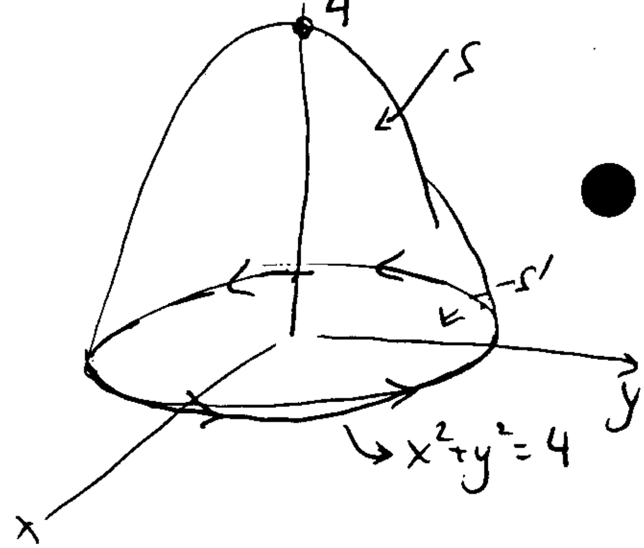
$$= \int_{x=0}^{2}\int_{y=0}^{2} -1\,dx\,dy = -4$$

$$\oint_C A \cdot d\vec{R} = \int_{\substack{x=0 \\ (y=0,\ z=0)}}^{1} (y-z+2)\,dx + \int_{\substack{y=0 \\ (x=2,\ z=0)}}^{1} (yz+4)\,dy + \int_{\substack{x=1 \\ (y=2,\ z=0)}}^{0} (y-z+2)\,dx + \int_{\substack{y=1 \\ (x=0,\ z=0)}}^{0} (yz+4)\,dy$$

$$= \int_{x=0}^{2} 2\,dx + \int_{y=0}^{2} 4\,dy - \int_{x=0}^{2} 4\,dx - \int_{0}^{2} 4\,dy = 4+8-8-8 = -4$$

p. 134, 65 Evaluate $\iint_S (\nabla \times A)\cdot n\,dS$ where $\vec{A} = (x^2+y-4)\vec{i} + 3xy\vec{j} + (3xz+z^2)\vec{k}$ and S is the surface of (a) the hemisphere $x^2+y^2+z^2=16$ above the plane, (b) the paraboloid $z = 4-(x^2+y^2)$ above the plane.





$$\iint_S (\nabla \times A)\cdot u\,dS = \iint_{S'} \nabla\times A \cdot u\, dS = \oint_C A\cdot dR$$

$$\nabla\times\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix} = \vec{i}(0) - \vec{j}(2z) + \vec{k}(3y-1)$$

$$\iint_{S'} \nabla\times A\cdot u\,dS = \iint_{S'} (3y-1)\,dx\,dy = -\underbrace{\iint_{S'} dS}_{\text{(area of circle on xy plane)}} = \begin{cases} -16\pi \\ -4\pi \end{cases}$$

Idea in both problems (63, 65):

$$\iint_S (\nabla\times A)\cdot u\,dS = \oint_C A\cdot d\vec{R}$$

, so surface integral is the same for any surface S bound by the same closed loop C.

(of course this can also be deduced from divergence thm. since $\nabla\cdot(\nabla\times\vec{A}) \equiv 0$)